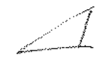
16.6 Surface area

Recall arc length:

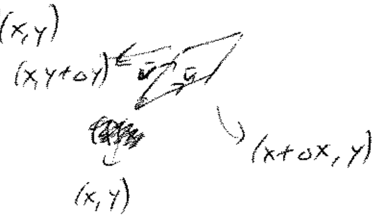
$$ds = \text{(formally)} \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$



$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

For surface area, instead of infinitesimal line segments, have infinitesimal parallelograms.

$z = f(x,y)$



$$\bar{u} = \langle \Delta x, 0, f_x \Delta x \rangle$$
$$\bar{v} = \langle 0, \Delta y, f_y \Delta y \rangle$$

$$\text{area of parallelogram} = |\bar{u} \times \bar{v}|$$

$$= \left| \det \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix} \right| = \Big| \bar{i}\,[-f_x \Delta x \Delta y] - \bar{j}\,[f_y \Delta x \Delta y] + \bar{k}\,[\Delta x \Delta y] \Big|$$

$$= \left[1 + (f_x)^2 + (f_y)^2\right]^{1/2} \Delta x \Delta y$$

Surface area of surface $z = f(x,y)$ over region S

$$= \iint_S \left[1 + (f_x)^2 + (f_y)^2\right]^{1/2} dx\,dy$$

Sanity check:

Suppose $z = \text{constant}$,
then surface area = area of S.

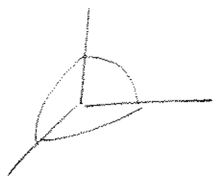
<u>Check</u>:

$$f(x,y) = \text{constant} \Rightarrow f_x = f_y = 0$$

$$\text{surface area} = \iint_S \left[1 + (f_x)^2 + (f_y)^2\right]^{1/2} dA$$

$$= \iint_S (1)\, dA = \text{area of } S$$

(Classic) **Ex** Find surface area of a hemisphere of radius R.

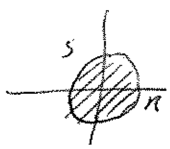


$\to$ half of a sphere

$$z = +\sqrt{R^2 - x^2 - y^2} = f(x,y)$$

Surface area $= \iint_S \left[1 + (f_x)^2 + (f_y)^2 \right]^{1/2} dA$

where S is the disk in the plane that the hemisphere projects onto.



$$f_x = \tfrac{1}{2}(R^2 - x^2 - y^2)^{-1/2}(-2x)$$

$$f_y = \tfrac{1}{2}(R^2 - x^2 - y^2)^{-1/2}(-2y)$$

$$1 + (f_x)^2 + (f_y)^2 = 1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}$$

$$= (R^2 - x^2 - y^2)^{-1}\left[(R^2 - \cancel{x^2} - \cancel{y^2}) + \cancel{x^2} + \cancel{y^2} \right]$$

$$= \frac{R^2}{R^2 - x^2 - y^2}$$

Surface area

$$= \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \frac{R}{[R^2 - x^2 - y^2]^{1/2}} \, dy \, dx$$

Convert to polar:

$$= \int_0^{2\pi} \int_0^R \frac{R}{(R^2 - r^2)^{1/2}} \, r \, dr \, d\theta$$

$$= R \int_0^{2\pi} (-\tfrac{1}{2})(2)(R^2 - r^2)^{1/2} \Big|_0^R \, d\theta = R \int_0^{2\pi} (-)\left[0 - R \right] d\theta$$

$$= +R^2 \int_0^{2\pi} d\theta = 2\pi R^2$$

Recall surface area of sphere of radius $R = 4\pi R^2 = 2 \times$ result done ✓

Ex Find surface area of the part of the paraboloid $z = R^2 - x^2 - y^2$ that is above the xy plane.

$$f(x,y) = R^2 - x^2 - y^2$$

Compute surface area over region S

$$S.A. = \iint_S \left[1 + (f_x)^2 + (f_y)^2\right]^{1/2} dA$$

$$= \int_{-R}^{R} \int_{-\sqrt{R-x^2}}^{+\sqrt{R-x^2}} \left[1 + (-2x)^2 + (-2y)^2\right]^{1/2} dy\, dx$$

Switch to polar:

$$= \int_0^{2\pi} \int_0^R \left[1 + 4r^2\right]^{1/2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8} \frac{2}{3} \left[1 + 4r^2\right]^{3/2} \Big|_0^R \, d\theta$$

$$= \underline{\frac{1}{4} \frac{1}{3} (2\pi) \left[(1 + 4R^2)^{3/2} - 1 \right]}$$

Start here Mon
Wed

17.2 Line integrals

We'll generalize ordinary definite integral $\int_a^b f(x)\,dx$

to an integral along a curve C : $\int_C f(x,y)\,ds$

→ "line integral"

Let C be a smooth plane curve:

$$x = x(t), \quad y = y(t) \qquad a \le t \le b$$

where $x'(t), y'(t)$ are continuous & not simultaneously 0.

~~Suppose C is positively oriented &~~

Partition $[a,b]$ into n int's, define

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s$$

Using $ds = \sqrt{(x')^2+(y')^2}\,dt$, in 2D,
have

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{(x')^2+(y')^2}\,dt$$

so $\int_C (1)\,ds$ = arc length

In 3D, $\displaystyle\int_a^b f(x(t), y(t), z(t))\left[(x')^2+(y')^2+(z')^2\right]^{1/2} dt$

Ex $\int_C x^2\,ds$, where C is $x = R\cos t,\ y = R\sin t,\quad 0 \le t \le \pi$

$$ds = \left[(-R\sin t)^2 + (R\cos t)^2\right]^{1/2} dt \quad = R\,dt$$

$$= \int_0^{\pi} (R\cos t)^2 (R\,dt) \quad = \quad R^3 \int_0^{\pi} \cos^2 t\,dt$$

$$= \tfrac{1}{2} R^3 \int_0^{\pi} [1 + \cos 2t)\,dt$$

$$= \tfrac{\pi}{2} R^3$$

Above ex = mass of a ~~disc~~ wire bent in half-circle
w/ density $= x^2$.

Ex A wire is bent in the shape of a helix:
$x = R\cos t,\quad y = R\sin t,\qquad z = ht$

& has mass density $\delta(x,y,z) = \alpha z$.
Compute ~~total~~ mass between $t = 0,\ T$.

$$\text{Mass} = \int_C \delta(x,y,z)\,ds \quad = \int_0^T (\alpha h t)\ \left[(-R\sin t)^2 + (R\cos t)^2 + h^2\right]^{1/2} dt$$

$$= \int_0^T \alpha h t \left[R^2 + h^2\right]^{1/2} dt \quad = \quad \alpha h \left[R^2 + h^2\right]^{1/2} \left(\tfrac{1}{2} T^2\right)$$

Work in physics

$$\equiv \int_C \bar{F} \cdot \bar{T}\, ds \quad \text{where } \bar{T} = \text{unit tangent vector}$$
$$= \frac{d\bar{r}/dt}{|d\bar{r}/dt|}$$

Since $ds = \left|\frac{d\bar{r}}{dt}\right| dt$,

$$\bar{T}\, ds = \left(\frac{d\bar{r}/dt}{\cancel{|d\bar{r}/dt|}}\right) \cancel{|d\bar{r}/dt|}\, dt$$
$$= \left(\frac{d\bar{r}}{dt}\right) dt$$

$$\text{Work} = \int_C \bar{F} \cdot \left(\frac{d\bar{r}}{dt}\right) dt = \int_C \bar{F} \cdot d\bar{r}$$

So, if $\bar{F} = M\bar{i} + N\bar{j} + P\bar{k}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

so $$W = \int_C \bar{F} \cdot d\bar{r} = \int_C \left(M\,dx + N\,dy + P\,dz\right)$$

Ex Consider $F = x\bar{i} + y\bar{j}$
pushed along the helix
$x = R\cos t, \quad y = R\sin t, \quad z = ht$

$\Rightarrow dx = -R\sin t\, dt, \quad dy = R\cos t\, dt, \quad dz = h\, dt$

so $$\bar{F} \cdot d\bar{r} = x\,dx + y\,dy$$
$$= (R\cos t)(-R\sin t\, dt) + (R\sin t)(R\cos t\, dt)$$
$$= 0$$

so work $= 0$.

Ex Consider the curve $\quad x = t, \ y = \frac{1}{2}t^2, \quad t \in [0,1]$

& on that curve evaluate

$$\int_C \left[(x^2+y^2)dx + y\,dy\right]$$
$$= \int_0^1 \Big[(t^2 + \tfrac{1}{4}t^2)\underbrace{(dt)}_{dx} + (\tfrac{1}{2}t^2)\underbrace{(t\,dt)}_{dy} \Big]$$
$$= \Big[\tfrac{1}{3}t^3 + \tfrac{1}{4}\tfrac{1}{3}t^3 + \tfrac{1}{2}\tfrac{1}{4}t^4 \Big]_0^1$$
$$= \tfrac{1}{3}(1 + \tfrac{1}{4}) + \tfrac{1}{8} = \tfrac{1}{3}\tfrac{5}{4} + \tfrac{1}{2}\tfrac{1}{4} = \tfrac{1}{4}\left(\tfrac{5}{3} + \tfrac{1}{2}\right) = \tfrac{1}{4}\left(\tfrac{10}{6} + \tfrac{3}{6}\right)$$
$$= \tfrac{1}{4}\left(\tfrac{13}{6}\right)$$

§ 17.3 Independence of path

Recall 2nd fundamental thm of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

There is an analogue for line integrals, as we'll see here.

Thm Fundamental theorem for line integrals

Let C be a piecewise smooth curve given parametrically by $\bar{r} = \bar{r}(t)$, $t \in [a,b]$.
If f is continuously differentiable on an open set containing C, then

$$\int_C \nabla f \cdot d\bar{r} = f(\bar{b}) - f(\bar{a}) \quad \text{where } \begin{array}{l} \bar{a} = \bar{r}(a) \\ \bar{b} = \bar{r}(b) \end{array}$$

Check

$$\int_C \nabla f \cdot d\bar{r} = \int_a^b (\nabla f \cdot \bar{r}')\,dt = \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{df}{dt}\,dt = f(b) - f(a)$$

Note $\int_C \nabla f \cdot d\bar{r}$ depends only on endpts,

not on the path itself.

→ "independence of path"
"path independent"

<u>Criteria for path independence:</u>

Let $\bar{F}(\bar{r})$ be continuous on some open connected set D. Then the line integral

$$\int_C \bar{F} \cdot d\bar{r} \quad \text{is independent of path}$$

if & only if

$$\bar{F} = \nabla f \quad \text{for some } f \quad (\text{ie, } \bar{F} \text{ is a conservative vector field})$$

We've already seen $\Leftarrow$.

$\Rightarrow$: Build f as follows.

Let (x_0, y_0) be any fixed pt of D.

Define $f(x,y) = \int_{C\,(x_0,y_0)}^{(x,y)} \bar{F} \cdot d\bar{r}$ for <u>any</u> path C (recall path-ind, so well-defined)

In particular, for a "right-angle path",

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y_0)} \bar{F} \cdot d\bar{r} + \int_{(x,y_0)}^{(x,y)} \bar{F} \cdot d\bar{r}$$

$$= \int_{(x_0,y_0)}^{(x,y_0)} F_1\, dx + \int_{(x,y_0)}^{(x,y)} F_2\, dy = \int_{x_0}^{x} F_1(t, y_0)\, dt + \int_{y_0}^{y} F_2(x,t)\, dt$$

Show $\partial_x f = F_1$:

Pick a pt (x_1, y_1).

$$f(x,y) = \int_{(x_0,y_0)}^{(x_1,y)} \bar{F} \cdot d\bar{r} + \int_{(x_1,y)}^{(x,y)} \bar{F} \cdot d\bar{r} = \underbrace{\int_{(x_0,y_0)}^{(x_1,y)} \bar{F} \cdot d\bar{r}}_{\text{ind' of } x} + \int_{x_1}^{x} F_1(t,y)\, dt$$

so $\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int^{x} F_1(t,y)\, dt = F_1(x,y)$ ✓ & analogously for $\partial_y f(x,y) = F_2$

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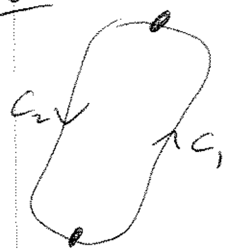
Note ~~this means~~ path independence

$\Rightarrow$ if C is any closed oriented curve then $\int_C \bar{F} \cdot d\bar{r} = 0$.

Check



$C_1 \cup C_2 = C$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r}$$

$$= \int_{C_1} \bar{F} \cdot d\bar{r} - \int_{-C_2} \bar{F} \cdot d\bar{r}$$

but these two are the same,

$\Rightarrow \; = 0.$

Alternately, shrink C to a point.

Conversely, can show if $\int_C \bar{F} \cdot d\bar{r} = 0$ for every closed C, then $\int \bar{F} \cdot d\bar{r}$ is path independent.

Summary TFAE:

- $\bar{F} = \nabla f$
- $\int_C \bar{F} \cdot d\bar{r}$ is independent of path
- $\int_C \bar{F} \cdot d\bar{r} = 0$ for every closed path

Thm Let $\bar{F} = \langle M, N, P \rangle$ where M, N, P & their 1^{st} deriv's are continuous on an open connected simply-connected set D.

Then $\bar{F}$ is conservative $(\bar{F} = \nabla f)$ iff curl $\bar{F} = 0$,

ie, iff $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}, \quad \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}, \quad \dfrac{\partial N}{\partial z} = \dfrac{\partial P}{\partial y}$

As a special case, if $\bar{F} = M\bar{i} + N\bar{j}$,
then F conservative iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

$\Rightarrow$: will check on homework
(follows from equality of 2^{nd} partial)

$\Leftarrow$: We'll learn how to construct f's next....

Ex $\quad \bar{F} = (9xy^2)\bar{i} + (9x^2y)\bar{j}$ is conservative?

$$\frac{\partial}{\partial y}(9xy^2) = 18xy = \frac{\partial}{\partial x}(9x^2y)$$

Find f:

$$\frac{\partial f}{\partial x} = 9xy^2$$

$$\Rightarrow f(x,y) = \int dx\,(9xy^2) = \tfrac{9}{2}x^2y^2 + C_1(y)$$

Demand $\dfrac{\partial f}{\partial y} = 9x^2y$

$$= 9x^2y + C_1'(y) \quad \Rightarrow C_1'(y) = 0$$

so $f(x,y) = 9x^2y + \underset{\hookrightarrow \text{constant}}{C}$

Then $\displaystyle\int_{(0,0)}^{(1,1)} \bar{F}\cdot d\bar{r} = f(1,1) - f(0,0)$

$$= [9(1)^2(1) + \cancel{C}] - [(0) + \cancel{C}] = \underline{9}$$

Ex let $\bar{F} = \langle 8x^3y, \; 2x^4 + y^4 \rangle$, show conservative, find f & calc' $\int \nabla F \cdot d\bar{r}$.

$$\frac{\partial}{\partial y}(8x^3y) = 8x^3$$

$$\frac{\partial}{\partial x}(2x^4 + y^4) = 8x^3$$

conservative ✓

Find f:

$$\frac{\partial f}{\partial x} = 8x^3y \Rightarrow f(x,y) = \int dx\, (8x^3y) = 2x^4y + C_1(y)$$

$$\frac{\partial f}{\partial y} = 2x^4 + C_1'(y)$$

also $= 2x^4 + y^4$

$$\Rightarrow C_1'(y) = y^4 \Rightarrow C_1(y) = \frac{1}{5}y^5 + \underbrace{C_2}_{\text{const}}$$

$$\Rightarrow f(x,y) = 2x^4 + \frac{1}{5}y^5 + C_2$$

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \bar{F} \cdot d\bar{r} = f(x_1,y_1) - f(x_0,y_0)$$

$$= 2x_1^4 + \frac{1}{5}y_1^5 - 2x_0^4 - \frac{1}{5}y_0^5$$

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§17.4 Green's theorem in the plane

We're going to generalize the 2nd fundamental theorem of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Green's theorem

Let C be a piecewise smooth, simple closed curve that forms the boundary of a region S in the xy plane.
If $M(x,y)$, $N(x,y)$ are continuous w/ continuous partial derivatives, then

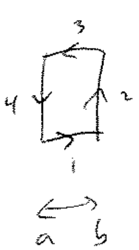
$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C \left[M\,dx + N\,dy\right]$$

Note Orientation of C important – traverse counterclockwise

~~Check for a rectangle:~~

Note if $\bar{F} = \langle M, N\rangle$ is conservative, then LHS $= 0$,
as expected $\to \int_C \bar{F}\cdot d\bar{r}$ should $= 0$

Check for a rectangle $[a,b]\times[c,d]$



$$\oint_C (M\,dx + N\,dy) = \int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} M\,dx + \int_{C_4} N\,dy$$ (using $dx = 0$ on vert, $dy = 0$ on hor)

$$= \int_a^b M(x,c)\,dx + \int_b^a M(x,d)\,dx + \int_c^d N(b,y)\,dy + \int_d^c N(a,y)\,dy$$

$$= \int_a^b \left[M(x,c) - M(x,d)\right] dx + \int_c^d \left[N(b,y) - N(a,y)\right] dy$$

$$= -\int_a^b \int_c^d \frac{\partial M}{\partial y}\,dy\,dx + \int_c^d \int_a^b \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$ ✓

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Ex $\cancel{\oint_C x^2\,dx + \ldots\,dy}$

$\int_C (y\,dx - x\,dy)$ on the circle $x = R\cos t$, $y = R\sin t$ $\quad t \in [0, 2\pi]$

Evaluate in 2 ways:

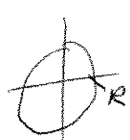
① Directly:

$$dx = -R\sin t\,dt, \qquad dy = R\cos t\,dt$$

$$\int_C (y\,dx - x\,dy) = \int_0^{2\pi} dt\left[(R\sin t)(-R\sin t) - (R\cos t)(R\cos t)\right]$$

$$= -R^2\int_0^{2\pi} dt\left[\sin^2 t + \cos^2 t\right] = -2\pi R^2$$

② Green's theorem:

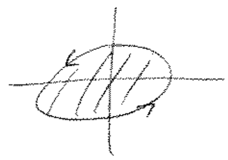


$$\iint_S\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_S\big((-1) - (1)\big)\,dA$$

$$= -2\iint_S dA$$

$= -2\,(\pi R^2)$ since double integral calculates area.

Classic Ex Use Green's theorem to compute the area of an ellipse.



$$x = a\cos t$$
$$y = b\sin t$$
$$t \in [0, 2\pi]$$

Consider $\oint_C (-y\,dx + x\,dy)$

By Green's thm, $= \iint_S \left(\frac{\partial x}{\partial x} - \frac{\partial}{\partial y}(-y) \right) dA = 2\iint_S dA = 2(\text{area})$

$$\Rightarrow \text{Area} = \frac{1}{2}\oint_C (-y\,dx + x\,dy)$$

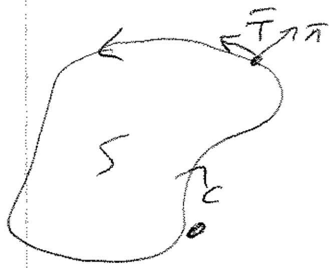
$$= \frac{1}{2}\int_0^{2\pi} \left[-(b\sin t)(-a\sin t\,dt) + (a\cos t)(b\cos t)\,dt \right]$$

$$= \frac{1}{2}ab\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \frac{1}{2}ab\,(2\pi) \qquad = \underline{\pi a b}$$

Note reduces to area of circle when $a = b$.

Let's restate Green's theorem in a more invariant-looking form.



$$\bar{T} = \frac{dx}{ds}\,\bar{\imath} + \frac{dy}{ds}\,\bar{\jmath}$$

$$\bar{n} = \frac{dy}{ds}\,\bar{\imath} - \frac{dx}{ds}\,\bar{\jmath}$$ is a unit normal (negative reciprocal slope; also, note $\bar{T}\cdot\bar{n} = 0$)

Let $\bar{F} = M(x,y)\,\bar{\imath} + N(x,y)\,\bar{\jmath}$ be a vector field.

$$\oint_C \bar{F}\cdot\bar{n}\,ds = \oint_C \left[M\frac{dy}{ds} - N\frac{dx}{ds}\right] ds$$

$$= \oint_C (M\,dy - N\,dx)$$

$$= \iint_S \left(\frac{\partial}{\partial x}(+M) - \frac{\partial}{\partial y}(-N)\right) dA$$

$$= \iint_S \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{=\nabla\cdot F} dA$$

$$\Rightarrow \boxed{\oint_C \bar{F}\cdot\bar{n}\,ds = \iint_S (\operatorname{div}\bar{F})\,dA = \iint_S (\nabla\cdot\bar{F})\,dA}$$

"Gauss's divergence theorem in the plane"
→ just a rephrasing of Green's theorem

Interpretation:

S'pose $\bar{F}$ = velocity vector of a fluid,
then
$$\oint_C \bar{F}\cdot\bar{n}\,ds = \text{"flux" of } \bar{F} \text{ across } C$$
→ measures amount of fluid leaving S

[after all, if $\bar{F}\cdot\bar{n} = 0$ at each pt on C,
then no fluid is moving across boundary,
only parallel to it.]

$$\oint \bar{F}\cdot\bar{n}\,ds = \iint_A (\operatorname{div}\bar{F})\,dA$$

⇝ div $\bar{F}$ measures sinks/sources

Another invariant form.

Write $\bar{F} = M\bar{i} + N\bar{j} + 0\bar{k}$

$$\oint_C \bar{F}\cdot\bar{T}\,ds = \oint \left(M\frac{dx}{ds} + N\frac{dy}{ds}\right)ds$$
$$= \oint (M\,dx + N\,dy)$$
$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \cancel{\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ M & N & 0 \end{vmatrix}} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$
$$= \bar{i}\left(-\frac{\partial N}{\partial z}\right) - \bar{j}\left(-\frac{\partial}{\partial z}M\right) + \bar{k}\left(\frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M\right)$$ but M, N ind' of z
$$= \bar{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

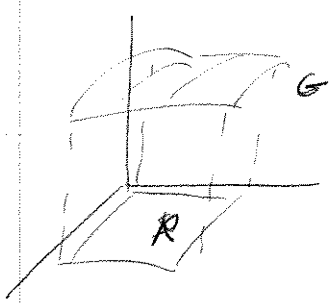
$$\cancel{\oint_C (M\,dx + N\,dy) = \iint_S (\nabla\times\bar{F})\cdot\bar{k}\,dA}$$

$$\oint_C \bar{F}\cdot\bar{T}\,ds = \iint_S (\nabla\times\bar{F})\cdot\bar{k}\,dA$$ → so the circulation of $\bar{F}$, ie, flow ∥ tangent, is measured by curl of $\bar{F}$.

"Stoke's Theorem in the plane"

§17.5 Surface integrals

Line integrals generalize ordinary definite integrals;
"surface integrals" generalize double integrals.



Line integral : $\int_C f(x,y,z)\underbrace{ds}_{\text{arc length}}$

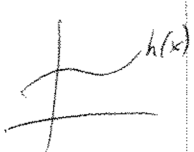
Surface integral $= \iint_G g(x,y,z)\underbrace{dS}_{\text{surface area}}$

S'pose G is the graph of $f(x,y)$ over a region R

then $dS = \left[1+f_x^{\,2}+f_y^{\,2}\right]^{1/2} dx\,dy$

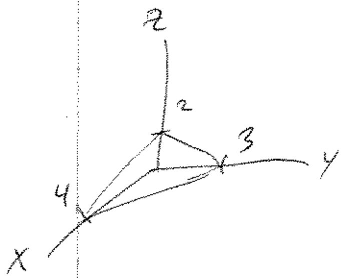
$$\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\left[1+f_x^{\,2}+f_y^{\,2}\right]^{1/2} dx\,dy$$

just as



$$\int_C f(x,y)\,ds = \int_C f(x,h(x))\,(1+(h')^2)\,dx$$

Ex Evaluate $\iint_S xyz\,dS$ where S = the part of the plane $3x+4y+6z=12$ that is ~~above the rectangle in the xy plane w/ vertices (0,0), (2,0), (2,1), (0,1)~~ in 1st octant



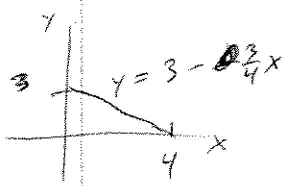
$$z = 2 - \tfrac{1}{2}x - \tfrac{2}{3}y$$

$$z_x = -\tfrac{1}{2}, \qquad z_y = -\tfrac{2}{3}$$

$$dS = \left[1 + z_x^2 + z_y^2\right]^{1/2} dA = \left[1 + (-\tfrac{1}{2})^2 + (-\tfrac{2}{3})^2\right]^{1/2} dA$$

$$= \left(1 + \tfrac{1}{4} + \tfrac{4}{9}\right)^{1/2} dx\,dy = \tfrac{1}{6}(36+9+16)^{1/2}\,dx\,dy$$

$$= \frac{\sqrt{61}}{6}\,dx\,dy$$



$$\iint_S xyz\,dS = \int_0^4\int_0^{3-\frac{3}{4}x} xy\left(2 - \tfrac{1}{2}x - \tfrac{2}{3}y\right)\frac{\sqrt{61}}{6}\,dy\,dx$$

$$= \frac{\sqrt{61}}{6}\int_0^4\int_0^{3-\frac{3}{4}x}\left[2xy - \tfrac{1}{2}x^2y - \tfrac{2}{3}xy^2\right]dy\,dx$$

$$= \frac{\sqrt{61}}{6}\int_0^4\left[xy^2 - \tfrac{1}{4}x^2y^2 - \tfrac{2}{9}xy^3\right]_{y=0}^{y=3-\frac{3}{4}x} dx$$

$$= \frac{\sqrt{61}}{6}\int_0^4\left[\left(x - \tfrac{1}{4}x^2\right)\left(3 - \tfrac{3}{4}x\right)^2 - \tfrac{2}{9}x\left(3 - \tfrac{3}{4}x\right)^3\right]dx$$

$$= \frac{\sqrt{61}}{6}\int_0^4\left[\left(x - \tfrac{1}{4}x^2\right)\left(9 - \tfrac{18}{4}x + \tfrac{9}{16}x^2\right) - \tfrac{2}{9}x\left(3^3 + 3(3)^2(-\tfrac{3}{4}x) + 3(3)(-\tfrac{3}{4}x)^2 + (-\tfrac{3}{4}x\right.\right.$$

$$= \frac{\sqrt{61}}{6}\int_0^4\left[9x - \tfrac{18}{4}x^2 + \tfrac{9}{16}x^3 - \tfrac{9}{4}x^2 + \tfrac{18}{16}x^3 - \tfrac{9}{64}x^4 - 6x - 6\left(-\tfrac{3}{4}x\right)x - 2\left(-\tfrac{3}{4}\right)^2x^3 - \tfrac{2}{9}\tfrac{9}{16}\left(-\tfrac{3}{4}\right)x^4\right]dx$$

(cont'd)

Ex, cont'd

$$= \frac{\sqrt{61}}{6}\int_0^4 \left[3x - \frac{9}{4}x^2 + \frac{9}{16}x^3 + \left(-\frac{9}{64} + \frac{6}{64}\right)x^4\right]dx$$

$$= \frac{\sqrt{61}}{6}\left[\frac{3}{2}x^2 - \frac{9}{4}\frac{1}{3}x^3 + \frac{9}{16}\frac{1}{4}x^4 - \frac{3}{64}\frac{1}{5}x^5\right]_0^4$$

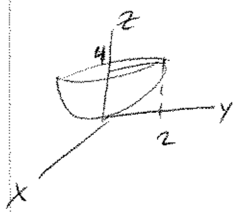
$$= \frac{\sqrt{61}}{6}\left[\frac{3}{2}(16) - 3(16) + 9(4) - \frac{3}{\cancel{64}}\frac{1}{5}\,4^{\cancel{5}\,2}\right]$$

$$= \frac{\sqrt{61}}{6}\left[24 - 48 + 36 - \frac{48}{5}\right]$$

$$= \frac{\sqrt{61}}{6}\left[12 - \frac{48}{5}\right] = \sqrt{61}\left[2 - \frac{8}{5}\right] = \frac{\sqrt{61}}{5}\left[10 - 8\right]$$

$$= \frac{2\sqrt{61}}{5}$$

<u>Ex</u> Evaluate $\iint_S (z)\, dS$ for S the part of the paraboloid $z = x^2+y^2$ below $z=4$.



$$z(x,y) = x^2+y^2$$
$$z_x = 2x, \quad z_y = 2y$$
$$dS = (1+z_x^2+z_y^2)^{1/2}\, dx\, dy$$
$$= (1+4x^2+4y^2)^{1/2}\, dx\, dy$$

$$\iint_A (x^2+y^2)\left(1+4(x^2+y^2)\right)^{1/2} dx\, dy$$ where $A =$ 

Convert to polar:

$$= \int_0^{2\pi}\int_0^2 r^2(1+4r^2)^{1/2}\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 r^3(1+4r^2)^{1/2}\, dr$$ after do $d\theta$ 1st

$u = r^2 \quad dv = r(1+4r^2)^{1/2}$

$du = 2r\,dr, \quad v = \frac{1}{8}\,\frac{2}{3}(1+4r^2)^{3/2}$

$$= 2\pi\left[\frac{1}{12} r^2 (1+4r^2)^{3/2}\Big|_0^2 - \int_0^2 \frac{1}{12}\, 2r\,(1+4r^2)^{3/2}\, dr\right]$$

$$= 2\pi\left[\frac{1}{12}(4)(1+16)^{3/2} - \frac{1}{6}\,\frac{1}{8}\,\frac{2}{5}(1+4r^2)^{5/2}\Big|_0^2\right]$$

$$= 2\pi\left[\frac{1}{3}(17)^{3/2} - \frac{1}{3}\,\frac{1}{8}\,\frac{1}{5}\left((17)^{5/2}-1\right)\right]$$

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Flux of a vector through a surface

Flux of $\bar{F}$ across a surface G $= \iint_G \bar{F}\cdot\bar{n}\, dS$

(just as flux across a line was, $\int_C \bar{F}\cdot\bar{n}\, ds$)

Ex Find the flux of $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ through the surface
$z = \sqrt{R^2 - x^2 - y^2}$

$$z_x = \tfrac{1}{2}(-2x)\frac{1}{z} = -\frac{x}{z}, \qquad z_y = \tfrac{1}{2}(-2y)\frac{1}{z} = -\frac{y}{z}$$

$$dS = \left[1 + (z_x)^2 + (z_y)^2\right]^{1/2} dx\,dy = \left[1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right]^{1/2} dx\,dy$$
$$= \frac{1}{z}(x^2+y^2+z^2)^{1/2}\, dx\,dy = \frac{R}{z}\, dx\,dy$$

Need a unit normal vector $\bar{n}$.
Define $H(x,y,z) = z - \sqrt{R^2 - x^2 - y^2}$

$$\nabla H = \left< -\left(\tfrac{1}{2}\right)(-2x)\frac{1}{\sqrt{\cdots}},\ -\left(\tfrac{1}{2}\right)(-2y)\frac{1}{\sqrt{\cdots}},\ 1 \right>$$
$$= \left< \frac{x}{z}, \frac{y}{z}, 1 \right>$$

$$|\nabla H| = \left(\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1\right)^{1/2} = \frac{1}{z}(x^2+y^2+z^2)^{1/2} = \frac{R}{z}$$

$$\bar{n} = \frac{\nabla H}{|\nabla H|}$$

$$\bar{F}\cdot\bar{n} = \frac{1}{|\nabla H|}\left[(x)\left(\frac{x}{z}\right) + (y)\left(\frac{y}{z}\right) + (z)(1)\right] = \frac{1}{|\nabla H|}\frac{1}{z}(x^2+y^2+z^2)$$
$$= \frac{1}{R/z}\frac{R^2}{z} = R$$

$$\iint \bar{F}\cdot\bar{n}\, dS = \int_0^{2\pi}\int_0^R R\left(\frac{R}{z}\right) r\, dr\, d\theta = R^2(2\pi)\int_0^R r(R^2-r^2)^{-1/2}\, dr$$
$$= 2\pi R^2\left(-\tfrac{1}{2}\right)(2)(R^2-r^2)^{1/2}\Big|_0^R = -2\pi R^2\left[0 - R\right]$$
$$= +2\pi R^3$$

More generally:

$$\iint_G \bar{F}\cdot\bar{n}\, dS = \iint_R (-Mf_x - Nf_y + P)\, dx\, dy$$

where $\bar{F} = \langle M, N, P\rangle$ & surface is defined by $z = f(x,y)$.

Check:
Write $H(x,y,z) = z - f(x,y)$

$$\nabla H = \langle -f_x, -f_y, 1\rangle$$

$$|\nabla H| = \left(1 + f_x^2 + f_y^2\right)^{1/2} \qquad \text{\& take } \bar{n} = \frac{\nabla H}{|\nabla H|}$$

So

$$\bar{F}\cdot\bar{n}\, dS = \underbrace{\frac{-Mf_x - Nf_y + P}{\cancel{(1+f_x^2+f_y^2)^{1/2}}}}_{\bar{F}\cdot\bar{n}} \underbrace{\cancel{(1+f_x^2+f_y^2)^{1/2}}\, dx\, dy}_{dS}$$

$$= (-Mf_x - Nf_y + P)\, dx\, dy$$

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§17.6 Gauss's Divergence Theorem

Gauss's theorem

Let S be a closed bounded solid in 3-space that is completely enclosed by a piecewise smooth surface ∂S.
Let $\bar{F} = \langle M, N, P \rangle$ be a vector field.
If $\bar{n}$ denotes the outer unit normal to ∂S, then

$$\iint_{\partial S} \bar{F}\cdot\bar{n}\, dS = \iiint_{S} (\operatorname{div} \bar{F})\, dV$$

Check for S a rectangular box, $[a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$

Over the planes $\parallel$ xy plane,

$$\iint \bar{F}\cdot\bar{n}\, dS = \int_{a_1}^{a_2}\int_{b_1}^{b_2} (-P(x,y,c_1))\, dy\, dx + \int_{a_1}^{a_2}\int_{b_1}^{b_2} (+P)(x,y,c_2)\, dy\, dx$$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} \frac{\partial P}{\partial z}\, dz\, dy\, dx$$

& similarly for the other 2 pairs of planes.

Ex $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$, verify Gauss's thm on a sphere of radius R.

$\bar{n}$: $x^2+y^2+z^2 = R^2$

set $H = x^2+y^2+z^2-R^2$

$\Rightarrow \nabla H = \langle 2x, 2y, 2z \rangle$

$|\nabla H| = 2(x^2+y^2+z^2)^{1/2} = 2R$

$\Rightarrow \bar{n} = \frac{1}{R}\langle x, y, z \rangle$

$$\iint_{\partial S} \bar{F}\cdot\bar{n}\, dS = \iint_{\partial S} \frac{1}{R}(x^2+y^2+z^2)\, dS = R\iint_{\partial S} dS$$

$$= R(4\pi R^2) = \underline{4\pi R^3}$$

$$\iiint_S (\nabla\cdot\bar{F})\, dV = \iiint_S (3)\, dV = 3\iiint_S dV = 3\left(\tfrac{4}{3}\pi R^3\right)$$

$$= \underline{4\pi R^3}$$

so $\displaystyle\iint_{\partial S} \bar{F}\cdot\bar{n}\, dS = \iiint_S (\nabla\cdot\bar{F})\, dV$ in this case.

Ex let's repeat the last ex but w/ a different $\bar{F}$.

Take $\bar{F} = x\bar{\imath}$.

Recall $\bar{n} = \frac{1}{R}\langle x, y, z\rangle$ on sphere.

$$\iint_{\partial S} \bar{F}\cdot\bar{n}\,dS = \iint_{\partial S} \frac{1}{R}x^2\,dS = \frac{1}{R}\iint_{\partial S} x^2\,dS$$

Treat upper & lower halves of sphere symmetrically.

$$z = \sqrt{R^2 - x^2 - y^2}$$

$$z_x = \frac{1}{2}(-2x)z^{-1} = -\frac{x}{z}, \qquad z_y = -\frac{y}{z}$$

$$dS = \left[1 + z_x^2 + z_y^2\right]^{1/2} dx\,dy = \left[1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right]^{1/2} dx\,dy = \frac{R}{z}\,dx\,dy$$

$$\frac{1}{R}\iint_{\partial S} x^2\,dS = \frac{2}{R}\int_0^{2\pi}\int_0^R (r\cos\theta)^2 \frac{R}{(R^2 - r^2)^{1/2}}\, r\,dr\,d\theta$$

$$= \frac{2}{\cancel{R}}\left[\int_0^{2\pi}\cos^2\theta\,d\theta\right]\left[\int_0^R r^3 (R^2 - r^2)^{-1/2}\,dr\right] \qquad \begin{matrix} u = r^2 & dv = r(R^2 - r^2)^{-1/2}\,dr \\ du = 2r\,dr & v = \left(-\frac{1}{2}\right)(2)(R^2 - r^2)^{1/2} \end{matrix}$$

$$= \frac{2}{\cancel{R}}\left[\frac{1}{2}(2\pi)\right]\left[-r^2(R^2 - r^2)^{1/2}\Big|_0^R - \int_0^R (-2)r\,(R^2 - r^2)^{1/2}\,dr\right]$$

$$= \frac{2}{\cancel{R}}(\pi)\left[0 + 2\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(R^2 - r^2)^{3/2}\Big|_0^R\right]$$

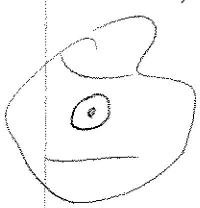
$$= \frac{2\pi}{\cancel{R}}\left(-\frac{2}{3}\right)\left(0 - R^3\right) = \frac{4\pi}{3}R^3 = \iint_{\partial S}\bar{F}\cdot\bar{n}\,dS$$

Compare: $\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x) = 1$

$$\iiint_S (\nabla\cdot\bar{F})\,dV = \iiint dV = \frac{4\pi}{3}R^3 = \iint_{\partial S}\bar{F}\cdot\bar{n}\,dS$$

<u>Classic ex</u> Let S be a solid region containing a point mass M at the origin in its interior & w/ field $\bar{F} = -cM\bar{r}/|\bar{r}|^3$.
Show that the flux of $\bar{F}$ across ∂S is $-4\pi c M$,
regardless of the shape of S.

Since $\bar{F}$ is discontinuous at origin, Gauss's law doesn't apply directly to S,
so surround the origin by a small ball S_a, centered at the origin,
of radius a, contained within S,
& apply Gauss to $S - S_a$.



$$\iint_{S-S_a} \bar{F}\cdot\bar{n}\, dS = \iint_{\partial S} \bar{F}\cdot\bar{n}\, dS + \iint_{-\partial S_a} \bar{F}\cdot\bar{n}\, dS$$

$$\underline{\text{also}} = \iiint_{S-S_a} (\text{div } \bar{F})\, dV$$

<u>Note</u> div $\bar{F} = 0$:

$$\frac{\partial}{\partial x}\left[x\,(x^2+y^2+z^2)^{-3/2}\right] = (x^2+y^2+z^2)^{-3/2} - \frac{3}{\cancel{2}}\,x\,(\cancel{2}x)\,(x^2+y^2+z^2)^{-5/2}$$

$$\text{div } \bar{F} = 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}$$
$$= 0 \checkmark$$

$$\Rightarrow \iint_{\partial S} \bar{F}\cdot\bar{n}\, dS = -\iint_{-\partial S_a} \bar{F}\cdot\bar{n}\, dS$$

Now, on $(-\partial S_a)$, $\bar{n} = -\frac{1}{a}(x\bar{i} + y\bar{j} + z\bar{k})$ (pts inward)
$= -\bar{r}/a$

so $\bar{F}\cdot\bar{n} = \dfrac{cM\bar{r}^2}{a|\bar{r}|^3} = \dfrac{cM}{a}\dfrac{1}{|\bar{r}|} = \dfrac{cM}{a^2}$ on ∂S_a

$$\Rightarrow \iint_{-\partial S_a} \bar{F}\cdot\bar{n}\, dS = \frac{cM}{a^2}\iint dS = \frac{cM}{a^2}(4\pi a^2) = 4\pi c M$$

$$\Rightarrow \boxed{\iint_{\partial S} \bar{F}\cdot\bar{n}\, dS = -4\pi c M}$$

§ 17.7 Stokes's theorem

We saw previously that Green's theorem could be written

$$\oint_{\partial S} \bar{F}\cdot\bar{T}\,ds = \iint_S (\operatorname{curl}\bar{F})\cdot\bar{k}\,dA$$

for S in the plane. This generalizes to other S:

Stokes's theorem

$$\oint_{\partial S} \bar{F}\cdot\bar{T}\,ds = \iint_S (\operatorname{curl}\bar{F})\cdot\bar{n}\,dS$$

(orientation: right hand rule)

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Ex $S =$ paraboloid $z = R^2 - x^2 - y^2$, bdd by $z = 0$

Parametrize ∂S as: $x = R\cos t,\ y = R\sin t,\ z = 0 \qquad t\in[0,2\pi)$

$\bar{F} = \langle y, -x, xy\rangle$

$$\oint_{\partial S}\bar{F}\cdot\bar{T}\,ds = \oint_{\partial S}\bar{F}\cdot d\bar{r} = \oint_{\partial S}(y\,dx - x\,dy) \quad \text{since } dz = 0$$

$$= \int_0^{2\pi}\left[(R\sin t)(-R\sin t\,dt) - (R\cos t)(R\cos t\,dt)\right]$$

$$= -R^2\int_0^{2\pi} dt = -2\pi R^2$$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & xy \end{vmatrix} = \bar{\imath}(x-0) - \bar{\jmath}(y-0) + \bar{k}(-1-1)$$

$$= \langle x, -y, -2\rangle$$

Recall $\iint_G \bar{F}\cdot\bar{n}\,dS = \iint_R (-M f_x - N f_y + P)\,dx\,dy \qquad$ (§17.5)

$$= \int_0^{2\pi}\int_0^R\left[-(x)(-2x) - (-y)(-2y) + (-2)\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^R\left[2x^2 \cancel{\theta} - 2y^2 - 2\right] r\,d\ldots$$

(cont'd)

Ex, cont'd

$$= \int_0^{2\pi}\int_0^R (2x^2-2y^2-2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^R (2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[2\frac{R^4}{4}\cos^2\theta - 2\frac{R^4}{4}\sin^2\theta - R^2\right]d\theta$$

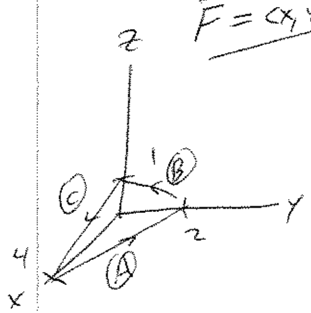
$$= \cancel{\frac{R^4}{2}\left(\frac{1}{2}2\pi\right)} - \cancel{\frac{R^4}{2}\left(\frac{1}{2}2\pi\right)} - 2\pi R^2$$

$$= -2\pi R^2 \qquad = \iint_S (\operatorname{curl}\overline{F})\cdot\overline{n}\,dS$$

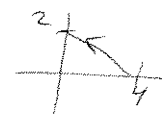
also $= \oint_{\partial S} \overline{F}\cdot\overline{T}\,ds$ ✓

Ex S = triangle given by portion of ~~$3x+2y+z=$~~ $x + 2y + 4z = 4$ in 1st octant.

$\overline{F} = \langle x, y, z\rangle$



First, evaluate $\oint_{\partial S} \overline{F}\cdot\overline{T}\,ds$.



Do this in 3 segments.

A) : $z = 0,\ y = 2 - \frac{x}{2}$

$$\cancel{\overline{T} = \frac{\langle -4, 2\rangle}{|\langle -4, 2\rangle|} = \frac{\langle -2, 1\rangle}{\sqrt{5}}}$$

$$\cancel{\overline{F}\cdot\overline{T} = \frac{1}{\sqrt{5}}(-2x + y)}$$

$$\int_A \overline{F}\cdot\overline{T}\,ds = \int_A (x\,dx + y\,dy) = \int_4^0 \left[x\,dx + \left(2 - \tfrac{x}{2}\right)\left(-\tfrac{1}{2}dx\right)\right]$$

$$= \int_4^0 \left[x - 1 + \frac{x}{4}\right]dx = \left[\frac{x^2}{2} - x + \frac{x^2}{8}\right]_4^0$$

$$= -\left(\frac{4^2}{2} - 4 + \frac{4^2}{8}\right) = -(8 - 4 + 2) = -6$$

B): $x = 0,\ z = 1 - \frac{y}{2}$

$$\int_B \overline{F}\cdot\overline{T}\,ds = \int_2^0 (y\,dy + z\,dz) = \int_2^0 \left(y\,dy + \left(1 - \tfrac{y}{2}\right)\left(-\tfrac{1}{2}dy\right)\right)$$

$$= \int_2^0 \left[y - \frac{1}{2} + \frac{y}{4}\right]dy = \left[\frac{y^2}{2} - \frac{1}{2}y + \frac{y^2}{8}\right]_2^0$$

$$= -\left[\frac{2^2}{2} - \frac{1}{2}2 + \frac{2^2}{8}\right] = -\left[2 - 1 + \frac{1}{2}\right] = -\frac{3}{2}$$

C): $y = 0,\ z = 1 - \frac{x}{4}$

$$\int_C \overline{F}\cdot\overline{T}\,ds = \int_0^4 (x\,dx + z\,dz) = \int_0^4 \left(x\,dx + \left(1 - \tfrac{x}{4}\right)\left(-\tfrac{1}{4}dx\right)\right)$$

$$= \int_0^4 \left[x - \frac{1}{4} + \frac{x}{16}\right]dx = \left[\frac{x^2}{2} - \frac{1}{4}x + \frac{x^2}{32}\right]_0^4$$

$$= \frac{16}{2} - \frac{1}{4}4 + \frac{16}{32} = 8 - 1 + \frac{1}{2} = 7 + \frac{1}{2}$$

(cont'd)

Ex, cont'd

$$\oint \bar{F}\cdot\bar{T}\,ds = \int_A \bar{F}\cdot\bar{T}\,ds + \int_B \bar{F}\cdot\bar{T}\,ds + \int_C \bar{F}\cdot\bar{T}\,ds$$

$$= -6 \quad -\tfrac{3}{2} \quad + 7 + \tfrac{1}{2} \qquad = \underline{0}$$

Now, the other half of Stokes:

$$\oint_{\partial S} \bar{F}\cdot\bar{T}\,ds = \iint_S (\operatorname{curl}\bar{F})\cdot\bar{n}\,dS$$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}\,(0) - \bar{j}\,(0) + \bar{k}\,(0)$$

$$= 0$$

so $\underline{\iint_S (\operatorname{curl}\bar{F})\cdot\bar{n}\,dS = 0}$

Interpretation:

$\oint_C \bar{F}\cdot\bar{T}\,ds$ is the "circulation" of $\bar{F}$ around C

so curl $\bar{F}$ acts as source/sink for circulation

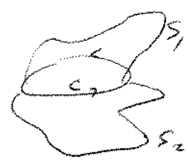
Consistency check

S'pose I want to use Stokes to compute $\oint_C \vec{F}\cdot\vec{T}\,ds$

Well, pick an S s.t. $C = \partial S$, then

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

But *which* S ?



If I glue together 2 S's, then ~~get a boundary~~

$$\iint_{S_1} (\text{curl}\,\vec{F})\cdot\vec{n}\,dS + \iint_{S_2} (\text{curl}\,\vec{F})\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot\vec{T}\,ds - \oint_C \vec{F}\cdot\vec{T}\,ds$$

orientation;
if $\vec{n}$ outwards-pting,
then one will be clockwise on C,
the other counterclockwise

$$= 0$$

So the choice doesn't matter *so long as*

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS = 0 \quad \text{whenever } S \text{ has no boundary.}$$

Why?

S'pose $S = \partial B$ for some 3D B.
Apply Gauss's divergence theorem:

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS = \iiint_B \text{div}\,(\text{curl}\,\vec{F})\,dV$$

(cont'd)

(cont'd)

Compute div (curl $\bar{F}$):

$$\bar{F} = \langle F_1, F_2, F_3 \rangle$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \langle F_{3y} - F_{2z},\ F_{1z} - F_{3x},\ F_{2x} - F_{1y} \rangle$$

$$\nabla \cdot (\nabla \times \bar{F}) = (\cancel{F_{3yx}} - \cancel{F_{2zx}}) + (\cancel{F_{1zy}} - \cancel{F_{3xy}}) + (\cancel{F_{2xz}} - \cancel{F_{1yz}})$$

$$= 0$$

$$\Rightarrow \iint_S (\text{curl}\,\bar{F}) \cdot \bar{n}\, dS = \iiint_B \nabla \cdot (\nabla \times \bar{F})\, dV$$

$$= 0$$

precisely as needed.