5 pts each part
10 pts total

2. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

taken counterclockwise around the circle

a) $|z-2| = 2$

Integrand has simple poles at $z = 1, \pm 3i$

Residue at $z=1$: $\dfrac{3(1)^3+2}{(1^2+9)} = \dfrac{5}{10} = \dfrac{1}{2}$

Residue at $z=-3i$: $\dfrac{3(-3i)^3+2}{(-3i-1)(-3i-3i)} = \dfrac{+i(81)+2}{(+6i)(1+3i)}$

Residue at $z=+3i$: $\dfrac{3(3i)^3+2}{(3i-1)(3i+3i)} = \dfrac{-81i+2}{(6i)(3i-1)}$

The circle $|z-2|=2$ encloses only the $z=1$ pole

$$\Rightarrow \int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

(cont'd)

b) $|z| = 4$

This contour encloses all 3 poles.

$$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left[\frac{1}{2} + \frac{81i+2}{(6i)(1+3i)} + \frac{-81i+2}{(6i)(3i-1)}\right]$$

$$= 2\pi i\left[\frac{1}{2} + \frac{(81i+2)(3i-1) + (-81i+2)(1+3i)}{(6i)(1+3i)(3i-1)}\right]$$

$$= 2\pi i\left[\frac{1}{2} + \frac{(\cancel{-243}+6i-81i-\cancel{2}) + (\cancel{+243}+6i+\cancel{2}-81i)}{(6i)(-9-1)}\right]$$

$$= 2\pi i\left[\frac{1}{2} + \frac{-150i}{(-10)(6i)}\right] = 2\pi i\left[\frac{1}{2}+\frac{15}{6}\right] = 2\pi i\left[\frac{1}{2}+\frac{5}{2}\right]$$

$$= 2\pi i\left[\frac{6}{2}\right] = \underline{6\pi i}$$

4 pts each part

12 pts total

3. Let C be the circle $|z| = 2$, described in the positive sense, and evaluate the integral

a) $\int_C \tan z \, dz$ Write $\tan z = \dfrac{\sin z}{\cos z}$, $\quad p(z) = \sin z$, $q(z) = \cos z$

$\cos z$ has zeros at $z = (2n+1)(\pi/2)$ $\rightsquigarrow$ simple poles

$$\text{Residue} = \frac{\sin(2n+1)\pi/2}{-\sin(2n+1)\pi/2} = -1$$

$\cancel{\Rightarrow \int_C \tan z \, dz = 2\pi i(-1) = -2\pi i}$

The contour encloses 2 poles, at $z = \pm \pi/2$

$$\Rightarrow \int_C \tan z \, dz = 2\pi i(-1-1) = \underline{-4\pi i}$$

---

b) $\displaystyle\int_C \frac{dz}{\sinh 2z}$

$\dfrac{1}{\sinh 2z}$ $\quad p(z) = 1$, $q(z) = \sinh 2z$ $\qquad q(0) = 0$, $q'(0) \neq 0$

$\rightarrow$ simple pole at $z = 0$, residue $= \dfrac{1}{2\cosh(2 \cdot 0)} = \dfrac{1}{2}$

~~Another~~ More poles at $z = \pm \dfrac{\pi i}{2}$, residue $= \dfrac{1}{2\cosh(\pm\pi i)} = \dfrac{1}{2\cos\pi} = -\dfrac{1}{2}$

$$\Rightarrow \int_C \frac{dz}{\sinh 2z} = 2\pi i\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right) = \underline{-\pi i}$$

c) $\int_C \frac{\cosh \pi z}{z(z^2+1)} dz$

Integrand has simple poles at $z = 0, \pm i$

Residue at $z=0$: $\frac{\cosh \pi(0)}{(0^2+1)} = 1$

Residue at $z=+i$: $\frac{\cosh(\pi i)}{(i)(i+i)} = -\frac{1}{2}\cosh \pi i = \frac{1}{2}$

Residue at $z=-i$: $\frac{\cosh(-\pi i)}{(-i)(-i-i)} = -\frac{1}{2}\cosh(-\pi i) = +\frac{1}{2}$

Contour encloses all 3 poles

$$\int_C \frac{\cosh \pi z}{z(z^2+1)} dz = 2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right) = \underline{4\pi i}$$

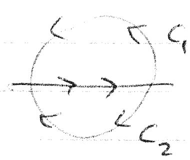
Churchill
p190
§59

5 pts each part
10 pts total

4. Compute the following improper integrals w/ residues:

a) $$\int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

$\frac{1}{z^2+1}$ has poles at $z = \pm i$



$$\int_C \frac{dz}{z^2+1} = \int \frac{Re^{i\theta}(i)\,d\theta}{R^2 e^{2i\theta}+1}$$

$$\rightarrow \int \frac{i\,d\theta}{Re^{i\theta}} \rightarrow 0 \text{ for both } C_1 \,\&\, C_2$$

Pick C_1. It contains $z = +i$.

Residue $= \frac{1}{+i+i} = \frac{1}{2i}$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}(2\pi i)\left(\frac{1}{2i}\right) = \frac{\pi}{2}$$

Check

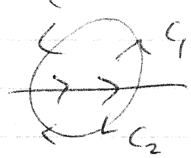
Pick C_2. It contains $z = -i$.

Residue $= \frac{1}{-i-i} = -\frac{1}{2i}$, Also: C_2 has negative orientation

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}(2\pi i)(-)\left(-\frac{1}{2i}\right) = \frac{\pi}{2} \checkmark$$

b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+1)^2}$

$\dfrac{1}{(z^2+1)^2}$ has poles at $z = \pm i$



Determine which converges:

$$\int_C \frac{dz}{(z^2+1)^2} = \int \frac{Re^{i\theta}(i)\,d\theta}{(R^2e^{2i\theta}+1)^2} \to \int \frac{i\,d\theta}{R^3e^{3i\theta}}$$

$\to 0$ for both C_1 & C_2

Pick C_1. It contains $z = +i$.

Residue: $\dfrac{1}{(z-i)^2}\dfrac{1}{(z+i)^2} = \dfrac{1}{(z-i)^2}\left[\dfrac{1}{(2i)^2} - \dfrac{2}{(z+i)^3}\Big|_{z=i}(z-i) + \cdots\right]$

so residue $= \dfrac{-2}{(2i)^3} = +\dfrac{1}{4i}$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+1)^2} = \frac{1}{2}(2\pi i)\left(\frac{1}{4i}\right) = \underline{\frac{\pi}{4}}$$

<u>Check</u> Pick C_2 instead, which contains $z = -i$.

Residue: $\dfrac{1}{(z+i)^2}\dfrac{1}{(z-i)^2} = \dfrac{1}{(z+i)^2}\left[\dfrac{1}{(z-i)^2}\Big|_{z=-i} - \dfrac{2}{(z-i)^3}\Big|_{z=-i}(z+i) + \cdots\right]$

$\Rightarrow$ residue $= \dfrac{-2}{(-2i)^3} = \dfrac{2}{(2i)^3} = \dfrac{-1}{4i}$, but C_2 also has negative orientation

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+1)^2} = \frac{1}{2}(2\pi i)(-)\left(-\frac{1}{4i}\right) = \frac{\pi}{4} \checkmark$$

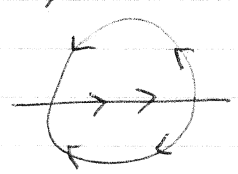
~~5 pts~~
10 pts

5. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} \qquad (a>b>0)$$

$$= \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iz} \, dz}{(z^2+a^2)(z^2+b^2)}$$

Completion:



$$\int_C \frac{e^{iRe^{i\theta}} \, Re^{i\theta} (i) \, d\theta}{(R^2 e^{2i\theta}+a^2)(R^2 e^{2i\theta}+b^2)}$$

$$\xrightarrow[R\to\infty]{} \int_C \frac{e^{iR\cos\theta - R\sin\theta}}{R^3 e^{3i\theta}} (i) \, d\theta$$

In order for this $\to 0$, need $\sin\theta > 0$
→ true for $0 \le \theta \le \pi$, <u>not</u> for $-\pi \le \theta \le 0$
→ so true for C_1 <u>not</u> C_2

Poles at $\pm ia$, $\pm ib$; C_1 contains $+ia$, $+ib$

Residue at $+ia$:

$$\frac{1}{z-ia} \, \frac{e^{iz}}{(z+ia)(z^2+b^2)} \Rightarrow \text{residue} = \frac{e^{-a}}{(2ia)(-a^2+b^2)}$$

Residue at $+ib$:

$$\frac{1}{z-ib} \, \frac{e^{iz}}{(z+ib)(z^2+a^2)} \Rightarrow \text{residue} = \frac{e^{-b}}{(2ib)(-b^2+a^2)}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iz} \, dz}{(z^2+a^2)(z^2+b^2)} = \mathrm{Re}\left[(2\pi i) \frac{1}{b^2-a^2} \left(\frac{e^{-a}}{2ia} - \frac{e^{-b}}{2ib} \right) \right]$$

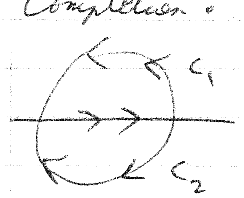
$$= \frac{\pi}{a^2-b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

~~1 pts~~
10 pts

6. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx \qquad (a>0)$

$$= \operatorname{Im} \int_{-\infty}^{\infty} \frac{z e^{iaz}}{z^4+4}\,dz$$

Completion:



$$\int_c \frac{Re^{i\theta}\, e^{iaRe^{i\theta}}\; Re^{i\theta}(i)\,d\theta}{R^4 e^{4i\theta}+1}$$

$$\xrightarrow[R\to\infty]{} \int_c \frac{e^{iaR\cos\theta - aR\sin\theta}}{R^2 e^{2i\theta}}\,(i)\,d\theta$$

In order for this $\to 0$, need $\sin\theta > 0$ (since $a>0$)
$\to$ True for $0 \le \theta \le \pi$, not for $-\pi \le \theta \le 0$
$\to$ so use C_1 not C_2

$$z^4 = -4 = 4e^{\pi i} \Rightarrow z = \sqrt{2}\, e^{\frac{0\pi i}{4}}, \sqrt{2}\, e^{\frac{\pi i}{4}+\frac{2\pi i}{4}}, \sqrt{2}\, e^{\frac{\pi i}{4}+\pi i}, \sqrt{2}\, e^{\frac{\pi i}{4}+\frac{6\pi i}{4}}$$

Simplify:
poles at $z = \sqrt{2}\,e^{\pi i/4}, \sqrt{2}\,e^{3\pi i/4}, \sqrt{2}\,e^{5\pi i/4}, \sqrt{2}\,e^{7\pi i/4}$

Of these, $\sqrt{2}\,e^{\pi i/4}$ & $\sqrt{2}\,e^{3\pi i/4}$ are enclosed within C_1.

(cont'd)

(cont'd)

$$f(z) = \frac{z e^{iaz}}{z^4+4}$$

Residue at $z = \sqrt{2}\, e^{\pi i/4} = 1+i$:

$$f(z) = \left(\frac{1}{z - \sqrt{2}\, e^{\pi i/4}}\right) \frac{z e^{iaz}}{(z^2+2i)(z - \sqrt{2}\, e^{5\pi i/2})}$$

$$\text{Residue} = \frac{(\sqrt{2}\, e^{\pi i/4})\, e^{ia(1+i)}}{(2e^{\pi i/2}+2i)(\sqrt{2}\, e^{\pi i/4} - \sqrt{2}\, e^{5\pi i/4})}$$

$$= \frac{e^{ia} e^{-a}}{(4i)(1-e^{4\pi i/4})} = \frac{e^{ia} e^{-a}}{8i}$$

Residue at $z = \sqrt{2}\, e^{3\pi i/4} = -1+i$:

$$f(z) = \frac{1}{(z - \sqrt{2}\, e^{3\pi i/4})} \frac{z e^{iaz}}{(z^2-2i)(z - \sqrt{2}\, e^{7\pi i/4})}$$

$$\text{Residue} = \frac{(\sqrt{2}\, e^{3\pi i/4})\, e^{ia(-1+i)}}{(2e^{3\pi i/2}-2i)(\sqrt{2}\, e^{3\pi i/4} - \sqrt{2}\, e^{7\pi i/4})}$$

$$= \frac{e^{-ia} e^{-a}}{(-4i)(1-e^{4\pi i/4})} = \frac{e^{-ia} e^{-a}}{(-4i)(2)} = -\frac{e^{-ia} e^{-a}}{8i}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\, dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{z e^{iaz}}{z^4+4}\, dz$$

$$= \operatorname{Im}\left[(2\pi i)\left(\frac{e^{ia} e^{-a}}{8i} - \frac{e^{-ia} e^{-a}}{8i}\right)\right]$$

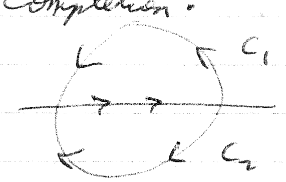
$$= \operatorname{Im}\left[\frac{\pi}{4} e^{-a} (e^{ia} - e^{-ia})\right] = \boxed{\frac{\pi}{2} e^{-a} \sin a}$$

~~5/5~~
10 pt

7. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2}$

$$= \int_{-\infty}^{\infty} \frac{z^2\,dz}{(z^2+1)^2}$$

Completion:



$$\int_c \frac{R^2 e^{2i\theta}\; Re^{i\theta}(i)\,d\theta}{(R^2 e^{2i\theta}+1)^2}$$

$$\xrightarrow{R\to\infty} \int_c \frac{i\,d\theta}{Re^{i\theta}} \to 0 \text{ for either } C$$

Pick C_1.

Poles at $z = \pm i$. $z = +i$ lies inside contour.

Residue at $z = +i$:

$$f(z) = \frac{1}{(z-i)^2}\,\frac{z^2}{(z+i)^2} = \frac{1}{(z-i)^2}\left[\left(\frac{z^2}{(z+i)^2}\right)\Bigg|_{z=i} + (z-i)\left(\frac{2z}{(z+i)^2} - \frac{2z^2}{(z+i)^3}\right)\Bigg|_{z=i} + \cdots \right]$$

$$\Rightarrow \text{residue} = \frac{2i}{(2i)^2} - \frac{2(i)^2}{(2i)^3} = -\frac{i}{2} + \frac{1}{-4i} = -\frac{i}{4}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = \int_{-\infty}^{\infty} \frac{z^2\,dz}{(z^2+1)^2} = 2\pi i\left(-\frac{i}{4}\right) = \boxed{\frac{\pi}{2}}$$

Check:

Residue at $z=-i$: $\displaystyle = \left(\frac{2z}{(z-i)^2} - \frac{2z^2}{(z-i)^3}\right)\Bigg|_{z=-i} = \frac{-2i}{(-2i)^2} - \frac{2(-i)^2}{(-2i)^3} = \frac{2i}{4} + \frac{2}{8i}$

$$= +\frac{i}{4}$$

but C_2 has opp' orientation

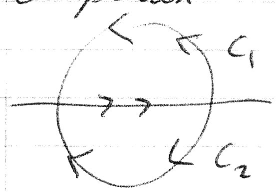
$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = 2\pi i\,(-)\left(+\frac{i}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2} \checkmark$$

~~5 pts~~
10 pts

8. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5}$

$$= \text{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+4z+5}\, dz$$

Completion:



$$\int_C \frac{e^{iRe^{i\theta}}}{R^2e^{2i\theta}+4Re^{i\theta}+5}\, Re^{i\theta} i \, d\theta$$

$$\xrightarrow[R\to\infty]{} \int \frac{e^{iR\cos\theta - R\sin\theta}}{Re^{i\theta}}\,(i)\,d\theta$$

(this $\to 0$ so long as $\sin\theta > 0$ $\Rightarrow$ use C_1.

Poles at $z = \frac{1}{2}\left(-4 \pm \sqrt{16-4(5)}\right) = -2 \pm i$

The pole $z = -2+i$ lies inside the contour.

Residue $= \left.\dfrac{e^{iz}}{(z+2+i)}\right|_{z=-2+i} = \dfrac{e^{-2i}e^{-1}}{(\cancel{2}+i+\cancel{2}+i)} = \dfrac{e^{-2i}e^{-1}}{2i}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+4x+5} = \text{Im}\int_{-\infty}^{\infty}\frac{e^{iz}}{z^2+4z+5}\,dz = \text{Im}\left[(2\pi i)\left(\frac{e^{-2i}e^{-1}}{2i}\right)\right]$$

$$\boxed{= -\frac{\pi}{e}\sin 2}$$

Churchill
p196 §61

10 pts

1. Compute $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

Write $z = e^{i\theta}$, so $\sin^2\theta = \left(\frac{z-z^{-1}}{2i}\right)^2$, $d\theta = \frac{dz}{iz}$

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_C \frac{dz/iz}{1+\left(\frac{z-z^{-1}}{2i}\right)^2} = \int_C \frac{(2i)^2}{(2i)^2(iz) + i(z^2-1)(z-z^{-1})}\,dz$$

$$= \int_C \frac{-4z}{-4iz^2 + i(z^2-1)(z^2-1)}\,dz \qquad (C = \text{unit circle})$$

$$= \int_C \frac{4iz}{-4z^2+z^4+1-2z^2}\,dz = \int_C \frac{4iz}{z^4-6z^2+1}\,dz$$

$(\sqrt{2}\pm 1)^2 = 3 \pm 2\sqrt{2}$

Poles at $z^2 = \frac{1}{2}\left(6 \pm \sqrt{36-4}\right) = 3 \pm 2\sqrt{2}$

The $z^2 = 3+2\sqrt{2}$ poles lie outside the unit circle.
The poles inside the unit circle are $z^2 = 3-2\sqrt{2}$.

Residues:

$$f(z) = \frac{4iz}{z^4-6z^2+1} = \frac{4iz}{\left(z-\sqrt{3-2\sqrt{2}}\right)\left(z+\sqrt{3-2\sqrt{2}}\right)\left(z^2-3-2\sqrt{2}\right)}$$

Residue at $z = +\sqrt{3-2\sqrt{2}}$

$$= \frac{1}{2\cancel{\sqrt{3-2\sqrt{2}}}}\;\frac{4i\cancel{\sqrt{3-2\sqrt{2}}}}{(\cancel{3}-2\sqrt{2}-\cancel{3}-2\sqrt{2})} = \frac{2i}{-4\sqrt{2}} = -\frac{i}{2\sqrt{2}}$$

Residue at $z = -\sqrt{3-2\sqrt{2}}$

$$= \frac{1}{-2\cancel{\sqrt{3-2\sqrt{2}}}}\;\frac{-4i\cancel{\sqrt{3-2\sqrt{2}}}}{(\cancel{3}-2\sqrt{2}-\cancel{3}-2\sqrt{2})} = \frac{2i}{-4\sqrt{2}} = -\frac{i}{2\sqrt{2}}$$

$$\Rightarrow \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = (2\pi i)\left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right) = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi$$

10 pts

2. Compute $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$ $\qquad (a>1)$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} \qquad \text{Write } \cos\theta = \tfrac{1}{2}(z+z^{-1}),\ d\theta = dz/iz$$

$$= \frac{1}{2}\int_C \frac{dz/iz}{(a+\frac{1}{2}z+\frac{1}{2}z^{-1})^2} \qquad C = \text{unit circle}$$

$$= \frac{4}{2i}\int_C \frac{dz}{(2az+z^2+1)(2a+z+z^{-1})} = \frac{2}{i}\int_C \frac{z\,dz}{(z^2+2az+1)^2}$$

Poles at $z = \frac{1}{2}\left(-2a \pm \sqrt{4a^2-4}\right) = -a \pm \sqrt{a^2-1}$
$= a\left[-1 \pm \sqrt{1-\frac{1}{a^2}}\right]$

Of these, only the pole at
$z = a\left[-1+\sqrt{1-\frac{1}{a^2}}\right]$
might lie within unit circle, other definitely outside.

Note
product of poles $= a^2\left[-1+\sqrt{1-\frac{1}{a^2}}\right]\left[-1-\sqrt{1-\frac{1}{a^2}}\right]$
$= a^2\left[1-\left(1-\frac{1}{a^2}\right)\right] = a^2\left(\frac{1}{a^2}\right) = 1$

so if one outside the unit circle, then other inside unit circle.
$\Rightarrow a\left[-1+\sqrt{1-\frac{1}{a^2}}\right]$ inside unit circle.

~~Residue there~~ $\cancel{= \frac{-a+\sqrt{a^2-1}}{(-a+\sqrt{a^2-1})-(-a-\sqrt{a^2-1})} = \frac{-a+\sqrt{a^2-1}}{2\sqrt{a^2-1}}}$

$$f'(z) = \frac{z}{(z^2+2az+1)^2} = \frac{1}{\left[z-(-a+\sqrt{a^2-1})\right]^2}\underbrace{\frac{z}{\left[z-(-a-\sqrt{a^2-1})\right]^2}}_{=\phi(z)}$$

So residue $= \phi'(z)\Big|_{z=-a+\sqrt{a^2-1}}$

$$\phi' = \frac{1}{\left[z-(-a-\sqrt{a^2-1})\right]^2} - \frac{2z}{\left[z-(-a-\sqrt{a^2-1})\right]^3}$$

(cont'd)

(cont'd)

$$\phi'\left(z = -a + \sqrt{a^2-1}\right)$$

$$= \frac{1}{\left[\left(-a+\sqrt{a^2-1}\right) - \left(-a - \sqrt{a^2-1}\right)\right]^2} - \frac{2\left[-a+\sqrt{a^2-1}\right]}{\left[\left(-a+\sqrt{a^2-1}\right) - \left(-a - \sqrt{a^2-1}\right)\right]^3}$$

$$= \frac{1}{\left[2\sqrt{a^2-1}\right]^2} - \frac{2\left[-a+\sqrt{a^2-1}\right]}{\left[2\sqrt{a^2-1}\right]^3}$$

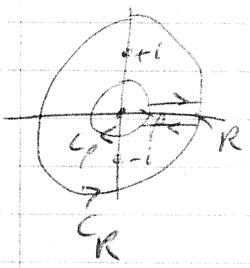
$$= \frac{1}{4(a^2-1)} - \frac{\left(-2a+2\sqrt{a^2-1}\right)}{8(a^2-1)\sqrt{a^2-1}}$$

$$= \cancel{\frac{1}{4(a^2-1)}} + \frac{a}{4(a^2-1)\sqrt{a^2-1}} - \cancel{\frac{1}{4(a^2-1)}}$$

$$\Rightarrow \boxed{\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i}\int_C \frac{z\,dz}{(z^2+2az+1)^2} = \frac{2}{\cancel{i}}(2\pi \cancel{i})\frac{a}{4\left(\sqrt{a^2-1}\right)^3} = \frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}}$$

10 pts

3. Show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}$



Put $\sqrt{x}$ branch cut along the ray $x > 0$.
Integrand just below branch cut:

$$\frac{1}{(-\sqrt{x})\,(x^2+1)} \qquad \left(\text{since } \left(e^{2\pi i}\right)^{1/2} = e^{\pi i} = -1\right)$$

Integrand just above branch cut:

$$\frac{1}{+\sqrt{x}\,(x^2+1)}$$

$$\int_{C_R} \frac{dz}{\sqrt{z}\,(z^2+1)} + \int_R^\rho \frac{dx}{(-\sqrt{x})(x^2+1)} + \int_{C_\rho} \frac{dz}{\sqrt{z}(z^2+1)} + \int_\rho^R \frac{dx}{\sqrt{x}\,(x^2+1)}$$

$$= (2\pi i)\Big[(\text{Residue at } z = +i) + (\text{Residue at } z = -i)\Big]$$

Residue at $z = +i$:

$$f(z) = \frac{1}{\sqrt{z}\,(z+i)(z-i)} \quad \text{so residue} = \left.\frac{1}{\sqrt{z}\,(z+i)}\right|_{z=i}$$

$$= \frac{1}{\left(e^{\pi i/2}\right)^{1/2}\,(2i)} = \frac{1}{\frac{1}{\sqrt{2}}(1+i)(2i)}$$

Residue at $z = -i$:

$$f(z) = \frac{1}{\sqrt{z}\,(z+i)(z-i)} \quad \text{so residue} = \left.\frac{1}{\sqrt{z}\,(z-i)}\right|_{z=-i}$$

$$= \frac{1}{\left(e^{3\pi i/2}\right)^{1/2}\,(-2i)} = \frac{1}{\frac{1}{\sqrt{2}}(1+i)(-2i)}$$

(cont'd)

(cont'd)

as $R \to \infty$,

$$\int_{C_R} \frac{dz}{\sqrt{z}\,(z^2+1)} = \int_{C_R} \frac{Re^{i\theta}(i)\,d\theta}{\sqrt{R}\,e^{i\theta/2}(R^2e^{2i\theta}+1)} \longrightarrow \int_{C_R} \frac{i\,d\theta}{R^{3/2}e^{3i\theta/2}} \longrightarrow 0$$

as $\rho \to 0$,

$$\int_{C_\rho} \frac{dz}{\sqrt{z}\,(z^2+1)} = \int_{C_\rho} \frac{\rho e^{i\theta}(i)\,d\theta}{\sqrt{\rho}\,e^{i\theta/2}(\rho^2e^{2i\theta}+1)} \longrightarrow \int_{C_\rho} \sqrt{\rho}\,e^{i\theta/2}(i)\,d\theta \to 0$$

So,

$$2\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \cancel{2\pi i\left[\frac{\sqrt{2}}{\sqrt{2}(1+i)(2i)}\right] = \frac{2\sqrt{2}\pi}{4+4}}$$

$$= 2\pi i\left(\frac{\sqrt{2}}{2i}\right)\left[\frac{1}{1+i} + \frac{1}{+1-i}\right] = (\pi\sqrt{2})\left[\frac{1-i}{2} + \frac{+1+i}{2}\right]$$

$$= \pi\sqrt{2}$$

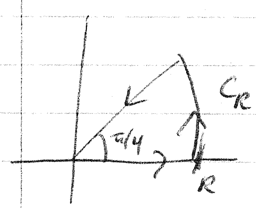
$$\Rightarrow \boxed{\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}}$$

5

4\. a) By considering the integral of $\exp(iz^2)$ around the positively-oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \pi/4$, show that

$$\int_0^R e^{ix^2}\,dx = e^{i\pi/4}\int_0^R e^{-r^2}\,dr - \int_{C_R} e^{iz^2}\,dz$$

where C_R is the arc $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$.



If C = entire boundary,

$$\int_C e^{iz^2}\,dz = 0 \quad \text{since no poles inside.}$$

Note $$\int_C e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx + \int_{C_R} e^{iz^2}\,dz + \int_R^0 e^{i(re^{i\pi/4})^2}\left(e^{i\pi/4}\,dr\right)$$

$$\Rightarrow \int_0^R e^{ix^2}\,dX = -\int_{C_R} e^{iz^2}\,dz + e^{i\pi/4}\int_0^R e^{i^2r^2}\,dr$$

$$= -\int_{C_R} e^{iz^2}\,dz + e^{i\pi/4}\int_0^R e^{-r^2}\,dr$$

5 pts

b) Show that the integral along C_R tends to zero as $R \to \infty$.

$z = Re^{i\theta}$

$$\int_{C_R} e^{iz^2}\,dz = \int_0^{\pi/4} e^{i(Re^{i\theta})^2} Re^{i\theta}(i)\,d\theta$$

$$= \int_0^{\pi/4} \left(e^{iR^2\cos 2\theta - R^2 \sin 2\theta}\right) Re^{i\theta}(i)\,d\theta$$

But for $0 \le \theta \le \pi/4$, $\sin 2\theta > 0$

so the $e^{-R^2 \sin 2\theta}$ ~~term~~ forces integral $\to 0$ as $R \to \infty$.

c) Use the results in parts (a) & (b), together with the known integration formula

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

to evaluate the Fresnel integrals

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

which are important in diffraction theory.

From (a) & (b):

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr = e^{i\pi/4}\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\| \qquad\qquad = \frac{1}{\sqrt{2}}(1+i)\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx$$

Putting these together:

$$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

$$\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$