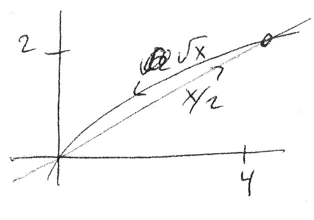
Use Green's theorem to evaluate the line integral.

$\oint_C (2xy\,dx + y^2 dy)$, where C is the closed curve formed by $y = x/2$ and $y = \sqrt{x}$ between $(0, 0)$ and $(4, 2)$



$$= \iint_S \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(2xy) \right) dA = \iint_S (-2x)\, dA$$

$$= \int_0^4 \int_{x/2}^{\sqrt{x}} (-2x)\, dy\, dx = \int_0^4 \Big[-2xy \Big]_{y=x/2}^{y=\sqrt{x}} dx$$

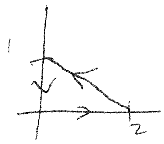
$$= \int_0^4 \left(-2x^{3/2} + \cancel{2} x^2/\cancel{2} \right) dx$$

$$= \left[-2\left(\frac{2}{5}\right) x^{5/2} + \frac{x^3}{3} \right]_0^4 = -\frac{4}{5}(4)^{5/2} + \frac{4^3}{3}$$

$$= -\frac{4}{5} 2^5 + \frac{2^6}{3} = 2^6\left[-\frac{2}{5} + \frac{1}{3} \right] = 2^6\left[-\frac{6}{15} + \frac{5}{15} \right]$$

$$= -\frac{2^6}{15} = \underline{\underline{-\frac{64}{15}}}$$

11. $\oint_C [xy\,dx + (x+y)\,dy]$ where C is the triangle w/ vertices $(0,0), (2,0), (0,1)$



$$= \iint_S \left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(xy) \right) dA$$

$$= \iint_S (1-x)\, dA = \int_0^2 \int_0^{1-x/2} (1-x)\, dy\, dx$$

$$= \int_0^2 (1-x)y \Big]_{y=0}^{y=1-x/2} dx = \int_0^2 (1-x)(1-\tfrac{x}{2})\, dx$$

$$= \int_0^2 \left[1 - \tfrac{3}{2}x + \tfrac{x^2}{2}\right] dx = \left[x - \tfrac{3}{4}x^2 + \tfrac{x^3}{6}\right]_0^2$$

$$= 2 - \tfrac{3}{\cancel{4}}(\cancel{4}) + \tfrac{8}{6} = -1 + \tfrac{4}{3} = \underline{\underline{\tfrac{1}{3}}}$$

10. III. Evaluate $\iint_G g(x,y,z)\,dS$ for: $g(x,y,z) = 2y^2 + z$
$G: z = x^2 - y^2, \quad 0 \le x^2 + y^2 \le 1$

$$z_x = 2x, \qquad z_y = -2y$$

$$dS = [1 + z_x^2 + z_y^2]^{1/2}\,dx\,dy = [1 + 4x^2 + 4y^2]^{1/2}\,dx\,dy$$

$$= \iint_R (2y^2 + x^2 - y^2)[1 + 4(x^2+y^2)]^{1/2}\,dx\,dy$$

$$= \iint_R (x^2+y^2)[1 + 4(x^2+y^2)]^{1/2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 r^2[1+4r^2]^{1/2}\,r\,dr\,d\theta = 2\pi\int_0^1 r^3[1+4r^2]^{1/2}\,dr$$

$u = r^2 \qquad dv = r(1+4r^2)^{1/2}\,dr$
$du = 2r\,dr \qquad v = \frac{1}{8}\frac{2}{3}(1+4r^2)^{3/2}$

$$= 2\pi\left[\frac{1}{12}r^2(1+4r^2)^{3/2}\Big|_0^1 - \frac{1}{6}\int_0^1 r(1+4r^2)^{3/2}\,dr\right]$$

$$= 2\pi\left[\frac{1}{12}(5)^{3/2} - \frac{1}{6}\frac{1}{8}\frac{2}{5}(1+4r^2)^{5/2}\Big|_0^1\right]$$

$$= 2\pi\left[\frac{5^{3/2}}{12} - \frac{1}{3}\frac{1}{8}\frac{1}{5}\left[5^{5/2} - 1\right]\right] = (2\pi)(5)^{3/2}\left[\frac{1}{12} - \frac{1}{24} + \frac{1}{24}\frac{1}{5^{5/2}}\right.$$

$$= 2\pi\left[\frac{5^{3/2}}{24} + \frac{1}{24}\frac{1}{5}\right) = \frac{\pi}{12}\left[5^{3/2} + \frac{1}{5}\right]$$

14. [redacted] Let G be the sphere $x^2+y^2+z^2=a^2$.
Evaluate each of the following:
(Hint: use symmetries to make each trivial.)

a) $\iint_G z\,dS = \underline{0}$ since for each positive z, there is a corresponding negative z

b) $\iint_G \frac{x+y^3+\sin z}{1+z^4}\,dS = \underline{0}$ since for each positive contribution, there is a corresponding negative contribution

c) $\iint_G (x^2+y^2+z^2)\,dS = a^2\iint_G dS = a^2(4\pi a^2) = \underline{4\pi a^4}$

since $\iint x^2 dS = \iint y^2 dS = \iint z^2 dS$

d) $\iint_G x^2\,dS = \frac{1}{3}\iint_G (x^2+y^2+z^2)\,dS = \underline{\frac{4\pi}{3}a^4}$

e) $\iint_G (x^2+y^2)\,dS = \frac{2}{3}\iint_G (x^2+y^2+z^2)\,dS = \underline{\frac{8\pi}{3}a^4}$

10. Use Gauss's divergence theorem to compute $\iint_{\partial S} \bar{F}\cdot\bar{n}\,dS$.

a. $\bar{F} = \langle z, x, y \rangle$; S is the hemisphere $0 \le z \le \sqrt{9-x^2-y^2}$

$$\iint_{\partial S} \bar{F}\cdot\bar{n}\,dS = \iiint_S (\nabla\cdot\bar{F})\,dV$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = 0$$

$$\Rightarrow \iint_{\partial S} \bar{F}\cdot\bar{n}\,dS = 0$$

6. 11. Let $F = \langle x, y, z \rangle$ & let S be a solid for which Gauss's divergence theorem applies. Show the volume of S is given by

$$Vol(S) = \frac{1}{3}\iint_{\partial S} F\cdot\vec{n}\, dS$$

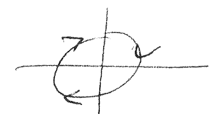
Gauss: $$\iint_{\partial S} F\cdot\vec{n}\, dS = \iiint_{S} (\nabla\cdot F)\, dV$$

but $\nabla\cdot F = 3$ so " $\iiint_{S} (3)\, dV = 3\, Vol(S)$

$$\Rightarrow Vol(S) = \frac{1}{3}\iint_{\partial S} F\cdot\vec{n}\, dS$$

III. Use Stokes' theorem to calculate $\iint_S (\text{curl } \bar{F}) \cdot \bar{n} \, dS$
where
$\bar{F} = \langle xz^2, x^3, \cos xz \rangle$, S is the part of the ellipsoid $x^2+y^2+3z^2=1$ below xy plane & $\bar{n}$ is lower normal

$$\iint_S (\text{curl } \bar{F}) \cdot \bar{n} \, dS = \oint_{\partial S} \bar{F} \cdot \bar{T} \, ds$$

∂S:  $z=0$

$x = \cos t$, $y = \sin t$

$$RHS = \int [x \cancel{z}^2 dx + x^3 dy]$$

$$= \int_{2\pi}^{0} (\cos t)^3 (\cos t) \, dt = -\int_0^{2\pi} \cos^4 t \, dt$$

$$= -\int_0^{2\pi} \left(\tfrac{1}{2}\right)^2 (1+\cos 2t)^2 \, dt = -\frac{1}{4}\int_0^{2\pi} [1 + 2\cos 2t + \cos^2 2t] \, dt$$

$$= -\frac{1}{4}\int_0^{2\pi} \left[1 + 2\cos 2t + \tfrac{1}{2}(1+\cos 4t)\right] dt$$

$$= -\frac{1}{4}\int_0^{2\pi} \left[\tfrac{3}{2} + 2\cos 2t + \tfrac{1}{2}\cos 4t\right] dt$$

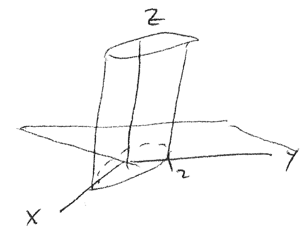
$$= -\frac{1}{4}\left[\tfrac{3}{2}(2\pi) + 0 + 0\right]$$

$$= -\frac{3}{4}\pi$$

III. Use Stokes' theorem to calculate $\oint F \cdot \bar{T}\, ds$ for

$\bar{F} = \langle 2z, x, 3y \rangle$, C is the ellipse that is the intersection of the plane $z = x$ & the cylinder $x^2 + y^2 = 4$, oriented clockwise as seen from above.

$$\oint_{\partial S} \bar{F} \cdot \bar{T}\, ds = \iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, dS$$



Let S = elliptical disk cut out by $z = x$.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & x & 3y \end{vmatrix} = \bar{i}(3) - \bar{j}(-2) + \bar{k}(1) = \langle 3, 2, 1 \rangle$$

$$z = x \Rightarrow z_x = 1, \quad z_y = 0$$

Using §17.5, §17.5 assumes upward normal, but clockwise ⇒ downward

$$\iint_S (\text{curl } \bar{F}) \cdot \bar{n}\, dS = -\iint_R \left[-3z_x - 2z_y + 1\right] dx\, dy$$

$$= -\int_0^{2\pi} \int_0^2 (-3 + 1)\, r\, dr\, d\theta = +2(2\pi)\left(\tfrac{1}{2} 2^2\right)$$

$$= \underline{+8\pi}$$

Compute $\displaystyle\int_{-\infty}^{\infty} \left(\frac{d^2}{dx^2} \delta(x) \right) f(x)\, dx$

Integrating by parts,

$$= -\int_{-\infty}^{\infty} \left(\frac{d}{dx} \delta(x) \right) f'(x)\, dx$$

$$= +\int_{-\infty}^{\infty} \delta(x)\; f''(x)\, dx$$

$$= \underline{f''(0)}$$

If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbb{C}^3$ are linear combinations of $(1,0,-1)$, $(0,1,1)$, $(1,1,1)$?

$$(1,0,0) = (1,1,1) - (0,1,1)$$

$$\begin{aligned}(0,0,1) &= (1,0,0) - (1,0,-1)\\ &= (1,1,1) - (0,1,1) - (1,0,-1)\end{aligned}$$

$$\begin{aligned}(0,1,0) &= (0,1,1) - (0,0,1)\\ &= (0,1,1) - (1,1,1) + (0,1,1) + (1,0,-1)\\ &= 2(0,1,1) - (1,1,1) + (1,0,-1)\end{aligned}$$

$\rightsquigarrow$ <u>all</u> vectors in $\mathbb{C}^3$ are linear combinations of the above

Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers.

Define
$$(x,y) + (x_1, y_1) = (x + x_1,\ y + y_1)$$
$$c(x,y) = (cx, y)$$

Is V, with these operations, a vector space over the field of real numbers?

Check:

- $\alpha + \beta = (x + x_1,\ y + y_1) = (x_1 + x,\ y_1 + y) = \beta + \alpha$
- $\alpha + (\beta + \gamma) = (x + (x_1 + x_2),\ y + (y_1 + y_2))$
  $= ((x + x_1) + x_2,\ (y + y_1) + y_2) = (\alpha + \beta) + \gamma$
- $0 = (0, 0)$ acts as additive identity
- for (x, y), $(-x, -y)$ acts as an additive inverse
- $1\,\alpha = \alpha$ trivially
- $(c_1 c_2)\,\alpha = ((c_1 c_2)x, y) = (c_1(c_2 x), y) = c_1(c_2 \alpha)$
- $c(\alpha + \beta) = (c(x + x_1),\ y + y_1) = (cx + cx_1,\ y + y_1)$
  $= c\alpha + c\beta$
- $(c_1 + c_2)\,\alpha = ((c_1 + c_2)x, y) = (c_1 x + c_2 x, y)$
  $\neq (c_1 x + c_2 x, 2y) = (c_1 x, y) + (c_2 x, y)$
  $= c_1 \alpha + c_2 \alpha$

$\rightarrow$ <u>not</u> a vector space

On $\mathbb{R}^n$, define the operations
$$\alpha \oplus \beta = \alpha - \beta$$
$$c \circ \alpha = -c\alpha$$
Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \circ)$?

- $\alpha \oplus \beta = \alpha - \beta = -(\beta - \alpha) = -(\beta \oplus \alpha)$
  so $\alpha \oplus \beta \neq \beta \oplus \alpha$

- $\alpha \oplus (\beta \oplus \gamma) = \alpha - (\beta - \gamma) = (\alpha - \beta) + \gamma \neq (\alpha \oplus \beta) \oplus \gamma$

- there exists unique additive identity

- there ~~exist~~ exist additive inverse: take "$-\alpha$" to be α
  then $\alpha \oplus \alpha = \alpha + (-\alpha) = 0$

- $1\alpha = -\alpha \neq \alpha$

- $(c_1 c_2) \circ \alpha = -(c_1 c_2)\alpha \neq -c_2(-c_1\alpha) = c_2 \circ (c_1 \circ \alpha)$

- $c \circ (\alpha \oplus \beta) = -c(\alpha - \beta) = -c\alpha - (-c\beta) = (c \circ \alpha) \oplus (c \circ \beta)$

- $(c_1 + c_2) \circ \alpha = -(c_1 + c_2)\alpha = (-c_1\alpha) + (-c_2\alpha)$
  $\neq (c_1 \circ \alpha) \oplus (c_2 \circ \alpha)$

Show that the only subspaces of $\mathbb{R}$ are $\mathbb{R}$ itself & the zero subspace.

$\{0\}$ is certainly a subspace.

Suppose a subspace contains $\alpha \neq 0$. Then it also contains all $c\alpha$, and so is the same as R.

Let V be the vector space of all functions from $\mathbb{R} \to \mathbb{R}$.
Let V_e be the subset of even functions, $f(-x) = f(x)$.
V_o be the subset of odd functions, $f(-x) = -f(x)$.

Show that V_e, V_o are subspaces of V.

V_e: Let $f, g \in V_e$.

$$(cf+g)(-x) = cf(-x) + g(-x)$$
$$= cf(x) + g(x) \Rightarrow cf+g \in V_e$$

Nonempty b/c $f(x) = x^2 \in V_e$

V_o: Let $f, g \in V_o$.

$$(cf+g)(-x) = cf(-x) + g(-x)$$
$$= -cf(x) - g(x) \Rightarrow cf+g \in V_o$$

Nonempty b/c $f(x) = x \in V_o$