(§ 3.2)

1. Find two linear operators T, U on $\mathbb{R}^2$ such that $TU = 0$, $UT \neq 0$.

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

<u>Check</u>

$$TU = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$UT = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$$

10. Let T be the (unique) linear operator on $\mathbb{C}^3$ for which

10 pt

$$T\epsilon_1 = (1, 0, i) \quad T\epsilon_2 = (0, 1, 1) \quad T\epsilon_3 = (i, 1, 0)$$

Is T invertible?

Recall T is invertible if & only if T is onto, so let's check whether the range of $T = \mathbb{C}^3$?

Try to solve

$$(a, b, c) = x_1(1, 0, i) + x_2(0, 1, 1) + x_3(i, 1, 0)$$

for x_1, x_2, x_3.

$$\Rightarrow \begin{cases} x_1 + ix_3 = a \\ x_2 + x_3 = b \\ ix_1 + x_2 = c \end{cases} \Rightarrow \begin{array}{l} ix_1 + x_2 = ia + b \\ \underline{\text{also}} \quad = c \end{array}$$

For general a, b, c, we have a ~~problem~~ contradiction.

$\Rightarrow T$ <u>not</u> onto

$\Rightarrow T$ <u>not</u> invertible

(§ 3.2)

11. Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, ~~and~~ and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Show that the linear transformation UT is not invertible.

10 pts

(UT) is invertible iff & only if null space $(UT) = \{0\}$

Now, null space of $UT \overset{?}{\supseteq}$ null space of T

$T: \mathbb{R}^3 \to \mathbb{R}^2$

so rank $T \leq 2$

but rank T + nullity $T = 3$

$\Rightarrow$ nullity $T = 3 -$ rank $T \geq 1$

Since nullity $T > 0$, null space of T contains more than $\{0\}$

$\Rightarrow$ null space of UT contains more than $\{0\}$

$\Rightarrow$ UT not invertible

(§ 3.3)

1\. Let V, W be vector spaces over a field F,
and let U be an isomorphism of V onto W.
10 pts Show that $T \mapsto UTU^{-1}$ is an isomorphism of $L(V,V)$ onto $L(W,W)$.

The map $T \mapsto UTU^{-1}$ is clearly a linear transformation:

$$(cT+S) \mapsto U(cT+S)U^{-1}$$
$$= c(UTU^{-1}) + (USU^{-1})$$

Need to show that it is one-to-one.
To do this, note if T is s.t. $UTU^{-1} = 0$, the zero transformation,

then $T = U^{-1} 0 U = 0$

$\Rightarrow$ null space $= \{0\}$

$\Rightarrow$ one-to-one,

& hence an isomorphism.

(§ 3.4)

2. Let θ be a real number. Show that the following two matrices are similar over the field of complex numbers:

10 pts

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Let $P = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$, so $P^{-1} = -\frac{1}{2i}\begin{bmatrix} -i & -i \\ -1 & 1 \end{bmatrix}$

$$P^{-1}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}P = \frac{1}{2i}\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & ie^{i\theta} \\ e^{-i\theta} & -ie^{-i\theta} \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} ie^{i\theta}+ie^{-i\theta} & -e^{i\theta}+e^{-i\theta} \\ e^{i\theta}-e^{-i\theta} & ie^{i\theta}+ie^{-i\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\Rightarrow$ the two matrices are similar

A-W 3.2.9

3\. For square matrices A, B, C,
verify the Jacobi identity

5 pts

$$[A, [B, C]] = [B, [A, C]] - [C, [A, B]]$$

where $[A, B] = AB - BA$.

$$RHS = B[A, C] - [A, C]B - C[A, B] + [A, B]C$$

$$= B(AC - CA) - (AC - CA)B - C(AB - BA) + (AB - BA)C$$

$$= A(BC - CB) + (CB - BC)A$$

$$= A[B, C] - [B, C]A$$

$$= [A, [B, C]]$$

A-W 3.2.13

4. The three Pauli spin matrices are

15 pts

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show that

a) $(\sigma_i)^2 = 1$

$$\sigma_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_2^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_3^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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b) $\sigma_j \sigma_k = i\sigma_l, \quad (j,k,l) = (1,2,3),\ (2,3,1),\ (3,1,2)$

$$\sigma_1 \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\sigma_3$$

$$\sigma_2 \sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i\sigma_1$$

$$\sigma_3 \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\sigma_2$$

(cont'd)

(cont'd)

c) $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbb{1}$

$\sigma_1^2 + \sigma_1^2 = 2(\mathbb{1})$, $\sigma_2^2 + \sigma_2^2 = 2(\mathbb{1})$, $\sigma_3^2 + \sigma_3^2 = 2(\mathbb{1})$
using (a)

$\sigma_1 \sigma_2$:

$$\sigma_2 \sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -i\sigma_3 = -\sigma_1 \sigma_2$$

$$\Rightarrow \underline{\sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0}$$

$\sigma_1 \sigma_3$:

$$\sigma_1 \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -i\sigma_2 = -\sigma_3 \sigma_1$$

$$\Rightarrow \underline{\sigma_1 \sigma_3 + \sigma_3 \sigma_1 = 0}$$

$\sigma_2 \sigma_3$:

$$\sigma_3 \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -i\sigma_1 = -\sigma_2 \sigma_3$$

$$\Rightarrow \underline{\sigma_2 \sigma_3 + \sigma_3 \sigma_2 = 0}$$

6. A-W 3.2.14

10 pts Using the Pauli σ_i of A-W 3.2.13, show that

$$(\bar{\sigma}\cdot\bar{a})(\bar{\sigma}\cdot\bar{b}) = \bar{a}\cdot\bar{b}(1) + i\,\bar{\sigma}\cdot(\bar{a}\times\bar{b})$$

where $\bar{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$

$$(\bar{\sigma}\cdot\bar{a})(\bar{\sigma}\cdot\bar{b}) = (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)(b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3)$$

$$= a_1b_1\sigma_1^2 + a_2b_2\sigma_2^2 + a_3b_3\sigma_3^2$$

$$+ (a_1b_2 - a_2b_1)\sigma_1\sigma_2 + (a_1b_3 - a_3b_1)\sigma_1\sigma_3 + (a_2b_3 - a_3b_2)\sigma_2\sigma_3$$

$$= (\bar{a}\cdot\bar{b})(1) + (a_1b_2 - a_2b_1)(i\sigma_3) + (a_1b_3 - a_3b_1)(-i\sigma_2) + (a_2b_3 - a_3b_2)(i\sigma_1)$$

Recall

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{x}(a_2b_3 - b_2a_3) - \hat{y}(a_1b_3 - b_1a_3) + \hat{z}(a_1b_2 - a_2b_1)$$

$$= (\bar{a}\cdot\bar{b})(1) + i\,\bar{\sigma}\cdot(\bar{a}\times\bar{b})$$