2. For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ and a matrix $\Lambda$ that is either diagonal or in Jordan normal form for $A = P\Lambda P^{-1}$.

10 pts

$$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

---

Find eigenvalues:

$$\det\begin{bmatrix} \frac{3}{2}-\lambda & -1/2 \\ -1/2 & 3/2-\lambda \end{bmatrix} = \left(\tfrac{3}{2}-\lambda\right)^2 - \tfrac{1}{4}$$

$$= \tfrac{9}{4} - 3\lambda + \lambda^2 - \tfrac{1}{4} = \lambda^2 - 3\lambda + 2$$

$$= (\lambda-1)(\lambda-2)$$

Find eigenvectors:

$\underline{\lambda = 1}$

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (1)\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{array}{l} \frac{3}{2}a - \frac{1}{2}b = a \\ -\frac{1}{2}a + \frac{3}{2}b = b \end{array} \Rightarrow \tfrac{1}{2}a = \tfrac{1}{2}b$$

$$\Rightarrow \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (2)\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{array}{l} \frac{3}{2}a - \frac{1}{2}b = 2a \\ -\frac{1}{2}a + \frac{3}{2}b = 2b \end{array} \Rightarrow \tfrac{1}{2}a = -\tfrac{1}{2}b$$

$$\Rightarrow \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(cont'd)

Define $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then $P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

<u>Check</u>

$$P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = A \checkmark$$

a

**3.** For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ & a matrix $\Lambda$ that is either diagonal or in Jordan normal form, for $A = P\Lambda P^{-1}$.

10 pts

$$A = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$$

---

Find eigenvalues:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ -4 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) + 4$$

$$= 5 - 6\lambda + \lambda^2 + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Find eigenvectors:

$\underline{\lambda = 3}$ $\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (3)\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{matrix} a+b = 3a \\ -4a+5b = 3b \end{matrix} \Rightarrow \begin{matrix} -2a+b = 0 \\ -4a+2b = 0 \end{matrix}$

$$\Rightarrow \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

But this is only <u>one</u> eigenvector, not enough to span $\mathbb{C}^2$.
So we must tree off another vector:

$(A - \lambda I) = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$, solve for $\bar{v}$ s.t. $(A - \lambda I)\bar{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} -2a+b = 1 \\ -4a+2b = 2 \end{matrix}$$

One sol'n: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(cont'd)

Define $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, then $P^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

<u>Check</u>

$$P \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} = A \checkmark$$

4. For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ and a matrix $\Lambda$ that is either diagonal or in Jordan normal form for $A = P\Lambda P^{-1}$.

10 pts

$$A = \begin{bmatrix} 5/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

---

Find eigenvalues:

$$\det \begin{bmatrix} \frac{5}{2}-\lambda & -1/2 & 0 \\ 1/2 & \frac{3}{2}-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} = (5-\lambda)\left[\left(\tfrac{5}{2}-\lambda\right)\left(\tfrac{3}{2}-\lambda\right) + \tfrac{1}{4}\right]$$

$$= (5-\lambda)\left[\tfrac{15}{4} - \tfrac{8}{2}\lambda + \lambda^2 + \tfrac{1}{4}\right]$$

$$= (5-\lambda)\left[4 - 4\lambda + \lambda^2\right]$$

$$= (5-\lambda)(2-\lambda)^2$$

Find eigenvectors:

$\underline{\lambda = 5}$

$$\begin{bmatrix} 5/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (5)\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{cases} \frac{5}{2}a - \frac{1}{2}b = 5a \\ \frac{1}{2}a + \frac{3}{2}b = 5b \\ 5c = 5c \end{cases}$$

$$\Rightarrow \begin{cases} -\frac{5}{2}a - \frac{1}{2}b = 0 \\ \frac{1}{2}a - \frac{7}{2}b = 0 \end{cases} \Rightarrow \begin{matrix} 5a + b = 0 \\ a - 7b = 0 \end{matrix} \Rightarrow 36b = 0$$

$$\Rightarrow \underline{a = b = 0}$$

$$\Rightarrow \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvectors, cont'd

$\underline{\lambda = 2}$

$$\begin{bmatrix} 5/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (2) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{cases} \frac{5}{2}a - \frac{1}{2}b = 2a \\ \frac{1}{2}a + \frac{3}{2}b = 2b \\ 5c = 2c \end{cases}$$

$$\Rightarrow \begin{cases} c = 0 \\ \frac{1}{2}a - \frac{1}{2}b = 0 \\ \frac{1}{2}a - \frac{1}{2}b = 0 \end{cases} \Rightarrow \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

But that's only one eigenvector; we need one more to form a basis. The matrix can't be diagonalized, but we can put it in Jordan normal form.

Solve $(A - 2I)\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for $\bar{v}$:

$$\begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \frac{1}{2}a - \frac{1}{2}b = 1 \\ \frac{1}{2}a - \frac{1}{2}b = 1 \\ 5c = 0 \end{matrix}$$

One solution: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(cont'd)

Let $P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

Using the expression in terms of cofactors,

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 0 & -2 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

<u>Check</u>

$$P\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -10 \\ -3 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -5 & 1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 5/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = A \checkmark$$

5. Compute $\exp\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$ :

10 pts

From previous results,

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}P^{-1} \quad \text{for } P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\exp\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = P\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}P^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e & e \\ e^2 & -e^2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e+e^2 & e-e^2 \\ e-e^2 & e+e^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e+e^2) & \frac{1}{2}(e-e^2) \\ \frac{1}{2}(e-e^2) & \frac{1}{2}(e+e^2) \end{bmatrix}$$

**6.** Compute $\exp\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$:

10 pts

From previous results,

$$\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} = P\underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}_{\equiv \Lambda} P^{-1} \quad \text{for } P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\Lambda^2 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 6 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 3^2 & 2\cdot 3 \\ 0 & 3^2 \end{bmatrix}$$

$$\Lambda^3 = \begin{bmatrix} 3^2 & 6 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 3^2+6\cdot 3 \\ 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 3^3 & 27 \\ 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 3^3 & 3\cdot 3^2 \\ 0 & 3^3 \end{bmatrix}$$

$$\Lambda^4 = \begin{bmatrix} 3^3 & 27 \\ 0 & 3^3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^4 & 3^3+27\cdot 3 \\ 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 3^4 & 4\cdot 3^3 \\ 0 & 3^4 \end{bmatrix}$$

<u>Claim</u> $\Lambda^n = \begin{bmatrix} 3^n & n\,3^{n-1} \\ 0 & 3^n \end{bmatrix}$

<u>Check</u> inductively:

$$\Lambda^n\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^n & n\,3^{n-1} \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^{n+1} & 3^n + n\,3^n \\ 0 & 3^{n+1} \end{bmatrix} = \begin{bmatrix} 3^{n+1} & (n+1)3^n \\ 0 & 3^{n+1} \end{bmatrix} = \Lambda^{n+1} \checkmark$$

$$\exp\Lambda = \begin{bmatrix} 1+3+\frac{1}{2!}3^2+\frac{1}{3!}3^3+\cdots & 0+1+\frac{1}{2!}(2\cdot 3)+\frac{1}{3!}(3\cdot 3^2)+\frac{1}{4!}(4\cdot 3^3)+\cdots \\ 0 & 1+3+\frac{1}{2!}3^2+\frac{1}{3!}3^3+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^3 & 0+1+3+\frac{1}{2!}3^2+\frac{1}{3!}3^3+\cdots \\ 0 & e^3 \end{bmatrix}$$

$$= \begin{bmatrix} e^3 & e^3 \\ 0 & e^3 \end{bmatrix} = \begin{bmatrix} e^3 & e^3 \\ 0 & e^3 \end{bmatrix}$$

(cont'd)

$$\exp\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} = P\left[\exp N\right]P^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^3 & e^3 \\ 0 & e^3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^3 - 2e^3 & e^3 \\ -2e^3 & e^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -e^3 & e^3 \\ -2e^3 & e^3 \end{bmatrix}$$

$$= \begin{bmatrix} -e^3 & e^3 \\ -4e^3 & 3e^3 \end{bmatrix} = e^3\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$$

(A-W 3.4.5)

1.6. If $A, B$ are Hermitian matrices, show that $(AB+BA)$, $i(AB-BA)$ are also Hermitian.

5 pts

$$(AB+BA)^\dagger = (AB)^\dagger + (BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger$$
$$= BA + AB$$
$$= (AB+BA) \checkmark$$

$$[i(AB-BA)]^\dagger = -i\left((AB)^\dagger - (BA)^\dagger\right) = -i\left(B^\dagger A^\dagger - A^\dagger B^\dagger\right)$$
$$= -i(BA - AB) = +i(AB-BA) \checkmark$$

(A-W 3.4.9)

2\. Two matrices A, B are each Hermitian.
Find a necessary and sufficient condition for their product AB to be Hermitian.

5 pts

$$(AB)^\dagger = B^\dagger A^\dagger$$
$$= BA$$

This $= AB$ if & only if $AB = BA$
or $[A, B] = 0$

(A-W 3.4.12(a))

30. Two matrices $U$, $H$ are related by $U = \exp(iaH)$ with $a$ real.

(a) If $H$ is Hermitian, show that $U$ is unitary.

5 pts

Claim $U^\dagger = U^{-1}$.

$$U^\dagger = \exp(-iaH^\dagger) = \exp(-iaH) \quad \text{since } H \text{ hermitian}$$

$$= U^{-1}.$$