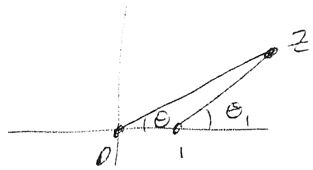
Riemann surfaces

4\. Describe a Riemann surface for the multiple-valued function,

10 pts

$$f(z) = \left(\frac{z-1}{z}\right)^{1/2}$$



To see where the branch cut lies, consider 3 cases:

- small circle enclosing $z=1$:
  $\theta \approx$ constant, but θ_1 varies $0 \leftrightarrow 2\pi$ & sees a cut
- small circle enclosing $z=0$:
  $\theta_1 \approx$ constant but θ varies $0 \leftrightarrow 2\pi$ & sees a cut
- big circle enclosing $z = 0$ & 1:
  $z-1 \approx z$, so $\frac{z-1}{z} \approx 1$ & no cut seen.

Thus, construct a Riemann surface for this function as follows.
Take 2 copies of the complex plane, call them A, B,
and cut them along the real axis between $z=0$ & $z=1$.

Sheet A has $0 \le \theta < 2\pi$
" B " $2\pi \le \theta < 4\pi$

so join the lower edge of the cut on A to the upper edge on B
& " ~~upper~~ " " " " " B " " " " " A.

Riemann surfaces

5\. Let C describe the positively-oriented circle $|z-2| = 1$ on the Riemann surface described for $z^{1/2}$, where the upper half of the circle lies on R_0 & the lower half on R_1.

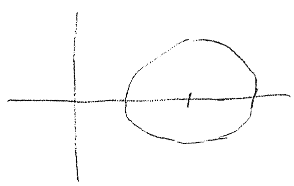
10 pts

Note that, for each point z on C, one can write

$$z^{1/2} = \sqrt{r}\, \exp\left(\pm \frac{i\theta}{2}\right), \quad \text{where } 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}$$

State why it follows that $\int_C z^{1/2}\, dz = 0$

Generalize this result to fit the case of other simple closed curves that cross from one sheet to another without enclosing the branch point.



For a path, $\int z^{1/2}\, dz = \frac{2}{3}\left[z^{3/2}(\text{end}) - z^{3/2}(\text{start})\right]$

Break up $\int_C z^{1/2}\, dz$ into 2 paths, one on each sheet.

R_0: start at $\theta = 0$, $z = 2$; end at $\theta = \cancel{\pi}\ 0$, $z = 1$ $\quad \int z^{1/2} dz = \frac{2}{3}\left[-2^{3/2} + 1^{3/2}\right]$

R_1: start at $\theta = 4\pi$, $z = 1$; end at $\theta = 4\pi$, $z = 2$ $\quad \int z^{1/2} dz = \frac{2}{3}\left[+2^{3/2} e^{4\pi \frac{3}{2}} - 1^{3/2} e^{4\pi\frac{3}{2}}\right]$

but $e^{4\pi \frac{3}{2}} = e^{2\pi \cdot 3} = 1$

so $\int_{R_0} z^{1/2} dz + \int_{R_1} z^{1/2} dz = \underline{\underline{0}}$

More generally, could put branch cut elsewhere, so that curve is on a single sheet & encloses no poles $\Rightarrow \int_C f(z)\, dz = 0$

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6. Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant in the interior of R.
10 pt Assuming $f(z) \neq 0$ anywhere in R, show that $|f(z)|$ has a minimum value in R which occurs on the boundary & ~~and~~ never in the interior. Hint: think about $g(z) = 1/f(z)$.

Recall from class:

S'pose a f'n f is continuous in a closed bounded region R & that it is analytic & not constant in the interior of R. Then the maximum value of $|f(z)|$ is reached on the boundary & never in interior.

$g(z) = \frac{1}{f(z)}$ satisfies all requirement

$\Rightarrow g(z)$ has maximum on boundary, not interior

$\Rightarrow f(z)$ has minimum on boundary, not interior.

7. Suppose that $f(z)$ is entire and the harmonic function

$$u(x,y) = \mathrm{Re}\,[f(z)]$$

10 pts has an upper bound, ie, $u(x,y) \le u_0 \;\; \forall z = x+iy$.

Show that $u(x,y)$ must be constant.

Hint: apply Liouville's theorem to $g(z) = e^{f(z)}$.

$g(z) = e^{f(z)}$ is entire

$|g(z)| = |e^{u}| = e^{u(x,y)} \le e^{u_0}$ so $g(z)$ is bounded

Liouville $\Rightarrow g(z)$ is constant

$\Rightarrow e^{f(z)}$ is constant

$\Rightarrow f(z)$ is constant.

Total pts: 140

10 points

1. Give two Laurent series expansions in powers of z for the function
$$f(z) = \frac{-1}{z^2(1-z)}$$

and specify the regions in which they are valid.

This function has singular points $z = 0, 1$,
& so is analytic in the regions
$$D_1: \ 0 < |z| < 1$$
$$D_2: \ |z| > 1$$

D_1: $f(z) = -z^{-2}\frac{1}{1-z} = -z^{-2}\sum_{n=0}^{\infty} z^n$ (valid for $|z| < 1$)
$$= -z^{-2} - z^{-1} - \sum_{n=0}^{\infty} z^n$$

D_2: $f(z) = +z^{-2}\frac{1}{z-1} = +z^{-2}\frac{1}{z}\frac{1}{1-1/z}$
$$= +z^{-3}\sum_{n=0}^{\infty}(1/z)^n$$
$$= +\sum_{n=3}^{\infty}\frac{1}{z^n}$$

10 points

2. Write two Laurent series expansions in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify the domains.

$f(z)$ is singular at $z = 0, \pm i$
so it is analytic in the domains

D_1: $0 < |z| < 1$

& D_2: $|z| > 1$

D_1: $f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} \sum_{n=0}^{\infty} (z^2)^n (-)^n$ (valid for $|z| < 1$)

$$= \frac{1}{z} + \sum_{n=1}^{\infty} z^{2n-1} (-)^n$$

D_2: $f(z) = \frac{1}{z} \frac{1}{z^2+1} = \frac{1}{z} \frac{1}{z^2} \frac{1}{1+1/z^2}$

$= \frac{1}{z^3} \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n (-)^n$ (valid for $|z| > 1$)

$$= \sum_{n=0}^{\infty} \frac{1}{z^{2n+3}} (-)^n$$

10 pts each part

30 pts total

3. a) Let z be any complex number, and let C denote the unit circle $w = e^{i\phi}$, $-\pi \le \phi \le \pi$, in the w plane.

Then, using a Laurent series of the form $\sum_{n=-\infty}^{\infty} c_n (w)^n = f(w)$

with $c_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}\,dw$ evaluated along the curve C above,

show that
$$\exp\left(\frac{z}{2}\left(w-\frac{1}{w}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z)\,w^n \qquad 0<|w|<\infty$$

where
$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(-i\left[n\phi - z\sin\phi\right]\right)d\phi \qquad n\in\mathbb{Z}$$

$J_n(x)$ is called the n^{th} Bessel function of the first kind, & $\exp\left(\frac{z}{2}\left(w-\frac{1}{w}\right)\right)$ is known as the generating functional for Bessel functions of the first kind.

$$f(w) = \sum_{n=-\infty}^{\infty} c_n w^n \quad \text{for} \quad c_n = \frac{1}{2\pi i}\oint \frac{e^{(z/2)(w-\frac{1}{w})}}{w^{n+1}}\,dw$$

For the curve C defined above,
$$c_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\exp\left[\frac{z}{2}\left(e^{i\phi}-e^{-i\phi}\right)\right]}{\left(e^{i\phi}\right)^{n+1}}\, e^{i\phi}\,(i)\,d\phi$$
$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{z\sin\phi}}{e^{in\phi}}\,(i)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp -i(n\phi - z\sin\phi)\,d\phi$$
$$= J_n(z)$$

$$\Rightarrow \boxed{f(w) = \exp \frac{z}{2}\left(w-\frac{1}{w}\right) = \sum_{n=-\infty}^{\infty} c_n w^n = \sum_{n=-\infty}^{\infty} J_n(z)\,w^n}$$

b) Show that $J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)\, d\phi$

Hint: this has nothing to do with Laurent series, but rather is merely a problem in symmetry properties of integrals.

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp -i(n\phi - z\sin\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\phi - z\sin\phi)\, d\phi \quad -\frac{i}{2\pi}\int_{-\pi}^{\pi} \sin(n\phi - z\sin\phi)\, d\phi$$

In the second integral,

$n\phi - z\sin\phi$ is odd under $\phi \mapsto -\phi$

$\sin(n\phi - z\sin\phi)$ is odd under $\phi \mapsto -\phi$

$\Rightarrow$ second integral vanishes

In the first integral,

$\cos(n\phi - z\sin\phi)$ is even under $\phi \mapsto -\phi$

$$\Rightarrow J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\phi - z\sin\phi)\, d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)\, d\phi$$

c) From the expression $e^{\frac{z}{2}(w-\frac{1}{w})} = \sum_{n=-\infty}^{\infty} J_n(z) w^n$,

show that for $n \geq 0$,

$$J_n(z) = \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,(n+s)!} \left(\frac{z}{2}\right)^{n+2s}$$

Hint: this also has nothing to do with Laurent series per se, though it is an exercise in infinite series.

$$e^{\frac{z}{2}(w-\frac{1}{w})} = e^{\frac{z}{2}w}\, e^{-\frac{z}{2}\frac{1}{w}}$$

$$= \left(\sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{z}{2}\right)^p w^p\right)\left(\sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{z}{2}\right)^r \left(\frac{1}{w}\right)^r\right)$$

When this is multiplied out, the coefficient of w^n, for $n \geq 0$, is

$$\frac{1}{n!}\left(\frac{z}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{z}{2}\right)^{n+1} \frac{1}{1!}\left(-\frac{z}{2}\right)^1 + \cdots$$

$$= \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,(n+s)!}\left(\frac{z}{2}\right)^{n+2s}$$

$$= J_n(z)$$

50 points per part

250 pt total

Churchill §55

4. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, an essential singular point, or a removable singular point.

a) $z \exp\left(\frac{1}{z}\right)$ $\rightarrow$ singular pt at $z = 0$

$$= z\left[1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= z + 1 + \underbrace{\frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots}_{\text{principal part}}$$

<u>essential singularity</u>

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b) $\frac{z^2}{1+z}$ $\rightarrow$ singular pt at $z = -1$

$$= \frac{(z+1-1)^2}{z+1} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1}$$

$$= (z+1) - 2 + \underbrace{\frac{1}{z+1}}_{\text{principal part}}$$

pole of order 1 - simple pole

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c) $\frac{\sin z}{z}$ $\rightarrow$ singular pt at $z = 0$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

Principal part $= 0$

Removable singularity

(cont'd)

(cont'd)

d) $\frac{\cos z}{z}$ → singular pt at $z = 0$

$$= \frac{1}{z}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right]$$

$$= \underbrace{\frac{1}{z}}_{\text{principal part}} - \frac{z}{2!} + \frac{z^3}{4!} + \cdots$$

<u>pole of order 1 - simple pole</u>

e) $\frac{1}{(2-z)^3}$ → singular pt at $z = 2$

It is its own Laurent series.

Principal part $= \frac{1}{(2-z)^3}$

→ pole of order 3

5 0 points per part
15 0 point total

5. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

a) $\dfrac{1-\cosh z}{z^3}$

$$= \frac{1}{z^3}\left[1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)\right]$$

$$= \frac{1}{z^3}\left[-\frac{z^2}{2!}-\frac{z^4}{4!}-\frac{z^6}{6!}\cdots\right]$$

$$= -\frac{1}{2!}\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}\cdots$$

→ pole of order 1, residue $= -\frac{1}{2!}$

b) $$\frac{1-\exp(2z)}{z^4} = z^{-4}\left[1-\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\cdots\right)\right]$$

$$= z^{-4}\left[-2z-\frac{(2z)^2}{2!}-\frac{(2z)^3}{3!}-\frac{(2z)^4}{4!}-\frac{(2z)^5}{5!}-\cdots\right]$$

$$= -2z^{-3}-\frac{2^2}{2!}z^{-2}-\frac{2^3}{3!}z^{-1}-\frac{2^4}{4!}-\frac{2^5}{5!}z-\cdots$$

→ pole of order 3, residue $= -\frac{2^3}{3!} = -\frac{2^2}{3} = -4/3$

c) $$\frac{\exp(2z)}{(z-1)^2} = (z-1)^{-2}\exp\left(2(z-1)+2\right)$$

$$= e^2(z-1)^{-2}\left[1+2(z-1)+\frac{(2(z-1))^2}{2!}+\cdots\right]$$

$$= e^2(z-1)^{-2}+2e^2(z-1)^{-1}+\frac{4e^2}{2!}+\cdots$$

→ pole of order 2, residue $= 2e^2$

5~~0~~ points per part
25~~0~~ ~~points total~~

6. Find the residue at $z=0$ of the functions:

a) $\frac{1}{z+z^2} = \frac{1}{z}\frac{1}{z+1} = \frac{1}{z}\sum_{n=0}^{\infty}(-)^n z^n = \frac{1}{z}[1 - z + z^2 \cdots]$

Residue $= 1$

b) $z\cos\left(\frac{1}{z}\right) = z\left[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right]$

$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} + \cdots$

Residue $= -\frac{1}{2!}$

c) $\frac{z - \sin z}{z} = \frac{1}{z}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\right]$

$= \frac{1}{z}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} \cdots\right] = \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} \cdots$

Residue $= 0$

d) $\frac{\cot z}{z^4} = \frac{1}{z^4}\left[\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} + \cdots\right]$

$= z^{-5} - \frac{1}{3}z^{-3} - \frac{1}{45}z^{-1} - \frac{2}{945}z + \cdots$

Residue $= -\frac{1}{45}$

e) $\frac{\sinh z}{z^4(1-z^2)} = z^{-4}\left[1 + z^2 + z^4 + z^6 + \cdots\right]\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$

$= z^{-4}\left[z + z^3\left(1 + \frac{1}{3!}\right) + z^5\left(1 + \frac{1}{3!} + \frac{1}{5!}\right) + \cdots\right]$

$= z^{-3} + z^{-1}\left(1 + \frac{1}{3!}\right) + z\left(1 + \frac{1}{3!} + \frac{1}{5!}\right) + \cdots$

Residue $= 1 + \frac{1}{3!} = 7/6$

5 points per part

15 points total

7. Use residues to evaluate the integrals of the following functions about the circle $|z|=3$ in the positive sense:

a) $\dfrac{\exp(-z)}{z^2}$ $\rightarrow$ singular pt at $z=0$

$$= z^{-2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} \cdots\right] = z^{-2} - z^{-1} + \frac{1}{2!} + \cdots$$

$$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{e^{-z}}{z^2} = 2\pi i(-1) = \underline{-2\pi i}$$

b) $z^{-2}\exp\left(\frac{1}{z}\right)$ $\rightarrow$ singular pt at $z=0$

$$= z^{-2}\left[1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right]$$

$$= \cancel{\left(z^2 + z + \frac{1}{2!} + \frac{1}{3!}z^{-1} + \cdots\right)} = z^{-2} + z^{-3} + \frac{1}{2!}z^{-4} + \cdots$$

$\rightarrow$ residue $= 0$

$$\int_C z^2 \exp\left(\frac{1}{z}\right)dz = 2\pi i\,(0) = \underline{0}$$

c) $\dfrac{z+1}{z^2-2z}$ $\rightarrow$ singular pts at $z = 0, 2$

Residue at $z=0$:

$$= \frac{z+1}{z}\,\frac{1}{z-2} = \left(1+\frac{1}{z}\right)\left(-\frac{1}{2}\right)\frac{1}{1-z/2} = \left(1+\frac{1}{z}\right)\left(-\frac{1}{2}\right)\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$= \left(-\frac{1}{2}\right)\left[\frac{1}{z} + \frac{3}{2} + \cdots\right] \Rightarrow \text{residue} = -1/2$$

Residue at $z=2$:

$$= \left(\frac{z-2+3}{z-2}\right)\frac{1}{z-2+2} = \left[1+\frac{3}{z-2}\right]\left(\frac{1}{2}\right)\left[\frac{1}{1+\frac{z-2}{2}}\right]$$

$$= \frac{1}{2}\left[1+\frac{3}{z-2}\right]\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{3}{z-2} + \left(-\frac{1}{2}\right) + \cdots\right] \Rightarrow \text{residue} = \frac{3}{2}$$

$$\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = \underline{2\pi i}$$

5 points each part
10 points total

8. Let $f(z)$ be a function which is analytic at z_0.

a) Show that if $f(z_0)=0$, then z_0 is a removable singular point of the function $g(z) = \dfrac{f(z)}{z-z_0}$

Taylor ~~expand~~: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots$
But $f(z_0)=0$

$$\Rightarrow g(z) = \frac{1}{z-z_0}\left[f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \right]$$
$$= f'(z_0) + \frac{f''(z_0)}{2!}(z-z_0) + \cdots$$

$\rightarrow$ principal part $\equiv 0$
$\Rightarrow$ removable singularity

b) Show that if $f(z_0) \neq 0$, then z_0 is a simple pole of the function $g(z)$, with residue $f(z_0)$.

Repeat the ~~above~~ above:

$$g(z) = \frac{1}{z-z_0}\left[f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots \right]$$

$\Rightarrow$ simple pole, residue $= f(z_0)$

Churchill
p180 § 56, 57

3 pts each part
13 pts total

1. In each case, show that the singular points of the function are poles. Determine the order of each pole, & find the corresponding residue.

a) $\frac{z^2+z}{z-1}$

$\rightarrow$ pole of order 1 at $z=1$, residue $= 2$

b) $\left(\frac{z}{2z+1}\right)^3$

$\rightarrow$ pole of order 3 at $z=-\frac{1}{2}$

Write $= \frac{\phi(z)}{(2z+1)^3} = \frac{1}{8}\frac{\phi(z)}{(z+1/2)^3} = \frac{1}{8}\frac{\phi(-\frac{1}{2}) + \phi'(-\frac{1}{2})(z+1/2) + \phi''(-1/2)(z+1/2)^2(\frac{1}{2!}) + \dots}{(z+1/2)^3}$

so residue $= \frac{1}{8}\frac{1}{2!}\phi''(-1/2)$, $\quad \phi(z) = z^3$

$\Rightarrow \phi' = 3z^2, \quad \phi'' = 6z$

$= \frac{1}{8}\frac{1}{2!}6(-1/2)$

$= -\frac{3}{16}$

c) $\coth z = \frac{\cosh z}{\sinh z}$ $\qquad p(z) = \cosh z, \ q(z) = \sinh z$

$q(0) = 0, \ q'(0) \neq 0$

$\Rightarrow$ simple pole (order 1) at $z=0$,

residue $= \frac{\cosh 0}{\cosh 0} = 1$

Additional poles at $z = n\pi i$ (since $\sinh iy = i \sin y$), which also have residue 1.

d) $\dfrac{\exp z}{z^2+\pi^2}$

$\to$ simple poles at $z = \pm \pi i$

at $z = +\pi i$: $= \dfrac{\exp z}{(z-\pi i)(z+\pi i)}$

so residue $= \dfrac{\exp(\pi i)}{2\pi i} = -\dfrac{1}{2\pi i}$

at $z = -\pi i$:

residue $= \dfrac{\exp(-\pi i)}{(-2\pi i)} = +\dfrac{1}{2\pi i}$

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e) $\dfrac{z}{\cos z}$ $\quad p(z) = z, \; q(z) = \cos z$

$q((2n+1)\pi) = 0, \quad q'((2n+1)\pi) \neq 0$

$\to$ simple (order 1) poles at $z = (2n+1)(\pi/2)$

Residue $= \dfrac{(2n+1)\pi/2}{-\sin((2n+1)\frac{\pi}{2})} = -(-)^n (2n+1)\frac{\pi}{2} = (-)^{n+1}(2n+1)\left(\frac{\pi}{2}\right)$

f) $\dfrac{z^{1/4}}{z+1}$ $\quad (|z|>0,\ 0<\arg z<2\pi)$

→ simple pole at $z=-1$,

residue is $\quad (-)^{1/4} = e^{\pi i (1/4)} = \frac{1}{\sqrt{2}}(1+i)$