Physics 5714: Methods of theoretical physics

Fall 2019

Test 1

September 30, 2019

NAME: ______________________________ Solutions

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has 11 pages, including this cover sheet. If it does not, see me.

1. (10 points) Construct a unit vector perpendicular to the surface

$$\frac{\sin(x)}{a^2} + \frac{y^3}{b^2} - \frac{z^2}{c^2} = 1$$

at any point (x, y, z) on the surface. (a, b, c) are fixed constants.

Define $H = \frac{\sin x}{a^2} + \frac{y^3}{b^2} - \frac{z^2}{c^2} - 1$

A vector $\perp$ surface $\{H = 0\}$ is

$$\nabla H = \langle \frac{\cos x}{a^2}, \frac{3y^2}{b^2}, -\frac{2z}{c^2} \rangle$$

We want a unit vector:

$$|\nabla H| = \left[\frac{\cos^2 x}{a^2} + \frac{9y^4}{b^4} + \frac{4z^2}{c^4} \right]^{1/2}$$

so we take

$$\boxed{\hat{n} = \frac{\nabla H}{|\nabla H|} = \left(\frac{\cos^2 x}{a^2} + \frac{9y^4}{b^4} + \frac{4z^2}{c^4} \right)^{-1/2} \langle \frac{\cos x}{a^2}, \frac{3y^2}{b^2}, -\frac{2z}{c^2} \rangle}$$

2. (10 points) Find the surface area of the part of the plane $3x + 4y + 6z = 12$ that is above the circle in the xy plane centered at the origin and with radius 1.

$$z = 2 - \frac{1}{2}x - \frac{2}{3}y$$

$$\frac{\partial z}{\partial x} = -\frac{1}{2}, \quad \frac{\partial z}{\partial y} = -\frac{2}{3}$$

$$dS = \left(1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)^{1/2} dA = \left(1 + \frac{1}{4} + \frac{4}{9}\right)^{1/2} dA = \left(\frac{61}{36}\right)^{1/2} dA$$

$$\text{Surface area} = \iint \frac{\sqrt{61}}{6}\, dA = \frac{\sqrt{61}}{6} \iint dA$$

$$\boxed{= \frac{\sqrt{61}}{6}\pi}$$

3. (10 points) Let f, g be any two functions on $\mathbb{R}^3$. Compute

$$\nabla \cdot (\nabla f \times \nabla g),$$

and simplify the result.

$$\nabla \cdot (\nabla f \times \nabla g) = \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix}$$

$$= \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - g_x f_z) + \frac{\partial}{\partial z}(f_x g_y - g_x f_y)$$

$$= \cancel{f_{xy} g_z} + \cancel{f_y g_{xz}} - \cancel{f_{xz} g_y} - \cancel{f_z g_{xy}}$$

$$- \cancel{f_{xy} g_z} - \cancel{f_x g_{yz}} + \cancel{g_{xy} f_z} + \cancel{g_x f_{yz}}$$

$$+ \cancel{f_{xz} g_y} + \cancel{f_x g_{yz}} - \cancel{g_{xz} f_y} - \cancel{g_x f_{yz}}$$

$$\boxed{= 0}$$

4. (10 points) Evaluate

$$\int\int_S g(x,y,z)dS$$

where

$$g(x,y,z) = x^2 + y^2 + z^2$$

and S is the surface $z = x+y+1$, over the square in the xy plane defined by $0 \le x \le 1$, $0 \le y \le 1$.

$$\frac{\partial z}{\partial x} = 1, \ \frac{\partial z}{\partial y} = 1 \Rightarrow dS = (1+1+1)^{1/2} dA = \sqrt{3}\, dA$$

$$\iint_S g(x,y,z)\,dS = \sqrt{3}\int_0^1 dx \int_0^1 dy \ \left[x^2+y^2+(x+y+1)^2\right] dA$$

$$= \sqrt{3}\int_0^1 dx \int_0^1 dy \left[x^2+y^2+x^2+y^2+1+2xy+2x+2y\right]$$

$$= \sqrt{3}\int_0^1 dx \left[\, 2x^2y\Big|_0^1 + \frac{2y^3}{3}\Big|_0^1 + y\Big|_0^1 + \frac{2xy^2}{2}\Big|_0^1 + 2xy\Big|_0^1 + \frac{2y^2}{2}\Big|_0^1\right]$$

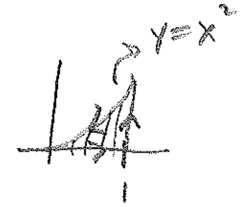
$$= \sqrt{3}\int_0^1 dx \left[2x^2 + \frac{2}{3} + 1 + x + 2x + 1\right]$$

$$= \sqrt{3}\int_0^1 dx \left[2x^2 + 3x + \frac{8}{3}\right]$$

$$= \sqrt{3}\left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + \frac{8}{3}x\right)\Big|_0^1 = \sqrt{3}\left[\frac{2}{3} + \frac{3}{2} + \frac{8}{3}\right]$$

$$\boxed{= \sqrt{3}\left[\frac{10}{3} + \frac{3}{2}\right] = \sqrt{3}\left[\frac{29}{6}\right]}$$

5. (10 points) Use Green's theorem to evaluate the line integral



$$\oint_C [(2x + y^2)dx + (x^2 + 2y)dy]$$

where C is the closed curve formed by $y = 0$, $x = 1$, $y = x^2$, traversed counterclockwise. For full credit, you must use Green's theorem, and *not* directly evaluate the line integral.

$$\int_C [(2x+y^2)dx + (x^2+2y)dy] = \iint_S (\underbrace{2x}_{\frac{\partial}{\partial x}(x^2+2y)} - \underbrace{2y}_{\frac{\partial}{\partial y}(2x+y^2)})\, dA$$

$$= \int_0^1 dx \int_0^{x^2} dy\, (2x-2y) = \int_0^1 [2xy - y^2]_0^{x^2} dx$$

$$= \int_0^1 dx \, [2x^3 - x^4] = \left[\frac{2}{4}x^4 - \frac{1}{5}x^5\right]_0^1$$

$$\boxed{= \frac{1}{2} - \frac{1}{5} = \frac{5}{10} - \frac{2}{10} = \frac{3}{10}}$$

6. (10 points) Let S be a closed surface in three-dimensional space, bounding some volume. Compute the surface integral

$$\int\int_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS$$

where $\vec{F}$ is any vector field.

Gauss's theorem:

$$\iint_{S=\partial V} (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iiint_V \underbrace{\nabla \cdot (\nabla \times \vec{F})}_{0}\, dV$$

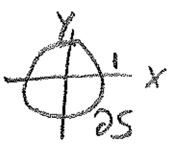
$= 0$

7. (12 points) Use Stokes' theorem to compute

$$\int\int_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} dS$$

where $\vec{F} = \langle -y, x, 0\rangle$, S is the hemisphere $z = \sqrt{1-x^2-y^2}$, and $\vec{n}$ is the upper normal. For full credit, you must use Stokes' theorem, and *not* directly evaluate the surface integral.

Stokes: $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \int_{\partial S} \vec{F} \cdot \vec{T} ds = \int_{\partial S} \vec{F} \cdot d\vec{r}$



$x = \cos\theta \qquad dx = -\sin\theta \, d\theta$

$y = \sin\theta \qquad dy = \cos\theta \, d\theta$

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int \left[(-y) dx + (x) dy\right]$$

$$= \int_0^{2\pi} d\theta \left[(-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta)\right]$$

$$= \int_0^{2\pi} d\theta$$

$$\boxed{= 2\pi}$$

8. (8 points) Show that the vector space of upper triangular 2×2 matrices (*i.e.* matrices whose nonzero entries lie only above or along the diagonal), is a subspace of the vector space of all 2×2 matrices.

Nonempty: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is upper-triangular

Let $\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$, $\begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}$ be upper-triangular.

Note

$$c\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} ca_1+b_1 & ca_2+b_2 \\ 0 & ca_3+b_3 \end{bmatrix}$$

is also upper-triangular.

$\Rightarrow$ <u>subspace</u>

9. (10 points) Let V be a vector space, and $\{W_\alpha\}$ a collection of subspaces of V. Show that the intersection of the subspaces, $\cap_\alpha W_\alpha$, is also a subspace of V.

Nonempty:

$0 \in W_\alpha$ for each $\alpha \Rightarrow 0 \in \cap_\alpha W_\alpha$

Let $x, y \in \cap_\alpha W_\alpha$. Then $x, y \in W_\alpha$ for all α.

Since each W_α is a subspace,

$cx+y \in W_\alpha$ for all α for any scalar c.

$\Rightarrow cx+y \in \bigcap_\alpha W_\alpha$

$\Rightarrow$ subspace

10. (10 points) Let V be the set $\mathbf{R}^2$, with the operation of vector addition defined by

$$(a,b) \oplus (c,d) = (a+c, b+d)$$

and the operation of scalar multiplication defined by

$$c \cdot (a,b) = (ca, -cb)$$

(Put simply, $\oplus$ is the usual vector addition, but the scalar multiplication $\cdot$ is different from the usual notion.) Is V together with $\oplus$, $\cdot$, a vector space? Explain.

Not a vector space.

It fails the following axioms:

- $1 \cdot x \neq x$: $\quad 1 \cdot (a,b) = (a,-b) \neq (a,b)$
- $(c_1 c_2) \cdot x \neq c_1 \cdot (c_2 \cdot x)$:

$$\begin{aligned}(c_1 c_2) \cdot (a,b) &= (c_1 c_2 a, -c_1 c_2 b) \\ &= c_1 \cdot (c_2 a, c_2 b) \\ &\neq c_1 \cdot (c_2 \cdot x)\end{aligned}$$