Physics 5714: Methods of theoretical physics

Fall 2019

Test 2

November 4, 2019

NAME: ______________________ Solutions

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has 13 pages, including this cover sheet. If it does not, see me.

1. (12 points total; 4 pts apiece) In each case, determine whether the listed vectors in each vector space V are linearly independent, and also whether or not they span the vector space. (Only a brief justification is required.) In each case, do they form a basis?

(a) $V =$ polynomials of degree ≤ 3, $\{2x^3 + 5x^2, 3x + 5, 1\}$

LI, don't span, not a basis

(b) $V = \mathbf{R}^3$, $\{(5, 1, 1), (0, 0, 0), (1, 1, 2)\}$

~~LI~~, don't span, not a basis

(c) $V =$ real-valued functions on the real line, $\{\sin(x), \cos(x), \exp(ix)\}$

Badly-stated problem: e^{ix} not real valued.

Use $e^{ix} = \cos x + i \sin x$:

LI over $\mathbb{R}$, ~~LI~~ over $\mathbb{C}$

Definitely does not span

2. (3 points each, 12 points total) Let T be a linear transformation $\mathbf{C}^3 \to \mathbf{C}^2$ such that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

(a) Compute

$$T\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$= aT\begin{bmatrix}1\\0\\0\end{bmatrix} + bT\begin{bmatrix}0\\1\\0\end{bmatrix} + cT\begin{bmatrix}0\\0\\1\end{bmatrix} = a\begin{bmatrix}0\\2\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}b-c\\2a-2c\end{bmatrix}$$

(b) What are the range and rank of T ?

Range $= \mathbb{C}^2$, rank $= 2$

(c) What are the null space and nullity of T ?

Null space $= \{a=b=c\}$, nullity $= 1$

(d) Is T invertible?

No : nullity > 0

3. (10 points) Let A be an $m \times n$ matrix, and P an $m \times m$ matrix, not necessarily invertible. How is the row space of A related to the row space of PA ? Explain.

Row j of $(PA) = \sum_i P_{ji}$ (row i of A)

hence

row space of $PA \subseteq$ row space of A

4. (3 points each, 12 points total) State whether the following matrices are diagonalizable. (No computations are expected, required, or even desired; rather, you should be able to tell in each case nearly by inspection whether the matrix is diagonalizable, so give just a phrase to explain your reasoning.)

(a)

$$\begin{bmatrix} 5 & -i \\ i & 8 \end{bmatrix}$$

Hermitian $\Rightarrow$ diagonalizable

(b)

$$\begin{bmatrix} 5 & -i \\ -i & 8 \end{bmatrix}$$

symmetric $\Rightarrow$ diagonalizable

(c)

$$\begin{bmatrix} 5 & 4 & 3 \\ 4 & 8 & 2 \\ 3 & 2 & 9 \end{bmatrix}$$

Symmetric $\Rightarrow$ diagonalizable

(d)

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Jordan normal form $\Rightarrow$ <u>not</u> diagonalizable

5. (15 points) Let V be the vector space of polynomials of degree ≤ 1, *i.e.*

$$c_0 + c_1 x$$

with dot product defined by

$$f(x) \cdot g(x) = \int_0^1 x f(x) g(x)\, dx$$

(Note this dot product is a bit different from what you have seen on homework previously.) Use the Gram-Schmidt procedure to construct an orthonormal basis from the linearly independent vectors

$$\{5,\ ax + b\},$$

for constants a, b. (The next page is left blank as extra space for your solution.)

$$|5|^2 = \int_0^1 x (5)^2 dx = \frac{1}{2}(5)^2$$

$$\boxed{v_1 = \frac{5}{5/\sqrt{2}} = \sqrt{2}}$$

$$\alpha_2 = ax+b$$

$$w_2 = \alpha_2 - (v_1 \cdot \alpha_2) v_1$$

$$v_1 \cdot \alpha_2 = \sqrt{2} \int_0^1 x(ax+b)\, dx = \sqrt{2}\left(\frac{a}{3} + \frac{b}{2}\right)$$

$$(v_1 \cdot \alpha_2) v_1 = 2\left(\frac{a}{3} + \frac{b}{2}\right) = \frac{2}{3}a + b$$

$$w_2 = \alpha_2 - (v_1 \cdot \alpha_2) v_1 = (ax + \cancel{b}) - \left(\frac{2}{3}a + \cancel{b}\right)$$

$$= a\left[x - \frac{2}{3}\right]$$

$$|w_2|^2 = a^2 \int_0^1 x\left(x - \frac{2}{3}\right)^2 dx = a^2 \int_0^1 \left(x^3 - \frac{4}{3}x^2 + \frac{4}{9}x\right) dx$$

$$= a^2\left(\frac{1}{4} - \frac{4}{3}\frac{1}{3} + \frac{4}{9}\frac{1}{2}\right) = \frac{a^2}{9}\left(\frac{9}{4} - 4 + 2\right) = \frac{a^2}{9}\left(\frac{1}{4}\right) = \frac{a^2}{36}$$

$$|w_2| = \frac{a}{6}$$

$$\boxed{v_2 = \frac{w_2}{|w_2|} = 6\left(x - \frac{2}{3}\right)}$$

(This page left blank as extra space for your solution.)

6. (15 points) For the matrix A below, either diagonalize or put in Jordan normal form. In other words, compute a matrix P and a matrix Λ that is either diagonal or in Jordan normal form, such that $A = P\Lambda P^{-1}$. In addition, explicitly check that $A = P\Lambda P^{-1}$.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 1/2 & 1/2 & 2 \end{bmatrix}$$

(The next two pages are left blank as extra space for your solution.)

Eigenvalues:

$$\det(A - \lambda I) = \det\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 1/2 & 1/2 & 2-\lambda \end{bmatrix} = (2-\lambda)\left[(3-\lambda)^2 - 1\right]$$

$$= (2-\lambda)\left[9 + \lambda^2 - 6\lambda - 1\right]$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 8)$$

$$= +(2-\lambda)(\lambda-2)(\lambda-4).$$

$$\Rightarrow \lambda = 2, 4$$

Eigenvectors:

$\lambda = 4$
$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 1/2 & 1/2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 4\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{aligned} 3a - b &= 4a \\ -a + 3b &= 4b \\ \tfrac{1}{2}(a+b) + 2c &= 4c \end{aligned}$$

$$\Rightarrow \begin{aligned} a &= -b \\ \tfrac{1}{2}(a+b) &= 2c \end{aligned} \Rightarrow \begin{aligned} a &= -b \\ c &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(This page left blank as extra space for your solution to the last problem.)

Eigenvectors, cont'd

$\underline{\lambda = 2}$

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 1/2 & 1/2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{matrix} 3a - b = 2a \\ -a + 3b = 2b \\ \frac{1}{2}(a+b) + 2c = 2c \end{matrix} \Rightarrow \begin{matrix} a = b \\ a + b = 0 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspace only <u>one</u>-dim'l, but $\lambda = 2$ is double root.

Not enough eigenvectors – can <u>not</u> diagonalize.

Instead, put in Jordan normal form.

Let $\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solve $(A - 2I)v_1 = \alpha$ for v_1:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a - b = 0 \\ -a + b = 0 \\ \frac{1}{2}(a+b) = 1 \end{matrix} \Rightarrow \begin{matrix} a = b \\ a = 1 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$$

& we pick $c = 0$.

Check that this procedure terminates:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} a - b = 1 \\ -a + b = 1 \end{matrix}\right\} \text{contradiction } \checkmark$$
$$\frac{1}{2}(a+b) = 0$$

(This page left blank as extra space for your solution to the last problem.)

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & +1 & 0 \\ 0 & 0 & -2 \\ -1 & -1 & 0 \end{bmatrix}$$

<u>Check</u> $PP^{-1} = I \checkmark$

<u>Check</u> $P\Lambda P^{-1} = -\frac{1}{2} P \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ -1 & -1 & 0 \end{bmatrix}$

$$= -\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -4 & 4 & 0 \\ -1 & -1 & -4 \\ -2 & -2 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 0 \\ -1 & -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 1/2 & 1/2 & 2 \end{bmatrix} = A \checkmark$$

7. (10 points) Show that if A is a symmetric matrix, then eigenvectors associated to distinct eigenvalues are orthogonal.

$$Av = \lambda v, \quad Aw = \sigma w, \quad \lambda \neq \sigma$$

$$\lambda w^T v = w^T A v = w^T A^T v = (Aw)^T v$$

$$= \sigma w^T v$$

but $\lambda \neq \sigma \Rightarrow \underline{w^T v = 0}$

8. (14 points) Suppose $T : V \to V$ is a linear operator, and $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$ are ordered bases for the two-dimensional vector space $V \cong \mathbb{C}^2$. Suppose that in the basis $\{\alpha_1, \alpha_2\}$, T can be represented by the matrix

$$\begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}.$$

Suppose that with respect to the standard basis for $\mathbb{C}^2$,

$$\alpha_1 = (1,5)^T, \quad \alpha_2 = (3,2)^T,$$

and that

$$\begin{aligned} \beta_1 &= \alpha_1 + 2\alpha_2, \\ \beta_2 &= 3\alpha_1 + \alpha_2. \end{aligned}$$

Find the matrix representing T in the basis $\{\beta_1, \beta_2\}$. (The next page is left blank as extra space for your solution.)

$P = [\beta_1 | \beta_2] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $\quad P^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$[T]_{\{\beta_1,\beta_2\}} = P^{-1}AP = P^{-1}\begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$= -\frac{1}{5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 11 & 8 \\ 6 & 3 \end{bmatrix}$

$= -\frac{1}{5}\begin{bmatrix} -7 & -1 \\ -16 & -13 \end{bmatrix}$

(This page left blank as extra space for your solution.)

Check

$T(\alpha_1) = A\begin{bmatrix}1\\0\end{bmatrix} = \alpha_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad T(\alpha_2) = A\begin{bmatrix}0\\1\end{bmatrix} = 5\alpha_1 + 3\alpha_2 = \begin{bmatrix}5\\3\end{bmatrix}$

$T(\beta_1) = T(\alpha_1 + 2\alpha_2) = \alpha_1 + 2(5\alpha_1 + 3\alpha_2) = 11\alpha_1 + 6\alpha_2$

$T(\beta_2) = T(3\alpha_1 + \alpha_2) = 3\alpha_1 + (5\alpha_1 + 3\alpha_2) = 8\alpha_1 + 3\alpha_2$

$\beta_1 = \alpha_1 + 2\alpha_2$
$\beta_2 = 3\alpha_1 + \alpha_2$ $\Rightarrow$ $\alpha_1 = -\frac{1}{5}(\beta_1 - 2\beta_2)$
$\alpha_2 = -\frac{1}{5}(-3\beta_1 + \beta_2)$

$T(\beta_1) = 11\alpha_1 + 6\alpha_2 = -\frac{1}{5}[11(\beta_1 - 2\beta_2) + 6(-3\beta_1 + \beta_2)]$
$= -\frac{1}{5}[-7\beta_1 - 16\beta_2]$

$T(\beta_2) = 8\alpha_1 + 3\alpha_2 = -\frac{1}{5}[8(\beta_1 - 2\beta_2) + 3(-3\beta_1 + \beta_2)]$
$= -\frac{1}{5}[-\beta_1 - 13\beta_2]$

& these agree with matrix given.