Physics 5714 – Problem set 5

1. Find two linear operators T, U on $\mathbf{R}^2$ such that $TU = 0$, $UT \neq 0$.

2. Let T be the (unique) linear operator on $\mathbf{C}^3$ for which

$$T\epsilon_1 \;=\; (1,0,i), \quad T\epsilon_2 \;=\; (0,1,1), \quad T\epsilon_3 \;=\; (i,1,0)$$

   where the ϵ_i are the standard basis vectors. Is T invertible?

3. Let T be a linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^2$, and let U be a linear transformation from $\mathbf{R}^2$ into $\mathbf{R}^3$. Show that the linear transformation UT is not invertible.

4. Let V, W be vector spaces over a field F, and let U be an isomorphism of V onto W. Show that $T \mapsto UTU^{-1}$ is an isomorphism of $L(V,V)$ onto $L(W,W)$.

5. Let θ be a real number. Show that the following two matrices are similar over the field of complex numbers:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix}$$

6. (AWH 2.2.7) For square matrices A, B, C, verify the Jacobi identity

$$[A,[B,C]] \;=\; [B,[A,C]] \;-\; [C,[A,B]]$$

   where $[A,B] = AB - BA$.

7. (AWH 2.2.11) The three Pauli spin matrices are

$$\sigma_1 \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \;=\; \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Show that

   (a) $(\sigma_i)^2 = 1$

   (b) $\sigma_j\sigma_k = i\sigma_\ell$, $(j,k,\ell) = (1,2,3), (2,3,1), (3,1,2)$

   (c) $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}1$

8. Using the Pauli σ_i of the last exercise, show that

$$(\vec{\sigma}\cdot\vec{a})\left(\vec{\sigma}\cdot\vec{b}\right) \;=\; \left(\vec{a}\cdot\vec{b}\right)1 \;+\; i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right)$$

   where $\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$.