Start here Fri

Consider $\nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) = \nabla\cdot\left(\frac{\hat{r}}{|\bar{r}|^2}\right)$

We've seen already that (p 32 in these notes)

- $\nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) = 0$ away from $r=0$
- $\iint_S \left(\frac{\bar{r}}{|\bar{r}|^3}\right)\cdot\hat{n}\, dS = 4\pi$ for any surface S enclosing $\bar{r}=0$

From Gauss's divergence theorem,

$$\iint_S \left(\frac{\bar{r}}{|\bar{r}|^3}\right)\cdot\hat{n}\, dS = \iiint_{\substack{V \\ \text{st } \partial V = S}} \nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) dV$$

So even though $\nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) = 0$ away from the origin,

we see $\iiint \nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) dV \neq 0$

$$= 4\pi$$

Clearly there's some sort of delta function here.

$$\nabla\cdot\left(\frac{\bar{r}}{|\bar{r}|^3}\right) = \underbrace{-\nabla\cdot\nabla\left(\frac{1}{r}\right)}_{= -\nabla^2\left(\frac{1}{r}\right)} = 4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

$\rightsquigarrow$ This is the starting point for Jackson's E&M.

Linear algebra

A field = set of numbers, on which $+$, $-$, $\cdot$, & $/$ (except by 0) are defined.

Ex $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, ...

Not ex: $\mathbb{Z}$, $\mathbb{N}$

Vector space over a field F

= set V with operations $+: V \times V \to V$ & scalar multiplication (by an element of the field), such that

- $x+y = y+x \quad \forall x,y \in V$
- $x+(y+z) = (x+y)+z \quad \forall x,y,z \in V$
- $\exists!\ 0 \in V$ s.t. $x+0 = x \quad \forall x \in V$
- $\forall x \in V\ \exists!$ "$-x$" s.t. $x+(-x) = 0$
- $1x = x \quad \forall x \in V \quad (1 \in F)$
- $(ab)x = a(bx) \quad \forall x \in V,\ \forall a,b \in F$
- $a(x+y) = (ax)+(ay) \quad \forall a \in F,\ x,y \in V$
- $(a+b)x = (ax)+(bx) \quad \forall a,b \in F,\ x \in V$

Call elements of V, vectors.

Exs of things that form vector spaces:

vectors in the ordinary sense
matrices
tensors
functions (mention QM application)
polynomials

$\leadsto$ outline + operation for each of these.

Linear combination

abstract sense

A vector β in a vector space V is said to be a <u>linear combination</u> of the vectors $\alpha_1, \ldots, \alpha_n$ in V if $\exists$ scalars $c_1, \ldots, c_n \in F$ s.t.

$$\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$$

(give some exs)

~~[illegible]~~ Subspaces

Def'n: Let V be a vector space over the field F.
A subspace of V is a subset W of V which is itself a vector space with the operations of vector add'n & scalar multiplication inherited from V.

(give exs to build intuition)

Thm: A nonempty subset $W \subseteq V$ is a subspace of V iff for each $\alpha, \beta \in W$, each $c \in F$, $c\alpha + \beta \in W$

- explain to give intuition

Pf: S'pose W is a nonempty subset of V st $c\alpha + \beta \in W$.
Since W is nonempty, $\exists \rho \in W$, so $(-1)\rho + \rho = 0 \in W$.
Then, $\forall \alpha \in W, c \in F$ ~~$c\alpha = c\alpha + 0$~~ $c\alpha + 0 = c\alpha \in W$
In particular, $(-1)\alpha = -\alpha \in W$
& the rest follows sim'ly.
[Explain]

Check def'n of vector space:

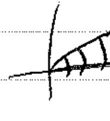
Conversely, s'pose ~~[illegible]~~ W is a subspace.
Then triv'ly, $c\alpha + \beta \in W$.

Ex $\{0\} \subseteq V$ is the zero subspace

Ex (symmetric matrices) $\subseteq$ matrices
↳ explain

Ex Hermitian matrices $\subseteq$ matrices
↳ explain

Ex polynomials $\subseteq$ functions

Ex not a subspace: vectors in  in 2D (~~[illegible]~~ $-v \notin$ region for $v \in$ region)

Ex Sol'n space of a system of homogeneous linear equ'ns.

Let A be $m \times n$ matrix,
then sol'n space $= \{x \mid Ax = 0\}$

If $Ax = 0$, then $A(cx+y) = cAx + Ay$
$= 0 + 0 = 0$

Thm Let V be a vector space over a field F.
The intersection of any collection of subspaces, is a subspace.

Pf Let $\{W_\alpha\}$ be a collection of subspaces of V,
set $W = \bigcap_\alpha W_\alpha$.

$0 \in W_\alpha \ \forall \alpha \Rightarrow 0 \in W$
S'pose $x, y \in W$, $c \in F$
then for all α, $x, y \in W_\alpha$ & $cx + y \in W_\alpha$
$\Rightarrow cx + y \in W$.

Def'n Let S be a set of vectors in a vector space V.
The <u>subspace spanned by S</u> is the intersection of all subspaces that contain S.
When $S =$ finite set $\{\alpha_1, \cdots \alpha_n\}$,
say $W =$ subspace spanned by $\{\alpha_1, \cdots \alpha_n\}$.

<u>Thm</u> The subspace spanned by a nonempty subset $S \subseteq V$
is the set of all linear combinations of vectors in S.

<u>Pf</u>

Let W = subspace spanned by S.
Note all linear combinations $\subseteq W$.
Furthermore, set of all lin' comb's is a subspace of V
$(x, y \in \{\text{lin comb's}\} \Rightarrow cx + y \in \{\text{lin comb's}\})$
$\Rightarrow W \subseteq \{\text{lin' comb's}\}$

$\Rightarrow W = \{\text{lin' comb's}\}$.

<u>Def'n</u> If $S_1, \cdots S_k$ are subsets of V,
the set of all sums $\alpha_1 + \cdots + \alpha_k$ $(\alpha_i \in S_i)$
is called <u>sum</u> of the subsets & is denoted $S_1 + \cdots + S_k$

Def'n Let V be a vector space over F.
A subset S of V is said to be <u>linearly dependent</u>
if $\exists$ distinct vectors $\alpha_1, \cdots \alpha_n$ & scalars $\underbrace{c_1 \neq 0, \cdots c_n \neq 0}_{\text{not all zero; some can vanish}}$
s.t.

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

A set which is not linearly dependent, is <u>linearly independent</u>.

° give some exs

<u>Facts</u>

- any set which contains a linearly dependent set, is itself linearly dependent
- any subset of a linearly independent set is linearly independent
- any set which contains the 0 vector is linearly dependent
- a set of vectors is linearly independent iff every finite subset is linearly independent

Def'n Let V be a vector space.
A basis for V is a linearly independent set of vectors in V which spans V.
V is said to be finite-dimensional if it has a finite basis.

- give some exs

Standard basis for $F^n = \{ (1,0,\cdots 0), (0,1,\cdots 0), \cdots, (0,0,\cdots 1) \}$

Ex Let P be an invertible $n \times n$ matrix.
The columns of P are a basis for F^n.

$\to$ If X is a column matrix,
$PX = x_1 P_1 + \cdots + x_n P_n$, $\quad x_i$ entries of X, P_i the columns of P

Since $PX = 0 \Rightarrow X = 0$,
the P_i are LI.
why does it span?
Let Y be any column vector.
If $X = P^{-1}Y$, then $Y = PX = x_1 P_1 + \cdots + x_n P_n$
& so spans.

Ex Consider $F[x]$ = poly's in x

A basis is $\{1, x, x^2, \cdots\}$
Check:
- spans: clear, all poly's are lin' comb' of monomials
- LI: ~~Know~~ each finite subset LI.
  Suffices to show, for each n, $\{1, x, \cdots x^n\}$ LI.
  Suppose $c_0 + c_1 x + \cdots + c_n x^n = 0$
  $\to$ would have to hold $\forall x \in F$,
  ie, every $x \in F$ a root of the poly' above,
  but can have no more than n distinct roots
  $\Rightarrow c_0 = \cdots = c_n = 0 \quad \Rightarrow$ LI.

~~[illegible]~~

Thm Let V be a v.s. spanned by a finite ~~#~~ set of vectors $\beta_1, \cdots \beta_n$. Then any LI set of vectors in V is finite & contains no more than n elements.

Pf Suffices to show any subset ~~containing~~ containing $> n$ vectors is lin' dep'.
Let S be such a subset, $\alpha_1, \cdots \alpha_m$ distinct vectors, $m > n$.
Since $\beta_1, \cdots \beta_n$ span V, $\exists$ scalars A_{ij} st

$$\alpha_i = \sum_j A_{ij} \beta_j$$

Then for any m scalars x_i,

$$\sum_i x_i \alpha_i = \sum_{ij} x_i A_{ij} \beta_j = \sum_j \Big(\sum_i x_i A_{ij} \Big) \beta_j$$

From (solving algebraic equ'ns), $\exists$ scalars x_i st $\sum_i x_i A_{ij} = 0 \ \forall j$

$$\Rightarrow \sum x_i \alpha_i = 0, \quad x_i \text{ not all } 0$$

$$\Rightarrow \alpha_i \ \cancel{\text{LI}}.$$

~~[illegible]~~

Cor If V is a finite-dim'l v.s., then any two bases of V have the same (finite) number of elements.

Pf Since V is finite-dim'l, it has a finite basis $\{\beta_1, \cdots \beta_n\}$.
By thm above, every basis is finite & has no more than n elements.
So if $\{\alpha_1, \cdots \alpha_m\}$ is a basis, $m \leq n$.
By same argument, $n \leq m \Rightarrow \underline{m = n}$.

Def'n <u>dimension</u> of a vector space = # of elements in a basis.

Cor Let V be a finite-dim'l v.s. & let $n = \dim V$. Then

a) any subset of V which contains more than n vectors is linearly dependent

b) no subset of V which contains fewer than n vectors can span V.

Lemma Let S be a LI subset of a vs V.
S'pose $\beta \in V$, $\beta \notin$ span of S.
Then the set obtained by adding β to S is LI.

Pf S'pose $\alpha_1, \dots, \alpha_n$ are distinct vectors in S,
and $c_1\alpha_1 + \dots + c_n\alpha_n + b\beta = 0$

Then $b = 0$, else $\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \dots + \left(-\frac{c_n}{b}\right)\alpha_n \Rightarrow \beta \in$ span of S.

Thus $c_1\alpha_1 + \dots + c_n\alpha_n = 0$, & since S is LI, all $c_i = 0$.

$\Rightarrow$ LI.

Thm If W is a subspace of a finite-dim'l v.s. V,
then every LI subset of W is finite & is part of a (finite) basis for W.

Pf see text

Cor If W is a proper subspace of a finite-dim'l vector space V, then W is finite-dim'l and $\dim W < \dim V$.

Pf S'pose $W \ni \alpha \neq 0$. By Thm above, there is a basis of W containing α which has $\leq (\dim V)$ elements.
$\Rightarrow$ W is finite-dim'l, & $\dim W \leq \dim V$.
Since W is a proper subspace,
there is a vector $\beta \in V$ which is not in W.
Adjoining β to any basis of W,
we get a LI subset of V $\Rightarrow$ $\dim W < \dim V$.

Cor In a finite-dim'l v.s. V, every nonempty LI ~~subset~~ set of vectors is part of a basis.

Cor Let A be an $n \times n$ matrix over a field F,
& s'pose the row vectors of A form a LI set of vectors in F^n.
Then A is invertible.

Pf Let $\alpha_1, \cdots \alpha_n$ be the row vectors of A,
& s'pose W is the subspace of F^n spanned by $\alpha_1, \cdots \alpha_n$.
Since they are LI, $\dim W = n$, so $W = F^n$.
If $\varepsilon_1, \cdots \varepsilon_n$ is std basis for F^n,
we then have $\varepsilon_i = \sum_j B_{ij} \alpha_j$ for some scalars B_{ij}.

Thus, for the matrix B w/entries B_{ij}, we have

$$BA = I.$$

~~[illegible]~~

Thm If W_1, W_2 are finite-dim'l subspaces of a vector space V, then $W_1 + W_2$ is finite-dim'l and

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2).$$

– explain intuition, refer to text for pf.