A-twisted Landau-Ginzburg models

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J. Guffin, ES, arXiv: 0801.3836, 0803.3955

M. Ando, ES, to appear

A **Landau-Ginzburg model** is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

$$S = \int_\Sigma d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\jmath} D_z \psi_-^i + \cdots \right.$$
$$\left. + g^{i\bar{\jmath}} \partial_i W \partial_{\jmath} \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar{\imath}} \psi_-^{\bar{\jmath}} D_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W} \right)$$

The superpotential $W : X \longrightarrow \mathbf{C}$ is holomorphic,

(so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....

In the past, people have mostly only considered Landau-Ginzburg models in which X = $\mathbf{C}^n$ for some n, (or an orbifold thereof,) with a quasi-homogeneous superpotential, and only one topological twist.

In the case X = $\mathbf{C}^n$, states are elements of

$$\mathbf{C}[x_1, \cdots, x_n]/(dW)$$

with correlation functions (at genus 0)

$$\begin{aligned}
\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_X d^2\phi \int \prod_i d\chi_+^{\bar{\imath}} d\chi_-^{\bar{\imath}} \mathcal{O}_1 \cdots \mathcal{O}_n \exp\left(-|dW|^2 \, + \, \chi_+^{\bar{\imath}} \chi_-^{\bar{\jmath}} \partial_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W} \right) \\
&= \sum_{dW=0} \mathcal{O}_1 \cdots \mathcal{O}_n \left(\det \partial^2 W \right)^{-1}
\end{aligned}$$

For nonlinear sigma models,
there are 2 topological twists: the A, B models.

1) A model

$$\psi_+^i \in \Gamma(\phi^*(T^{1,0}X)) \to \chi^i \qquad \psi_-^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X)) \to \chi^{\bar{\imath}}$$

$$Q \cdot \phi^i = \chi^i, \;\; Q \cdot \phi^{\bar{\imath}} = \chi^{\bar{\imath}}, \;\; Q \cdot \chi = 0, \;\; Q^2 = 0$$

$$Q \sim d$$

States $\quad b_{\mu \cdots \nu} \chi^\mu \cdots \chi^\nu \leftrightarrow H^{\cdot,\cdot}(X)$

Correlation functions

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_d \int_{\mathcal{M}_d} \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \wedge \mathrm{Obs}$$

2) B model

$$\psi_{\pm}^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X))$$

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} \qquad \theta_i = g_{i\bar{\jmath}}\left(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}}\right)$$

$$Q\cdot\phi^i = 0,\ \ Q\cdot\phi^{\bar{\imath}} = \eta^{\bar{\imath}},\ \ Q\cdot\eta^{\bar{\imath}} = 0,\ \ Q\cdot\theta_j = 0,\ \ Q^2 = 0$$

Identify $\quad \eta^{\bar{\imath}} \leftrightarrow d\bar{z}^{\bar{\imath}} \qquad \theta_j \leftrightarrow \dfrac{\partial}{\partial z^j} \qquad Q \leftrightarrow \bar{\partial}$

States:

$$b^{j_1\cdots j_m}_{\bar{\imath}_1\cdots\bar{\imath}_n}\eta^{\bar{\imath}_1}\cdots\eta^{\bar{\imath}_n}\theta_{j_1}\cdots\theta_{j_m} \leftrightarrow H^n\left(X,\Lambda^m TX\right)$$

We can also talk about A, B twists of LG models over nontrivial spaces,
generalizing earlier notions on LG TFT....

LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

$$b(\phi)^{j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where η, θ are linear comb's of ψ

$$Q \cdot \phi^i = 0, \;\; Q \cdot \phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \;\; Q \cdot \eta^{\bar{\imath}} = 0, \;\; \underline{Q \cdot \theta_j = \partial_j W}, \;\; Q^2 = 0$$

Identify $\quad \eta^{\bar{\imath}} \leftrightarrow d\overline{z}^{\bar{\imath}}, \;\; \theta_j \leftrightarrow \dfrac{\partial}{\partial z^j}, \;\; Q \leftrightarrow \overline{\partial}$

so the states are hypercohomology

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

Quick checks:

1) W=0, standard B-twisted NLSM

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

$$\mapsto \; H^{\cdot}\left(X, \Lambda^{\cdot} TX \right) \checkmark$$

2) X=$\mathbf{C}^n$, W = quasihomogeneous polynomial

Seq' above resolves fat point {dW=0}, so

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

$$\mapsto \; \mathbf{C}[x_1, \cdots, x_n]/(dW) \checkmark$$

Correlation functions (schematic):

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_X d\phi \int d\eta^{\overline{\imath}} d\theta_j \, \omega_1 \wedge \cdots \wedge \omega_n \exp\left(-|dW|^2 - \eta^{\overline{\imath}} \theta_j g^{j\overline{k}} \partial_{\overline{\imath}} \partial_{\overline{k}} \overline{W} \right.$$

$$= \int_{dW=0} d\phi \, \omega^j_{1\overline{\imath}} \wedge \cdots \wedge \omega^j_{n\overline{\imath}} (\Omega_X)^{\overline{\imath}\cdots\overline{\imath}\overline{k}\cdots\overline{k}} (\Omega_X)_{j\cdots jm\cdots m} \left(D_{\overline{k}} \partial^m \overline{W} \right) \cdots \left(D_{\overline{k}} \partial^m \overline{W} \right) \left| \det \partial^2 W \right|^{-2}$$

where Ω_X = holomorphic top-form

ω = rep' of class given in forms on X

Math: this is a product on hypercohomology; elegant understanding?

LG A model:

Defining the A twist of a LG model is more interesting.

(Fan, Jarvis, Ruan) (Ito; J Guffin, ES)

Producing a TFT from a NLSM involves changing what bundles the ψ couple to, e.g.

$$\psi \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* TX) \mapsto \Gamma(\Sigma, \phi^* TX), \Gamma(\Sigma, K_\Sigma \otimes \phi^* TX)$$

The two inequivalent possibilities are the A, B twists.

To be consistent, the action must remain well-defined after the twist.

True for A, B NLSM's & B LG, but not A LG....

LG A model:

The problem is terms in the action of the form

$$\psi_+^i \psi_-^j D_i \partial_j W$$

If do the standard A NLSM twist,
this becomes a 1-form on Σ,
which can't integrate over Σ .

Fix: modify the A twist.

LG A model:

There are several ways to fix the A twist,
and hence, several different notions of a LG A model.

One way: multiply offending terms in the action
by another 1-form.

Another way: use a different prescription for
modifying bundles.

The second is advantageous for physics, so I'll use it,
but,
disadvantage: not all LG models admit A twist
in this prescription.

To twist, need a U(1) isometry on X w.r.t. which the superpotential is quasi-homogeneous.

Twist by "R-symmetry + isometry"

Let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) \;=\; \lambda W(\phi_i)$$

then twist: $\psi \;\mapsto\; \Gamma\left(\text{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K_\Sigma}^{-(1/2)Q_L}\right)$

where $$Q_{R,L}(\psi) \;=\; Q(\psi) \;+\; \begin{cases} 1 & \psi = \psi_+^i, R \\ 1 & \psi = \psi_-^i, L \\ 0 & \text{else} \end{cases}$$

Example: $X = \mathbf{C}^n$, W quasi-homog' polynomial

Here, to A twist, need to make sense of e.g. $K_\Sigma^{1/r}$

where r = 2(degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)
-- but then usually can't make sense of $K_\Sigma^{1/r}$

I'll work with the latter case.

LG A model:

A twistable example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with $W = p\pi^* s, \; s \in \Gamma(B, \mathcal{E})$

Accessible states are Q-closed (mod Q-exact) prod's:

$$b(\phi)_{\bar{\imath}_1 \cdots \bar{\imath}_n j_1 \cdots j_m} \psi_-^{\bar{\imath}_1} \cdots \psi_-^{\bar{\imath}_n} \psi_+^{j_1} \cdots \psi_+^{j_m}$$

where

$$\phi \sim \{s = 0\} \subset B \qquad \psi \sim TB|_{\{s=0\}}$$

$$Q \cdot \phi^i = \psi_+^i, \;\; Q \cdot \phi^{\bar{\imath}} = \psi_-^{\bar{\imath}}, \;\; Q \cdot \psi_+^i = Q \cdot \psi_-^{\bar{\imath}} = 0, \;\; Q^2 = 0$$

Identify $\psi_+^i \leftrightarrow dz^i, \;\; \psi_-^{\bar{\imath}} \leftrightarrow d\overline{z}^{\bar{\imath}}, \;\; Q \leftrightarrow d$

so the states are elements of $H^{m,n}(B)|_{\{s=0\}}$

Correlation functions:

B-twist:

Integrate over X, weight by

$$\exp\left(-|dW|^2 + \text{ fermionic}\right)$$

and then perform transverse Gaussian,
to get the standard expression.

A-twist:

Similar: integrate over $\mathcal{M}_X$

and weight as above.

Witten equ'n in A-twist:

BRST: $$\delta\psi^i_- = -\alpha\left(\overline{\partial}\phi^i - i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W}\right)$$

implies localization on sol'ns of

$$\overline{\partial}\phi^i - i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W} = 0$$ ("Witten equ'n")

On complex Kahler mflds, there are 2 independent BRST operators:

$$\delta\psi^i_- = -\alpha_+\overline{\partial}\phi^i + \alpha_- i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W}$$

which implies localization on sol'ns of

$$\begin{aligned} \overline{\partial}\phi^i &= 0 \\ g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W} &= 0 \end{aligned}$$

which is what we're using.

Sol'ns of Witten equ'n:

$$\int_\Sigma \left|\overline{\partial}\phi^i - i g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \overline{W}\right|^2 = \int_\Sigma \left(\left|\overline{\partial}\phi^i\right|^2 + \left|\partial_i W\right|^2 \right)$$

LHS = 0 iff RHS = 0

hence sol'ns of Witten equ'n
same as the moduli space we're looking at.

LG A model, cont'd

In prototypical cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 \;-\; \chi^p dz^i D_i s \;-\; \mathrm{c.c.} \;-\; F_{i\overline{\jmath}} dz^i d\overline{z}^{\overline{\jmath}} \chi^p \chi^{\overline{p}} \right.}_{\text{Mathai}-\text{Quillen form}}$$

The MQ form rep's a Thom class, so

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \mathrm{Eul}(N_{\{s=0\}/\mathcal{M}}) \\ &= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \end{aligned}$$

-- same as A twisted NLSM on {s=0}

Not a coincidence, as we shall see shortly.

Example:

LG model on Tot(O(-5) --> **P**⁴),
W = p s

Twisting: $p \in \Gamma(K_\Sigma)$

Degree 0 (genus 0) contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \int_{\mathbf{P}^4} d^2\phi^i \int \coprod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-|s|^2 \,-\, \chi^i \chi^p D_i s \,-\, \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} \,-\, R_{ip\overline{p}\overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

(curvature term ~ curvature of O(-5))

(cont'd)

Example, cont'd

In the A twist (unlike the B twist),
the superpotential terms are BRST exact:

$$Q \cdot \left(\psi_-^i \partial_i W \; - \; \psi_+^{\overline{\imath}} \partial_{\overline{\imath}} \overline{W} \right) \; \propto \; -|dW|^2 \; + \; \psi_+^i \psi_-^j D_i \partial_j W \; + \; \text{c.c.}$$

So, under rescalings of W by a constant factor λ,
physics is unchanged:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathbf{P}^4} d^2\phi^i \int \prod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-\lambda^2 |s|^2 \; - \; \lambda \chi^i \chi^p D_i s \; - \; \lambda \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} \; - \; R_{i p \overline{p} \overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

Example, cont'd

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathbf{P}^4} d^2\phi^i \int \coprod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-\lambda^2 |s|^2 - \lambda \chi^i \chi^p D_i s - \lambda \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} - R_{i p \overline{p} \overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

Limits:

1) $\lambda \longrightarrow 0$

Exponential reduces to purely curvature terms; bring down enough factors to each up χ^p zero modes.

Equiv to, inserting Euler class.

2) $\lambda \longrightarrow \infty$

Localizes on $\{s = 0\} \subset \mathbf{P}^4$

Equivalent results, either way.

RG preserves TFT's.

If two physical theories are related by RG,
then, correlation functions in a top' twist of one
=
correlation functions in corresponding twist of other.

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p s

$\Big\downarrow$ Renormalization group flow

NLSM on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

This is why correlation functions match.

Another example:

LG model on Tot($\mathcal{O}(-1)^2 \dashrightarrow \mathbf{P}^1 \times \mathbf{C}^4$)

$\Big\downarrow$ RG flow

NLSM on small res'n of conifold

Degree > 0 contributions to NLSM corr' f'ns include multicovers.

In the LG model, those multicovers arise from same localization as before on vanishing locus of (induced) section -- no obstruction bundle, etc.

Another way to associate LG models to NLSM.

S'pose, for ex, the NLSM has target space
= hypersurface $\{G=0\}$ in $\mathbf{P}^n$ of degree d

Associate LG model on $[\mathbf{C}^{n+1}/\mathbf{Z}_d]$
with $W = G$

* **Not** related by RG flow

* But, related by Kahler moduli,
so have same B model

Relations between LG models

LG model on
Tot(O(-5) --> $\mathbf{P}^4$)
with W = p s

(Same TFT) (RG flow)

NLSM on $\{s=0\} \subset \mathbf{P}^4$

(Kahler)
(Only B twist same)

LG model on
$[\mathbf{C}^5/\mathbf{Z}_5]$
with W = s

Analogous results exist for elliptic genera.

Recall: elliptic genus = 1-loop partition function.

$$\mathrm{Tr}\,(-)^{F_R}\exp(i\gamma J_L)q^{L_0}\overline{q}^{\overline{L}_0}$$

For a heterotic NLSM on X, dim n,
with bundle E, rank r, this is

$$q^{-(1/24)(2n+r)}\int_X \hat{A}(TX)\wedge \mathrm{ch}\left(\otimes S_{q^n}((TX)^{\mathbf{C}})\otimes \Lambda_{q^n}((e^{i\gamma}\mathcal{E})^{\mathbf{C}})\right)$$

-- TX contributions from left-moving bosonic modes
-- E contribution from left-moving fermionic modes
-- right-moving contributions cancel

Elliptic genera of LG models on $\mathbf{C}^n$ -- Witten

R-symmetries are NLSM + $\mathbf{C}^*$ action,
just as in A twist

Result (for n=1) proportional to

$$\int_{\mathrm{pt}} \hat{A}(\mathrm{pt}) \mathrm{ch} \left(\otimes S_{q^n}((e^{i\gamma} L)^{\mathbf{C}}) \otimes \Lambda_{q^n}(-(e^{i(k+1)\gamma} L)^{\mathbf{C}}) \right)$$

where pt is fixed point of $\mathbf{C}^*$ action
L is line bundle over pt

W is homogeneous of degree k+2

Elliptic genus of LG model over
X = Tot(E* ---> B)
with W = p s
(s a section of E)

is proportional to

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\left(\otimes S_{q^n}((TB)^{\mathbf{C}}) \otimes S_{-q^n}((e^{-i\gamma}\mathcal{E})^{\mathbf{C}}) \otimes \Lambda_{q^n}((e^{i\gamma}TB)^{\mathbf{C}}) \otimes \Lambda_{-q^n}((\mathcal{E})^{\mathbf{C}} \right.$$

-- **C*** rotates fibers, so B is the fixed-point locus

It can be shown that the elliptic genus just outlined, for a LG model on Tot(E* --> B), matches an elliptic genus of NLSM on {s = 0} in B.

Physics: the two theories are related by RG flow.

Math: the two genera are related by a Thom class.

This is closely related to A-twisted LG, where we saw that theories related by RG had correlation functions related by Mathai-Quillen, a rep' of a Thom class.

Suggests: RG flow interpretation in twisted theories as Thom class, perhaps from underlying Atiyah-Jeffrey description of the TFT.

Possible mirror symmetry application:

Part of what we've done is to replace NLSM's with LG models that are `upstairs' in RG flow.

Then, for example, one could imagine rephrasing mirror symmetry as a duality between the `upstairs' LG models.

-- P. Clarke, 0803.0447

Another application: generalizing GLSM's

Given a standard GLSM, integrate out the gauge field to get a LG model over a nontrivial space/stack.

Example: quintic in $\mathbf{P}^4$

LG on Tot($O(-5) \dashrightarrow \mathbf{P}^4$) $\xleftrightarrow{\text{Kahler}}$ LG on Tot($O(-1)^5 \dashrightarrow B\mathbf{Z}_5$)

with $W = p\,s$ $\qquad\qquad\qquad\qquad$ $= [\mathbf{C}^5/\mathbf{Z}_5]$

The two spaces/stacks are birational.

Could imagine, working with families of LG models on birational spaces -- &, if don't have to come from GLSM, then don't have to be toric.

Hybrid LG models

In the same vein, we should also take a moment to explain the `hybrid LG models' appearing in GLSM's.

These are LG models on stacks, rather than spaces.

Example: GLSM for complete int' $\mathbf{P}^n[w_1,\ldots,w_k]$

r >> 0: LG on Tot($O(-w_1) + \ldots + O(-w_k)$ ---> $\mathbf{P}^n$)

$\Updownarrow$ birational

r << 0: LG on Tot($O(-1)^{n+1}$ ---> $\mathbf{P}^{k-1}{}_{[w1\ldots wk]}$)

-- ``hybrid''

Special case: $\mathbf{P}^0{}_{[n]} = B\mathbf{Z}_n$, Tot($O(-1)^n$ --> $B\mathbf{Z}_k$) = $[\mathbf{C}^n/\mathbf{Z}_k]$

Open problem:

Analyses of the form described here,
for LG models on stacks,
instead of spaces.

Answers known for special case of
orbifolds of vector spaces,
but, more general cases look more difficult.

Another direction:

Heterotic Landau-Ginzburg models

We'll begin with heterotic nonlinear sigma models....

Heterotic nonlinear sigma models:

Let X be a complex manifold,

$\mathcal{E} \longrightarrow X$ a holomorphic vector bundle

such that $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

Action:

$$S = \int_\Sigma d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i h_{a\bar{b}} \lambda_-^{\bar{b}} D_z \lambda_-^a + \cdots \right.$$

ϕ, ψ_+ as before $\qquad \lambda_-^a \in \Gamma\left(\mathcal{E} \otimes \sqrt{K_\Sigma} \right)$

Reduces to ordinary NLSM when $\mathcal{E} = TX$

Heterotic Landau-Ginzburg model:

$$S = \int_\Sigma d^2x \Big(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i h_{a\bar{b}} \lambda_-^{\bar{b}} D_z \lambda_-^a + \cdots$$
$$+ h^{a\bar{b}} F_a \overline{F}_{\bar{b}} + \psi_+^i \lambda_-^a D_i F_a + \text{c.c.}$$
$$+ h_{a\bar{b}} E^a \overline{E}^{\bar{b}} + \psi_+^i \lambda_-^{\bar{a}} D_i E^b h_{\overline{a}b} + \text{c.c.} \Big)$$

Has two superpotential-like pieces of data

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^\vee)$$

such that $\displaystyle \sum_a E^a F_a = 0$

(2,2) locus: $\mathcal{E} = TX, \ E^a \equiv 0, \ F_a = \partial_i W$

Pseudo-topological twists:

* If $E^a = 0$, then can perform std B twist

$$\psi_+^{\bar{\imath}} \in \Gamma((\phi^* T^{1,0} X)^\vee) \qquad \lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$$

Need $\Lambda^{\mathrm{top}} \mathcal{E} \cong K_X, \ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States $\mathbf{H}^{\cdot}\left(\cdots \longrightarrow \Lambda^2 \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{O}_X \right)$

* If $F_a = 0$, then can perform std A twist

$$\psi_+^{i} \in \Gamma(\phi^* T^{1,0} X) \qquad \lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$$

Need $\Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X, \ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States $\mathbf{H}^{\cdot}\left(\cdots \longrightarrow \Lambda^2 \mathcal{E}^\vee \xrightarrow{i_{E^a}} \mathcal{E}^\vee \xrightarrow{i_{E^a}} \mathcal{O}_X \right)$

* More gen'ly, must combine with **C*** action.

Heterotic LG models are related to heterotic NLSM's via renormalization group flow.

Example:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$

with $\mathcal{E}' = \pi^* \mathcal{F}_2$ & $F_a \equiv 0, \quad E^a \neq 0$

Renormalization group

A heterotic NLSM on B

with $\mathcal{E} = \mathrm{coker}\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Example:

Corresponding to NLSM on $\mathbf{P}^1 \times \mathbf{P}^1$ with E′ as cokernel

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$* = \left[\begin{array}{cc} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{array} \right]$$

have (upstairs in RG) LG model on

$$X = \mathrm{Tot} \left(\mathcal{O} \oplus \mathcal{O} \stackrel{\pi}{\longrightarrow} \mathbf{P}^1 \times \mathbf{P}^1 \right)$$

with $$\mathcal{E} = \pi^* \mathcal{O}(1,0)^2 \oplus \pi^* \mathcal{O}(0,1)^2$$

$$F_a \equiv 0 \qquad \begin{array}{ll} E^1 = x_1 p_1 + \epsilon_1 x_1 p_2 & E^3 = \tilde{x}_1 p_1 \\ E^2 = x_2 p_1 + \epsilon_2 x_2 p_2 & E^4 = \tilde{x}_2 p_2 \end{array}$$

Example, cont'd

Since $F_a = 0$, can perform std A twist.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathbf{P}^1 \times \mathbf{P}^1} d^2x \int d\chi^i \int d\lambda^{\overline{a}} \mathcal{O}_1 \cdots \mathcal{O}_n \left(\lambda^{\overline{a}} \tilde{E}_1^{\overline{a}} \right) \left(\lambda^{\overline{b}} \tilde{E}_2^{\overline{b}} \right) f(\tilde{E}_1^{\overline{a}}, \tilde{E}_2^{\overline{a}})$$

which reproduces std results for
(0,2) quantum cohomology in this example.

One can also compute elliptic genera in these models.

For the given example,
elliptic genus proportional to

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\left(\otimes S_{q^n}((TB)^{\mathbf{C}}) \otimes S_{q^n}((e^{-i\gamma}\mathcal{F}_1)^{\mathbf{C}}) \otimes \Lambda_{-q^n}((e^{-i\gamma}\mathcal{F}_2)^{\mathbf{C}}\right)$$

and there is a Thom class argument that
this matches a corresponding elliptic genus
of the NLSM related by RG flow.

Another class:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_2^\vee \xrightarrow{\pi} B\right)$
with $\mathcal{E}' = \pi^* \mathcal{F}_1$ & $E^a \equiv 0, \quad F_a \neq 0$

$$\Big\downarrow \text{Renormalization group}$$

A heterotic NLSM on B
with $\mathcal{E} = \ker\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Example: deformation of the (2,2) quintic

Consider LG with $$X = \mathrm{Tot}\left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbf{P}^4 \right)$$

$$\mathcal{E} = TX \qquad E^a \equiv 0 \qquad F_i = p(D_i s + s_i)$$

$$s \in \Gamma(\mathcal{O}(5))$$

This RG flows to a heterotic NLSM describing a (0,2) deformation of a (2,2) quintic.

Example, cont'd

Can A twist -- must combine w/ **C*** action.

Result:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathbf{P}^4} \int d\chi^i \int d\lambda^i \int d\chi^p \int d\lambda^p \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp \left(-|s|^2 \; - \; \chi^i \lambda^p D_i s \; - \; \underline{\chi^{\overline{p}} \lambda^{\overline{\imath}} \left(D_{\overline{\imath}} s \; + \; s_{\overline{\imath}} \right)} \; - \; R_{i \overline{p} p \overline{k}} \chi^i \chi^{\overline{p}} \lambda^p \lambda^{\overline{k}} \right)$$

-- not quite Mathai-Quillen

-- superpotential terms not BRST exact

Most general case:

LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \oplus \mathcal{F}_3^\vee \xrightarrow{\pi} B \right)$

with gauge bundle $\mathcal{E}$ given by

$$0 \longrightarrow \pi^* \mathcal{G}^\vee \longrightarrow \mathcal{E} \longrightarrow \pi^* \mathcal{F}_2 \longrightarrow 0$$

$\Big\downarrow$ Renormalization group

NLSM on $Y \equiv \{ G_\mu = 0 \} \subset B \qquad G_\mu \in \Gamma(\mathcal{G})$

with bundle $\mathcal{E}'$ given by cohom' of the monad

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

(2,2) locus: $\mathcal{F}_1 = 0, \; \mathcal{F}_2 = TB, \; \mathcal{F}_3 = \mathcal{G}$

Spectators:

Possible complaint: X isn't CY in gen'l,
and E doesn't have c_1=0;
why should they have good IR fixed point?

Answer: add spectators.

$$X \;=\; \mathrm{Tot}\left(\mathcal{F}_1 \oplus \mathcal{F}_3^\vee \xrightarrow{\pi} B \right)$$

$$\mapsto\; \mathrm{Tot}\left(\mathcal{F}_1 \oplus \mathcal{F}_3^\vee \oplus \left(K_B \otimes \Lambda^{\mathrm{top}} \mathcal{F}_1^\vee \otimes \Lambda^{\mathrm{top}} \mathcal{F}_3 \right) \xrightarrow{\pi} B \right)$$

$$\mathcal{E} \;\mapsto\; \mathcal{E} \oplus \pi^* \left(K_B \otimes \Lambda^{\mathrm{top}} \mathcal{F}_1^\vee \otimes \Lambda^{\mathrm{top}} \mathcal{F}_3 \right)^\vee$$

plus a canonical term added to F_a to give mass.

Heterotic GLSM phase diagrams:

Heterotic GLSM phase diagrams are famously different from (2,2) GLSM phase diagrams;
however,
the analysis of earlier still applies.

A LG model on X, with bundle E,
can be on the same Kahler phase diagram as
a LG model on X', with bundle E',
if X birat'l to X', and E, E' match on the overlap.

(necessary, not sufficient)

Example:

NLSM on $\{G=0\} \subset W\mathbf{P}^4_{w_1,\cdots,w_5}$ $\quad G \in \Gamma(\mathcal{O}(d))$

with bundle $\mathcal{E}'$ given by

$$0 \longrightarrow \mathcal{E}' \longrightarrow \oplus\mathcal{O}(n_a) \longrightarrow \mathcal{O}(m) \longrightarrow 0$$

is described (upstairs in RG) by a LG model on

$$X \;=\; \mathrm{Tot}\left(\mathcal{O}(-m) \stackrel{\pi}{\longrightarrow} W\mathbf{P}^4\right)$$

with bundle $0 \longrightarrow \pi^*\mathcal{O}(d) \longrightarrow \mathcal{E} \longrightarrow \oplus\pi^*\mathcal{O}(n_a) \longrightarrow 0$

and is related to LG on

$$\mathrm{Tot}\left(\oplus\mathcal{O}(-w_i) \longrightarrow B\mathbf{Z}_m\right) \;=\; [\mathbf{C}^5/\mathbf{Z}_m]$$

with ~ same bundle.

Open problem:

In the Kahler duality:

A LG model on X, with bundle E,
can be on the same Kahler phase diagram as
a LG model on X′, with bundle E′,
if X birat'l to X′, and E, E′ match on the overlap.

How to uniquely determine E′ ?

What are sufficient conditions to be on same SCFT moduli space?

How does this compare to GLSM Kahler moduli space, in the special case of toric X, X′ ? Larger?

Bold, unjustified, conjecture:

Given a heteroTic LG model on X, with bundle E,
describing IR NLSM on Calabi-Yau Y,
if X′ is birat'l to X,
then E′ --> X is uniquely determined by:
1) E′ matches E on common Zariski open subset,
2) $ch_2(E') = ch_2(TX')$,
3) Witten indices of IR NLSM's match.

Summary:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

-- Mathai-Quillen, elliptic genera, Thom forms

* Heterotic LG models