Heterotic mirror symmetry, and quantum sheaf cohomology

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As everyone here is well aware,
the idea of mirror symmetry, ~ 26 years ago,
had amazing implications for math & physics,
and spawned important generalizations,
such as homological mirror symmetry.

Today, I want to *survey* developments in another generalization, known as heterotic mirror symmetry or (0,2) mirror symmetry.

Most of this talk will focus on one aspect,
a generalization of quantum cohomology,
but first,
in general terms, what is heterotic mirror symmetry?

Ordinary mirror symmetry is about, in `typical' cases
("type II strings"),
complex Kahler manifolds.

Heterotic mirror symmetry, in `typical' cases,
concerns pairs
(complex Kahler manifold X, hol' vector bundle $\mathcal{E} \to X$)
such that
$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$$

In `typical’ geometric cases, when spaces are Calabi-Yau,

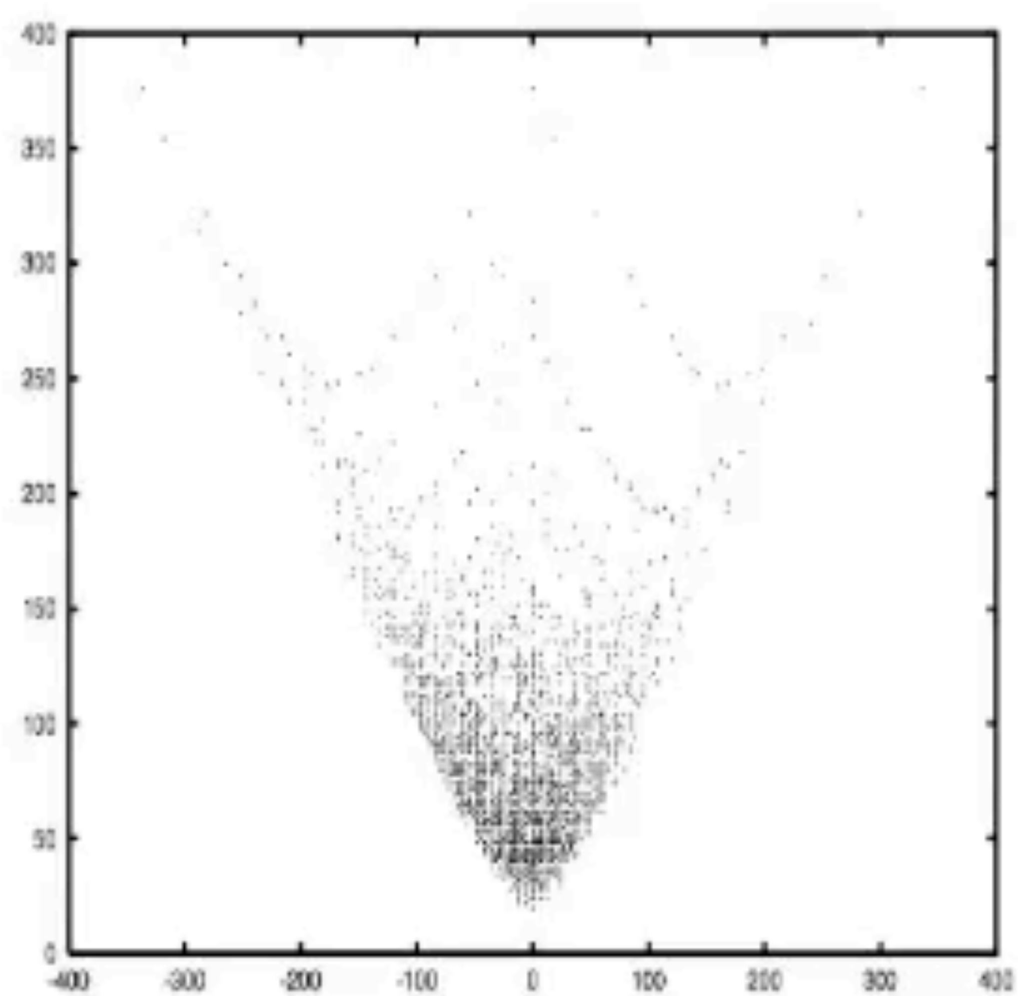
Ordinary mirror symm’	*Heterotic mirror symm’*
X, Y mirror	$(X, \mathcal{E}), (Y, \mathcal{F})$ mirror
$h^{p,q}(X) = h^{n-p,q}(Y)$	$h^p(X, \wedge^q \mathcal{E}^*) = h^p(Y, \wedge^q \mathcal{F})$
$\dim X = \dim Y$	$\dim X = \dim Y$ $\mathrm{rk}\, \mathcal{E} = \mathrm{rk}\, \mathcal{F}$
$\text{cpx moduli}\, X$ $= \text{Kähler moduli}\, Y$	$\{\text{cpx, Kähler, bdle moduli}\}(X, \mathcal{E})$ $= \{\text{cpx, Kähler, bdle moduli}\}(Y, \mathcal{F})$

In the special case $\mathcal{E} = TX$, heterotic becomes ordinary.

Why in the world would we believe this exists?

Why in the world would we believe this exists?

Numerical evidence for heterotic mirror symmetry:



Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

Unlike ordinary mirror symmetry,
which is now well-understood,
heterotic mirror symmetry is still under development.

Constructions include:

- Blumenhagen-Sethi '96 extended Greene-Plesser orbifold construction to heterotic models — handy but only gives special cases

- Adams-Basu-Sethi '03 repeated Hori-Vafa-style GLSM duality

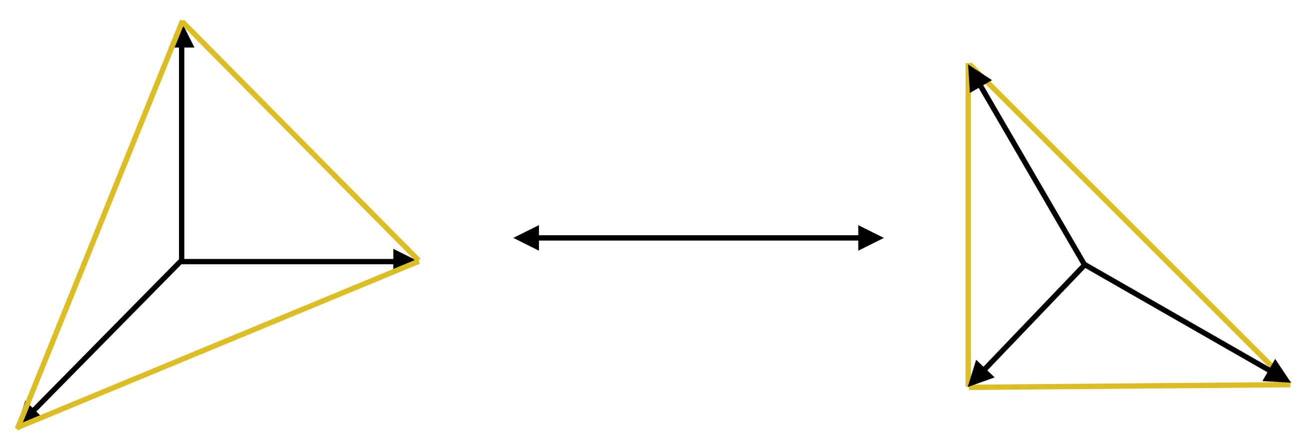
— but results must be supplemented by manual computations;
het' version does not straightforwardly generate examples

More recent progress includes a version of Batyrev's construction….

Current state-of-the-art in constructions of heterotic mirrors:

- Melnikov-Plesser '10 extended Batyrev's construction & monomial-divisor mirror map to include def's of tangent bundle, for special ('reflexively plain') polytopes

Dualize polytopes as before:



$$P^0 = \{y | \langle x, y\rangle \geq -1 \; \forall x \in P\}$$

& encode tangent bdle def's in a matrix:

$$A \longleftrightarrow A^T$$

Progress, but still don't have a general construction.

Quantum cohomology?

One of the driving developments behind interest in mirror symmetry were, of course, Gromov-Witten invariants and quantum cohomology.

There is a heterotic analogue of quantum cohomology, and significant progress has been made in understanding it.

For most of the rest of this talk, I want to focus on this particular aspect of heterotic mirror symmetry,
this heterotic analogue of quantum cohomology,
known as *quantum sheaf cohomology*.

Outline

- Outline def'n of quantum sheaf cohomology

- Computations on $\mathbb{P}^1 \times \mathbb{P}^1$

  & results for toric varieties

- (0,2) Toda-like Landau-Ginzburg mirrors

- Results for Grassmannians

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

(Katz-ES '04, ES '06, Guffin-Katz '07, ….)

When bundle = tangent bundle,
encodes Gromov-Witten invariants.

When \neq tangent bundle, Gromov-Witten inv'ts not relevant.

Mathematical GW computational tricks no longer apply.

No known analogue of periods, Picard-Fuchs equations.

New methods needed….

… and a few have been developed.

(A Adams, J Distler, R Donagi, J Guffin, S Katz, J McOrist, I Melnikov, R Plesser, ES, ….)

Strictly speaking, quantum cohomology & Gromov-Witten invariants arise in a topological field theory, the A model.

To describe the heterotic version of quantum cohomology,
I need to describe the heterotic analogues of the A, B models,
known as the A/2, B/2 models.

These are pseudo-topological field theories
— not quite TFT's in the usual sense,
but,
at least at genus zero, behave like TFT's.

Heterotic versions of topological field theories:

For a space X and bundle $\mathcal{E} \to X$

A/2 model: Exists when $\det \mathcal{E}^* \cong K_X$ & $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States counted by $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$

Reduces to A model when $\mathcal{E} = TX$.

B/2 model: Exists when $\det \mathcal{E} \cong K_X$ & $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States counted by $H^\bullet(X, \wedge^\bullet \mathcal{E})$

Reduces to B model when $\mathcal{E} = TX$.

New symmetries: $\mathrm{A}/2(X, \mathcal{E}) \cong \mathrm{B}/2(X, \mathcal{E}^*)$

In add'n, these pseudo-TFT's behave well under mirror symmetry:

Ordinary mirrors: If X, Y are mirror, then

$$\mathrm{A}(X) \cong \mathrm{B}(Y)$$

Heterotic mirrors: If $(X, \mathcal{E})$, $(Y, \mathcal{F})$ are mirror, then

$$\mathrm{A}/2(X, \mathcal{E}) \cong \mathrm{B}/2(Y, \mathcal{F})$$

Now, how to compute in these theories?

Classical A model computations:

For X a space,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle \; = \; \int_X \omega_1 \wedge \cdots \wedge \omega_k$$

for 'operators' $\mathcal{O}_i \; \sim \; \omega_i \; \in \; H^{p_i, q_i}(X)$

This (classical contribution to the) correlation function is nonzero when

$$\sum p_i \; = \; \dim X \; = \; \sum q_i$$

ie, when $\omega_1 \wedge \cdots \wedge \omega_k$ is a top-form.

Classical A/2 model computations:

For X a space, $\mathcal{E} \to X$ a hol' vector bundle s.t.

$$\wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X, \quad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle \; = \; \int_X \omega_1 \wedge \cdots \wedge \omega_k$$

for `operators' $\mathcal{O}_i \; \sim \; \omega_i \; \in \; H^{q_i} \left(X, \wedge^{p_i} \mathcal{E}^* \right)$

Now, $\omega_1 \wedge \cdots \wedge \omega_k \; \in \; H^{\sum q_i} \left(X, \wedge^{\sum p_i} \mathcal{E}^* \right)$

In order for $\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle$ to be a number,

require $\sum q_i = \dim X \qquad \sum p_i = \mathrm{rank}\, \mathcal{E}$

& use $\wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X$

A model computations:

Schematically: For X a space,

$\mathcal{M}_d$ a space of holomorphic $S^2 \longrightarrow X$

we compute a "correlation function" in A model TFT

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = q^d \int_{\mathcal{M}_d} \omega_1 \wedge \cdots \wedge \omega_k \, (\wedge c_{\mathrm{top}}(\mathrm{Obs}))$$

where $\mathcal{O}_i \sim \omega_i \in H^{p_i, q_i}(\mathcal{M}_d)$

$$= q^d \int_{\mathcal{M}_d} (\text{top form on } \mathcal{M}_d)$$

which encodes minimal area surface information.

Such computations are at the heart of Gromov-Witten theory.

A/2 model computations:

Schematically: For X a space, $\mathcal{E}$ a bundle on X,

$\mathcal{M}_d$ a space of holomorphic S² —> X

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle \;=\; q^d \int_{\mathcal{M}_d} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k \, (\wedge \Omega^n)$$

where $\mathcal{O}_i \;\sim\; \tilde{\omega}_i \;\in\; H^{q_i}\left(\mathcal{M}_d, \wedge^{p_i} \mathcal{F}^* \right)$ for $\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$

where $\pi : \Sigma \times \mathcal{M}_d \to \mathcal{M}_d$, $\alpha : \Sigma \times \mathcal{M}_d \to X$

$$\Omega \in H^1 \left(\mathcal{M}_d, \mathcal{F}^* \otimes \mathcal{F}_1 \otimes (\mathrm{Obs})^* \right) \qquad \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E}$$

$$\left. \begin{array}{c} \wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} \mathcal{F}^* \otimes \wedge^{\mathrm{top}} \mathcal{F}_1 \otimes \wedge^{\mathrm{top}} (\mathrm{Obs})^* \cong K_{\mathcal{M}_d}$$

hence, again,

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

(S Katz, ES, 2004)

More succinctly, whereas the ordinary A model computes *intersection theory* on a moduli space of curves,

the A/2 model is computing *sheaf cohomology* on a moduli space of curves.

In the rest of this talk, I'm going to present results for correlation functions & quantum sheaf cohomology, but, it should be emphasized that computational methods for A/2 theories are still relatively primitive by comparison to what exists for GW theory.

Correlation functions are often usefully encoded in \`operator products' (OPE's), which encode ring rel'ns.

Physics: Say $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$ ("operator product")

if all correlation functions preserved:

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_C \cdots \rangle$$

Math: if interpret correlation functions as maps

$$\mathrm{Sym}^\bullet W \longrightarrow \mathbb{C}$$

(where W is the space of $\mathcal{O}$'s)

then OPE's are the kernel, of form $\mathcal{O}_A \mathcal{O}_B - \sum_i \mathcal{O}_i$

This discussion is getting a bit too abstract for my taste.

Next: Quickly outline results for correlation f'ns,
to show how ordinary quantum cohomology
& quantum sheaf cohomology arise.

Then: after outlined results,
I'll go back and outline computations.

Examples:

Ordinary ("type II") case: $(\mathcal{E} = T)$

$$X \ = \ \mathbb{P}^1 \times \mathbb{P}^1$$

Space of operators = $W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}[\psi, \tilde{\psi}]$

Correlation functions:

$$\langle \psi \tilde{\psi} \rangle = 1 \qquad \langle \psi^2 \rangle = 0 = \langle \tilde{\psi}^2 \rangle$$

$$\langle \psi^3 \tilde{\psi} \rangle = q \qquad \langle \psi \tilde{\psi}^3 \rangle = \tilde{q}$$

$$\langle \psi^5 \tilde{\psi} \rangle = q^2 \qquad \langle \psi^3 \tilde{\psi}^3 \rangle = q\tilde{q} \qquad \langle \psi \tilde{\psi}^5 \rangle = \tilde{q}^2$$

$$\cdots$$

Pattern (`OPE'): $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Examples:

Ordinary ("type II") case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}[\psi, \tilde{\psi}]$$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

where $q, \tilde{q} \sim \exp(-\mathrm{area})$

$\longrightarrow 0$ in classical limit

Looks like a deformation of cohomology ring, hence called "quantum cohomology"

Examples:

Heterotic case:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \mathcal{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathcal{E}$: $\quad 0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$

where $\quad * = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ $\quad A, B, C, D$ const' 2x2 matrices

$x, \tilde{x}$ vectors of homog' coord's

and $W = \mathbb{C}^2$

Special case: $\quad \mathcal{E} = TX \quad$ when $\quad A = D = I_{2\times 2}, \quad B = C = 0$

Can show space of operators = $H^1(X, \mathcal{E}^*) = W = \mathbb{C}[\psi, \tilde{\psi}]$

Results for correlation functions….

Heterotic case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathcal{E} \text{ a deformation of } T(\mathbb{P}^1 \times \mathbb{P}^1)$$

2-pt correlation functions:

$$\langle \psi \psi \rangle = -\alpha^{-1} \Gamma_1 \qquad \langle \psi \tilde{\psi} \rangle = \alpha^{-1} \Delta \qquad \langle \tilde{\psi} \tilde{\psi} \rangle = -\alpha^{-1} \Gamma_2$$

where

$$\Gamma_1 = \gamma_{AB} \det D - \gamma_{CD} \det B \qquad \Gamma_2 = \gamma_{CD} \det A - \gamma_{AB} \det C$$

$$\gamma_{AB} = \det(A+B) - \det A - \det B$$

$$\gamma_{CD} = \det(C+D) - \det C - \det D$$

$$\Delta = \det A \det D - \det B \det C$$

$$\alpha = \Delta^2 - \Gamma_1 \Gamma_2$$

$\{\alpha = 0\}$ = locus where bundle degenerates

— messier than the ordinary non-deformed case

Heterotic case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathcal{E} \text{ a deformation of } T(\mathbb{P}^1 \times \mathbb{P}^1)$$

4-pt correlation functions:

$$\langle \psi^4 \rangle = q \frac{\Gamma_1}{\alpha^2} \left[\gamma_{CD} \Gamma_1 - 2\Delta \det D \right] + \tilde{q} \frac{\Gamma_1}{\alpha^2} \left[-\gamma_{AB} \Gamma_1 + 2\Delta \det B \right]$$

$$\langle \psi^3 \tilde{\psi} \rangle = q \frac{1}{\alpha^2} \left[-\Gamma_1^2 \det C + \Delta^2 \det D \right] + \tilde{q} \frac{1}{\alpha^2} \left[\Gamma_1^2 \det A - \Delta^2 \det B \right]$$

$$\langle \psi^2 \tilde{\psi}^2 \rangle = q \frac{\Delta}{\alpha^2} \left[-\Gamma_2 \det D + \Gamma_1 \det C \right] + \tilde{q} \frac{\Delta}{\alpha^2} \left[\Gamma_2 \det B - \Gamma_1 \det A \right]$$

$$\langle \psi \tilde{\psi}^3 \rangle = q \frac{1}{\alpha^2} \left[\Gamma_2^2 \det D - \Delta^2 \det C \right] + \tilde{q} \frac{1}{\alpha^2} \left[-\Gamma_2^2 \det B + \Delta^2 \det A \right]$$

$$\langle \tilde{\psi}^4 \rangle = q \frac{\Gamma_2}{\alpha^2} \left[-\gamma_{CD} \Gamma_2 + 2\Delta \det C \right] + \tilde{q} \frac{\Gamma_2}{\alpha^2} \left[-\gamma_{AB} \Gamma_2 - 2\Delta \det A \right]$$

— messier still …

Heterotic case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathcal{E} \text{ a deformation of } T(\mathbb{P}^1 \times \mathbb{P}^1)$$

Although these correlation functions are increasingly unwieldy, one can quickly detect some basic patterns, for example:

$$\langle \psi^2 \det(A\psi + B\tilde{\psi}) \rangle = q \langle \psi^2 \rangle \qquad \langle \psi^2 \det(C\psi + D\tilde{\psi}) \rangle = \tilde{q} \langle \psi^2 \rangle$$

& more gen'ly in 4-pt functions,

$$\langle f_2(\psi, \tilde{\psi}) \det(A\psi + B\tilde{\psi}) \rangle = q \langle f_2(\psi, \tilde{\psi}) \rangle$$

$$\langle f_2(\psi, \tilde{\psi}) \det(C\psi + D\tilde{\psi}) \rangle = \tilde{q} \langle f_2(\psi, \tilde{\psi}) \rangle$$

which (correctly) suggests that the OPE's (ring relations) are

$$\det(A\psi + B\tilde{\psi}) = q, \quad \det(C\psi + D\tilde{\psi}) = \tilde{q}$$

— These are the quantum sheaf cohomology rel'ns.

Summary so far:

Ordinary ("type II") case: $X = \mathbb{P}^1 \times \mathbb{P}^1$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Heterotic case:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \mathcal{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathcal{E}$: $\quad 0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$

where $\quad * = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ $\quad A, B, C, D$ const' 2x2 matrices; $x, \tilde{x}$ vectors of homog' coord's

Here, $W = H^1(X, \mathcal{E}^*) = \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$

OPE's: $\quad \det\left(A\psi + B\tilde{\psi}\right) = q, \quad \det\left(C\psi + D\tilde{\psi}\right) = \tilde{q}$

Check: $\quad \mathcal{E} = TX \quad$ when $\quad A = D = I_{2\times 2}, \quad B = C = 0$

& in this limit, OPE's reduce to those of ordinary case

quantum sheaf cohomology

So far:

Outlined results for correlation functions in ordinary & heterotic cases, to illustrate how in general terms quantum corrected cohomology rings arise.

However, I have not yet explained how to compute those correlation functions, or derive q.s.c. more systematically.

That, I'll do next….

Quantum sheaf cohomology

Let's consider the

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \to H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $\underbrace{H^0(Z \otimes W)}_{0} \to H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \to \underbrace{H^1(Z \otimes W)}_{0}$

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$

$\searrow 0$

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow[\delta]{\sim} H^1(S_1) \xrightarrow[\delta]{} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\text{top}} E^* \cong K_X \\ \mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX}) \end{array}\right\} \overset{\mathrm{GRR}}{\Longrightarrow} \wedge^{\text{top}} F^* \cong K_M$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet} W / (Q^{a+1}, \tilde{Q}^{b+1})$$

where $Q = \det(A\psi + B\tilde{\psi})$, $\tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => OPE's $Q = q$, $\tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Review of quantum sheaf cohomology

General result:

(Math: Donagi, Guffin, Katz, ES, '11)
(Physics: McOrist, Melnikov '08)

For any toric variety, and any def' E of its tangent bundle,

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \to \mathrm{E} \to 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} = \mathrm{q}_a$$

where the M's are matrices of chiral operators built from *.

Generalizes Batyrev's ring $\displaystyle \prod_i \left(\sum_b Q_i^b \psi_b \right)^{Q_i^a} = q_a$

So far I’ve outlined quantum sheaf cohomology for toric varieties with deformations of their tangent bundles.

Next: proposals for heterotic Toda duals

Toda duals

The mirror to the A model on $\mathbb{P}^n$ is a B-twisted Landau-Ginzburg model, defined by a superpotential

$$W \; = \; X_1 + \cdots + X_n + \frac{q}{X_1 \cdots X_n}$$

often referred to as the `Toda dual.'

Analogous statements are known for heterotic theories, which we'll review, but first let's review how this works.

Toda duals (to $\mathbb{P}^n$)

$$W \; = \; X_1 + \cdots + X_n + \frac{q}{X_1 \cdots X_n}$$

Genus zero correlation functions:

$$\langle f(X_1, \cdots, X_n) \rangle \; = \sum_{d_{\ln X} W = 0} \frac{f(X_1, \cdots, X_n)}{\det\left(\partial^2_{\ln X} W\right)}$$

$$dW = 0 \Longrightarrow \quad X_1 = X_2 = \cdots = X_n \equiv X$$

& $\quad X = q X^{-n}$ or $\quad X^{n+1} = q \qquad$ (q.c. rel'n!)

Can show $\quad \det(\partial^2 W) = (n+1) X^n$

hence $\quad \langle X^m \rangle = \sum_{X^{n+1} = q} \frac{X^m}{(n+1) X^n}$

thus $\quad \langle X^{n + d(n+1)} \rangle = q^d \qquad$ matching A model.

What's the heterotic analogue?

A heterotic Landau-Ginzburg model is defined by

- complex Kahler manifold X
- holomorphic vector bundle $\mathcal{E} \to X$
- holomorphic section $(J_a) \in \Gamma(\mathcal{E}^*)$

Recover ordinary Landau-Ginzburg models when

$$\mathcal{E} = TX, \quad J_a = \partial_a W$$

(Z Chen, ES, R Wu, in progress)

The mirror to the (A/2) theory on $\mathbb{P}^n \times \mathbb{P}^m$,

with def' of tangent bundle param'd by matrices A, B, C, D,

is a Landau-Ginzburg theory on $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^m$ with $\mathcal{E} = T$

$$J_i \;=\; a^{(1-n)/n} \left(a X_i + b \frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^{n} \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} - \frac{q_1}{X_1 \cdots X_n} \right)$$

$$\tilde{J}_k \;=\; d^{(1-m)/m} \left(d \tilde{X}_k + c \frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^{m} \nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} - \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} \right)$$

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$$\det(Ax + By) \;=\; a x^{n+1} + b y^{n+1} + \sum_{i=1}^{n} \mu_i x^i y^{n+1-i},$$

$$\det(Cx + Dy) \;=\; c x^{m+1} + d y^{m+1} + \sum_{k=1}^{n} \nu_k x^k y^{n+1-k},$$

Let me outline correlation functions in these theories.

For heteroic LG models of the form just discussed,

at genus 0,

$$\langle f(X_i, \tilde{X}_k) \rangle = \sum_{J, \tilde{J}=0} \frac{f(X_i, \tilde{X}_k)}{\det(\partial(J, \tilde{J}))}$$

$$J, \tilde{J} = 0 \Longrightarrow \quad X_1 = \cdots = X_n \equiv X, \quad \tilde{X}_1 = \cdots = \tilde{X}_m \equiv \tilde{X}$$

& $\quad \det(AX + B\tilde{X}) = q, \quad \det(CX + D\tilde{X}) = \tilde{q}$

— the quantum sheaf cohomology ring rel'ns

Can show all (genus 0) correlation functions match those of the corresponding A/2 theory, which is how we've checked this proposal.

Grassmannians

Let me quickly outline results for q.s.c. rings for Grassmannians.

(J Guo, Z Lu, ES, 1512.08586 & to appear)

On G(k,n), the Grassmannian of k-planes in $\mathbf{C}^n$, for 1 < k < n-1, the tangent bundle has moduli:

$$h^1(G(k,n), \mathrm{End}\, T) \;=\; \begin{cases} n^2 - 1 & 1 < k < n-1 \\ 0 & \text{else} \end{cases}$$

We'll deform the tangent bundle, and describe the resulting q.s.c. ring.

Deformations of tangent bundle of G(k,n)

The tangent bundle itself can be represented as the cokernel

$$0 \longrightarrow S^* \otimes S \xrightarrow{*} \mathcal{O}^n \otimes S^* \longrightarrow T \longrightarrow 0$$

We can encode a deformation $\mathcal{E}$ of the tangent bundle by modifying the map *.

$$* : \omega_\alpha^\beta \mapsto A_j^i \omega_\alpha^\beta x_\beta^j + B_j^i \omega_\beta^\beta x_\alpha^j$$

where the x_α^i are Stiefel coordinates, and S is the universal subbundle.

The tangent bundle arises in the special case that A = I, B = 0.

So long as A invertible, can perform GL(n) rotation to eliminate, so moduli are in (traceless part of) B.

Given a deformation $\mathcal{E}$ of T,

- we don't have a mathematical derivation/def'n of the quantum sheaf cohomology ring,

but

- we can use physics computations to determine its form.

Since this is a mostly math audience, I'll spare you the physics details, and instead outline the results.

Structure of quantum sheaf cohomology ring for a generic deformation of T G(k,n)

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \rangle$$

where
$$D_m \; = \; \det \left(\sigma_{(1+j-i)} \right)_{1 \leq i,j \leq m}$$
$$R_{(r)} \; = \; \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$$

for I_i the char' poly's of B:
$$\det(tI + B) \; = \; \sum_{i=0}^{n} I_{n-i} t^i$$

Exs: $I_0 = 1, \quad I_1 = \mathrm{Tr}\, B, \quad I_n = \det B$

Quantum sheaf cohomology ring:

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\Big\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \Big\rangle$$

If we turn off the deformation (set B=0), then

$$R_{(n)} = \sigma_{(n)}$$

and with some work it can be shown that the ring above can be presented as

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(n-k)}]/\langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q\rangle$$

which is a standard presentation of the (ordinary) quantum cohomology ring of G(k,n).

(Buch, Kresch, Tamvakis, Bertram, Witten, Siebert, Tian, ….)

Example: G(1,3)

This has no nontrivial deformations, so any result should be equivalent to ordinary quantum cohomology ring of $\mathbb{P}^2$.

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\langle D_2, \cdots, R_{(3)} + q, R_{(4)} + q\sigma_{(1)}, \cdots \rangle$$

which $= \mathbb{C}[\sigma_{(1)}]/\langle R_{(3)} + q \rangle\rangle$

using $D_2 = \sigma_{(1)}^2 - \sigma_{(2)}, \cdots$ to eliminate $\sigma_{(m)}$ for m>1, and

the result $R_{(3+\ell)} + q\sigma_{(\ell)} = \sigma_{(\ell)}(R_{(3)} + q)$

Now, $R_{(3)} = \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma^i = \left(\sum_{i=0}^{3} I_i \right) \sigma^3 = (\det(I+B))\, \sigma^3$

so the qsc ring is $\mathbb{C}[\sigma]/\langle \det(I+B)\sigma^3 + q \rangle$

which is equivalent to std quantum cohomology ring.

G(1,n), G(n-1,n) admit no deformations and so their q.s.c. rings coincide with ordinary q.c. rings

However, for 1 < k < n-1,
the q.s.c. ring of a def' of G(k,n) is not the same as the ordinary q.c. ring.

The description of the q.s.c. ring given is valid generically.

Breaks down along discriminant locus, where bundle degenerates.

This turns out to be the locus where, on G(k,n), B has k eigenvalues whose sum is -1.

Summary

- Survey of heterotic mirror symmetry

- Outlined def'n of quantum sheaf cohomology

- Computations on $\mathbb{P}^1 \times \mathbb{P}^1$

  & results for toric varieties

- (0,2) Toda-like Landau-Ginzburg mirrors

- Results for Grassmannians

I hope to see you all at

June 27 - July 2, 2016

Paris, France