Recent developments in 2d (0,2) theories

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My talk today will focus on recent advances in understanding heterotic compactifications of string theory.

Some terminology:

Worldsheets of e.g. type II strings have (2,2) worldsheet susy typically (= dim'l reduction of 4d N=1 susy).

NLSM's with (2,2) susy are specified by cpx Kahler mfld X.

Worldsheets of heterotic strings — today's focus — have instead (0,2) worldsheet susy typically.

NLSM's with (0,2) susy are specified by cpx Kahler mfld X together with a hol' vector bundle E —> X.

(2,2) = special case of (0,2), in which E = TX.

One of the standard technical tools in string compactifications = gauged linear sigma model (GLSM).

(Witten, '93)

This is a 2d gauge theory.
It's not itself a SCFT, but will often RG flow to one.

Much of the progress in string compactifications over the last half dozen years has revolved around better understanding GLSM's and their RG flows.

So, what has been happening?….

Over the last half dozen years, there's been a *tremendous* amount of progress in perturbative string compactifications.

A few of my favorite examples:

- nonpert' realizations of geometry (Pfaffians, double covers)
  (Hori-Tong '06, Caldararu et al '07,…)
- perturbative GLSM's for Pfaffians (Hori '11, Jockers et al '12,…)
- non-birational GLSM phases - physical realization of homological projective duality (Hori-Tong '06, Caldararu et al '07, Ballard et al '12; Kuznetsov '05-'06,…)
- examples of closed strings on noncommutative res'ns
  (Caldararu et al '07, Addington et al '12, ES '13)
- localization techniques: new GW & elliptic genus computations, role of Gamma classes, …
  (Benini-Cremonesi '12, Doroud et al '12; Jockers et al '12, Halverson et al '13, Hori-Romo '13, Benini et al '13, ….)
- heterotic strings: nonpert' corrections, 2d dualities, non-Kahler moduli (many)

Far too much to cover in one talk! I'll focus on just one….

Today I'll restrict to

- heterotic strings: nonpert' corrections, 2d dualities,
  non-Kahler moduli

My goal today is to give a survey of some of the results in (0,2) over the last six years or so, both new results as well as some older results to help provide background & context.

So, what will I discuss?…

Outline:

Review of quantum sheaf cohomology

Dualities in 2d

- (0,2) mirror symmetry
- Gauge dualities — Seiberg(-like) dualities
  - — corresponding geometry
  - — 2d tricks one can't play in 4d
- Decomposition in 2d nonabelian gauge theories
  - Ex: $SU(2) = SO(3)_+ + SO(3)_-$

Brief overview of moduli

Review of quantum sheaf cohomology

Recall ordinary quantum cohomology:

- OPE ring in the A model TFT
- encodes nonperturbative corrections
- operators in the A model correspond to elements of cohomology of some space X
- OPE ring mostly looks like wedge products of differential forms, except that powers beyond top-forms need not vanish

Ex: ordinary quantum cohomology of $\mathbb{P}^n$

$$\mathbb{C}[x]/(x^{n+1}-q)$$

Looks like a deformation of cohomology ring, hence 'quantum' cohomology.

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

(Katz-ES '04, ES '06, Guffin-Katz '07, ….)

Encodes nonperturbative corrections to charged matter couplings.

Example: (2,2) compactification on CY 3-fold

Gromov-Witten invariants encoded in $\overline{\mathbf{27}}^{\mathbf{3}}$ couplings

Off the (2,2) locus, Gromov-Witten inv'ts no longer relevant.

Mathematical GW computational tricks no longer apply.

No known analogue of periods, Picard-Fuchs equations.

New methods needed….

… and a few have been developed.

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

Ex: ordinary quantum cohomology of $\mathbb{P}^n$

$$\mathbb{C}[x]/(x^{n+1}-q)$$

Compare: quantum sheaf cohomology of $\mathbb{P}^n \times \mathbb{P}^n$

with bundle

$$0 \to O \oplus O \xrightarrow{*} O(1,0)^{n+1} \oplus O(0,1)^{n+1} \to E \to 0$$

where

$$* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$$

$x, \tilde{x}$ homog' coord's on $\mathbb{P}^n$'s

is given by $\mathbb{C}[x,y]/(\det(Ax+By)-q_1, \det(Cx+Dy)-q_2)$

Check: When E=T, this becomes $\mathbb{C}[x,y]/(x^{n+1}-q_1, \mathrm{y}^{n+1}-q_2)$

Review of quantum sheaf cohomology

Ordinary quantum cohomology

= OPE ring of the A model TFT in 2d

The A model is obtained by twisting (2,2) NLSM along $U(1)_V$

In a heteroticc (0,2) NLSM, if $\det E^* \cong K_X$

then there is a nonanomalous U(1) we can twist along.

Result: a pseudo-topological field theory, “A/2 model”

Quantum sheaf cohomology

= OPE ring of the A/2 model

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

When does that OPE ring close into itself?

(2,2) susy **not** required.

For a SCFT, can use combination of

- worldsheet conformal invariance
- right-moving N=2 algebra

to argue closure on patches on moduli space.

(Adams-Distler-Ernebjerg, ’05)

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

A model:

Operators: $b_{i_1 \cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \cdots \chi^{i_1} \cdots \chi^{i_p} \leftrightarrow H^{p,q}(X)$

A/2 model:

Operators: $b_{\bar{\imath}_1 \cdots \bar{\imath}_q a_1 \cdots a_p} \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_q} \lambda_-^{a_1} \cdots \lambda_-^{a_p} \leftrightarrow H^q(X, \wedge^p E^*)$

On the (2,2) locus, A/2 reduces to A.

For operators, follows from

$$H^q(X, \wedge^p T^* X) = H^{p,q}(X)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Schematically:

A model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n = \int_X (\text{top-form})$$

$$\omega_i \in H^{p_i, q_i}(X)$$

A/2 model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n$$

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\text{top}}(X, \wedge^{\text{top}} E^*) = \mathrm{H}^{\text{top}}(X, K_X)$

using the anomaly constraint $\det E^* \cong K_X$

Again, a top form, so get a number.

Review of quantum sheaf cohomology

To make this more clear, let's consider an

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \to H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

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where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \rightarrow \wedge^2 E^* \rightarrow \wedge^2 Z \rightarrow Z \otimes W \rightarrow \mathrm{Sym}^2 W \otimes O \rightarrow 0$$

Break into short exact sequences:

$$0 \rightarrow \wedge^2 E^* \rightarrow \wedge^2 Z \rightarrow S_1 \rightarrow 0$$

$$0 \rightarrow S_1 \rightarrow Z \otimes W \rightarrow \mathrm{Sym}^2 W \otimes O \rightarrow 0$$

Examine second sequence:

induces $\cancelto{0}{H^0(Z \otimes W)} \rightarrow H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \rightarrow \cancelto{0}{H^1(Z \otimes W)}$

Since Z is a sum of $O(-1,0)$'s, $O(0,-1)$'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$

0

Since Z is a sum of $O(-1,0)$'s, $O(0,-1)$'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \underset{\delta}{\overset{\sim}{\to}} H^1(S_1) \underset{\delta}{\to} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\mathrm{top}} E^* \cong K_X \\ \mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX}) \end{array}\right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} F^* \cong K_M$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet} \mathrm{W} / (Q^{a+1}, \tilde{Q}^{b+1})$$

where $Q = \det(A\psi + B\tilde{\psi})$, $\tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => OPE's $Q = q, \ \tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Review of quantum sheaf cohomology

General result: (Donagi, Guffin, Katz, ES, '11)

For any toric variety, and any def' E of its tangent bundle,

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \to \mathrm{E} \to 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} = \mathrm{q}_a$$

where the M's are matrices of chiral operators built from *.

Review of quantum sheaf cohomology

So far, I've outlined mathematical computations of quantum sheaf cohomology, but GLSM-based methods also exist:

- Quantum cohomology ((2,2)): Morrison-Plesser '94
- Quantum sheaf cohomology ((0,2)): McOrist-Melnikov '07, '08

Briefly, for (0,2) case:

One computes quantum corrections to effective action of form

$$L_{\mathrm{eff}} = \int d\theta^+ \sum_a \Upsilon_a \log\left[\prod\nolimits_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} / q_a \right]$$

from which one derives $\displaystyle\prod_\alpha \left(\det M_{(\alpha)}\right)^{Q^a_\alpha} = q_a$

— these are q.s.c. rel'ns — match math' computations

Review of quantum sheaf cohomology

State of the art: computations on toric varieties

To do: compact CY's

Intermediate step: Grassmannians (work in progress)

Briefly, what we need are better computational methods.

Conventional GW tricks seem to revolve around idea that A model is independent of complex structure, not necessarily true for A/2.

- McOrist-Melnikov '08 have argued an analogue for A/2
- Despite attempts to check (Garavuso-ES '13), still not well-understood

Outline:

Review of quantum sheaf cohomology

Dualities in 2d

- (0,2) mirror symmetry
- Gauge dualities — Seiberg(-like) dualities
  - — corresponding geometry
  - — 2d tricks one can't play in 4d
- Decomposition in 2d nonabelian gauge theories
  Ex: $SU(2) = SO(3)_+ + SO(3)_-$

Brief overview of moduli

(0,2) mirror symmetry ((0,2) susy)

We'll begin our discussion of dualities with a review of progress on a conjectured generalization of mirror symmetry, so first, let's review mirror symmetry.

Nonlinear sigma models with (2,2) susy defined by complex Kahler X. (Assume X CY.)

Mirror (typically) defined by complex Kahler CY Y.

$$\mathrm{CFT}(X) = \mathrm{CFT}(Y)$$

$$\dim X = \dim Y$$

$$A(X) = B(Y)$$

$$H^{p,q}(X) = H^{n-p,q}(Y)$$

cpx moduli of X = Kahler moduli of Y

Used to make predictions for e.g. Gromov-Witten invariants.

(0,2) mirror symmetry ((0,2) susy)

There exists a conjectured generalization of mirror symmetry: (0,2) mirrors.

Nonlinear sigma models with (0,2) susy defined by space X, with gauge bundle $E \to X$

(0,2) mirror defined by space Y, w/ gauge bundle F.

dim X = dim Y

rk E = rk F

A/2(X, E) = B/2(Y, F)

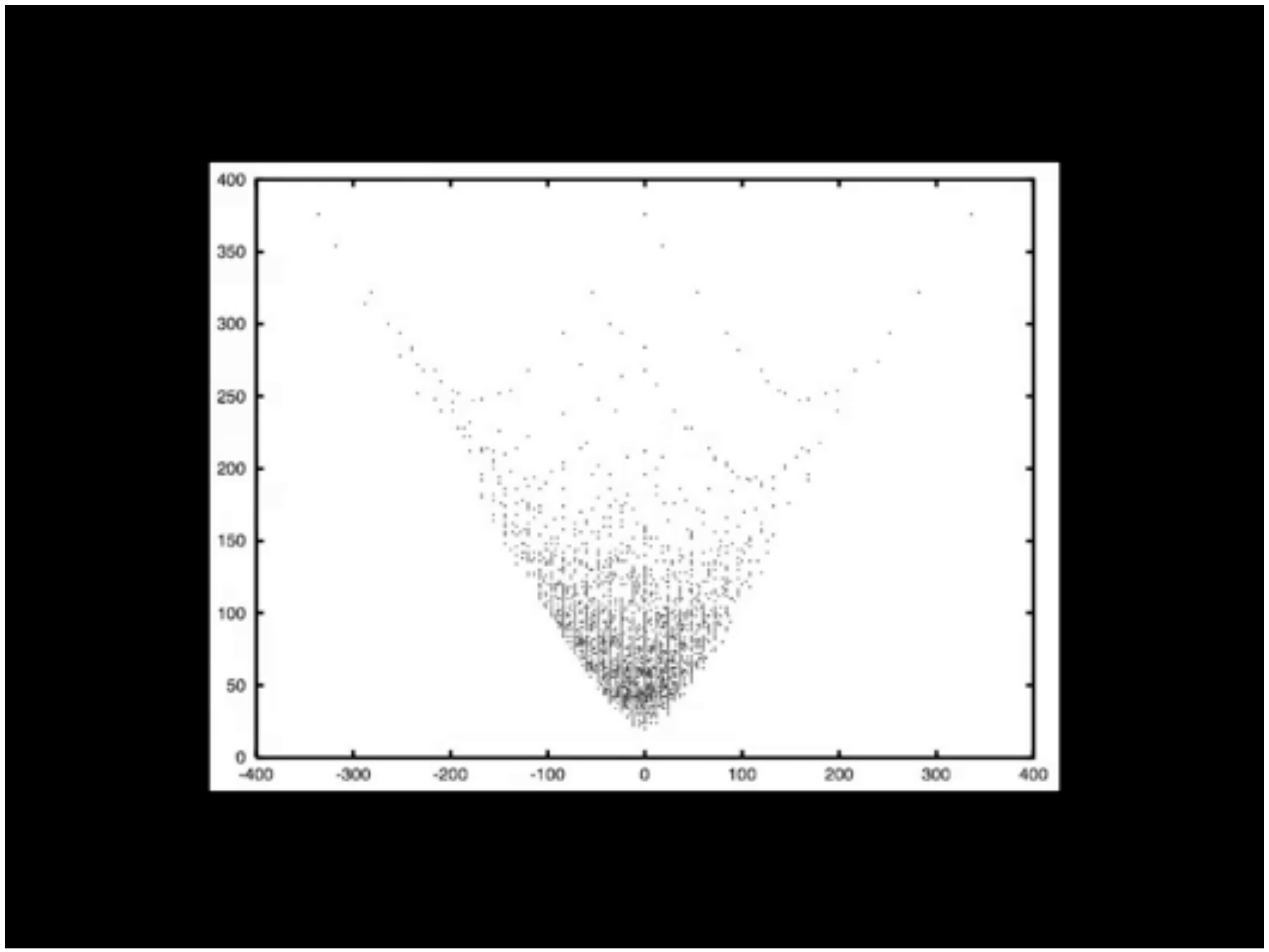
$$H^p(X, \wedge^q E^*) = H^p(Y, \wedge^q F)$$

(moduli) = (moduli)

When E=TX, should reduce to ordinary mirror symmetry.

(0,2) mirror symmetry **((0,2) susy)**

Numerical evidence:



Horizontal:

$$h^1(E) - h^1(E^*)$$

Vertical:

$$h^1(E) + h^1(E^*)$$

(E rank 4)

(Blumenhagen-Schimmrigk-Wisskirchen, NPB 486 ('97) 598-628)

(0,2) mirror symmetry ((0,2) susy)

Constructions include:

- Blumenhagen-Sethi '96 extended Greene-Plesser orbifold construction to (0,2) models — handy but only gives special cases
- Adams-Basu-Sethi '03 repeated Hori-Vafa-Morrison-Plesser-style GLSM duality in (0,2)
- Melnikov-Plesser '10 extended Batyrev's construction & monomial-divisor mirror map to include def's of tangent bundle, for special ('reflexively plain') polytopes

Progress, but still don't have a general construction.

Gauge dualities ((0,2) & (2,2) susy)

So far we've discussed dualities that act nontrivially on target-space geometries.

Next: gauge dualities in 2d
— different gauge theories which flow to same IR fixed point.

Such dualities are of long-standing interest in QFT, and there has been much recent interest in 2d dualities:
Hori '11, Benini-Cremonesi '12, Gadde-Gukov-Putrov '13, Kutasov-Lin '13, Jia-ES-Wu '14, …

In 2d, we'll see dualities can at least sometimes be understood as different presentations of **same** geometry.

This not only helps explain why these dualities work in 2d, but also implies a procedure to generate examples (at least for CY, Fano geometries) …..

Prototype: Seiberg duality in 4d:

Electric theory:	Magnetic theory:
$SU(N_c)$ gauge theory	$SU(N_f\text{-}N_c)$ gauge theory
N_f chiral mult's Q in fund'	N_f chiral mult's q in fund'
N_f chiral mult's $\tilde{Q}$ in antifund'	N_f chiral mult's $\tilde{q}$ in antifund'
	N_f^2 neutral chirals M
	$W = M q \tilde{q}$

Both have global symmetry $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$

Flow to same IR fixed point, at least for $3N_c/2 < N_f < 3\,N_c$

Check:
- 't Hooft anomalies
- compare baryons & mesons
- moduli spaces

Prototype: Seiberg duality in 4d:

Electric theory:

$SU(N_c)$ gauge theory

N_f chiral mult's Q in fund'

N_f chiral mult's $\tilde{Q}$ in antifund'

Magnetic theory:

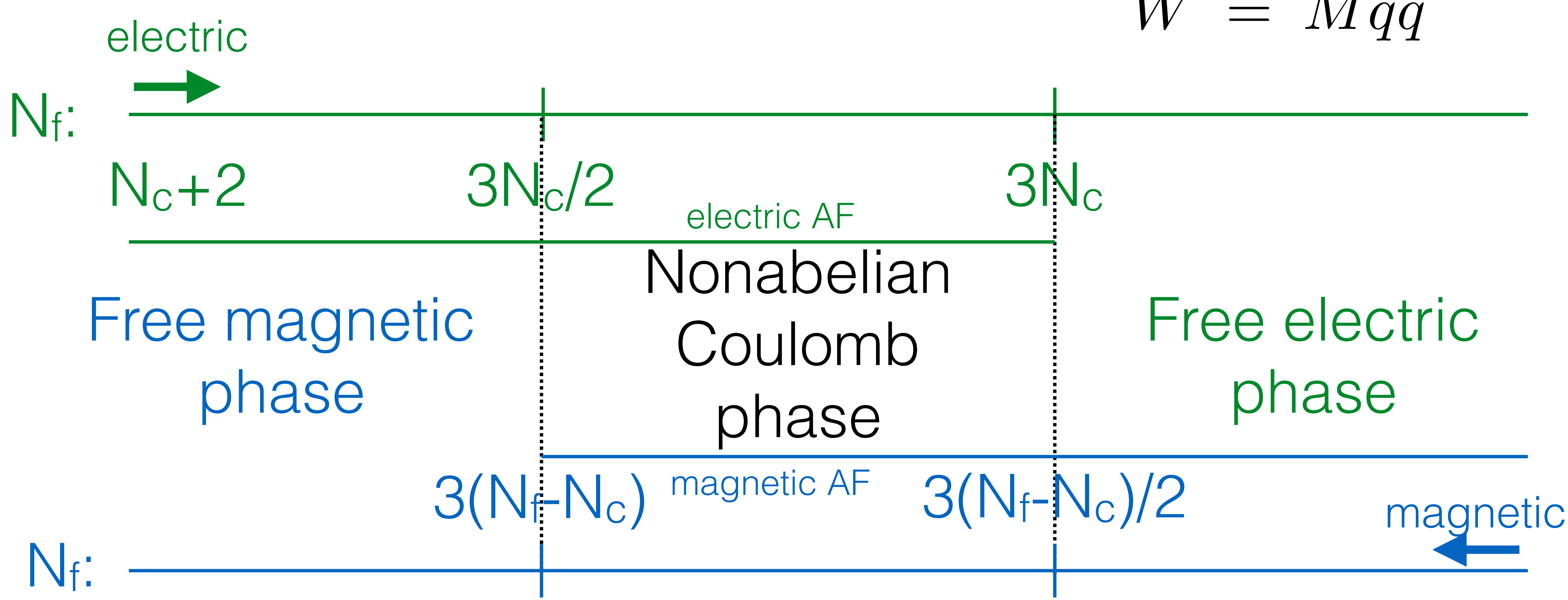
$SU(N_f\text{-}N_c)$ gauge theory

N_f chiral mult's q in fund'

N_f chiral mult's $\tilde{q}$ in antifund'

N_f^2 neutral chirals M

$$W = Mq\tilde{q}$$



Gauge dualities ((0,2) & (2,2) susy)

In principle, 2d theories are easier to work with, as the gauge field does not propagate, so it can be integrated out, turning the gauge theory into a NLSM on some geometry.

I'll outline a program of understanding 2d gauge dualities via corresponding geometries.

The geometries arise via standard methods;
let me take a moment to review them,
before discussing dualities….

Prototypical example: $\mathbb{CP}^n$ **model** **((2,2) susy)**

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Analyze semiclassical low-energy behavior:

Potential $V = D^2$

where $D = \sum_i |\phi_i|^2 - r$

r = Fayet-Iliopoulos parameter

When $r \gg 0$, $\{V = 0\} = S^{2n+1}$

so semiclassical Higgs moduli space is $\{V = 0\} / U(1) = \mathbb{CP}^n$

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Semiclassical Higgs moduli space is $\{V=0\}/U(1)=\mathbb{CP}^n$

Of course, that doesn't tell the whole story.

r is renormalized at one-loop:

$$\Delta r \propto \sum_i q_i \quad \text{here, } = n+1$$

so the $\mathbb{CP}^n$ shrinks to strong coupling under RG.

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Summary:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

RG

Nonlinear sigma model on $\mathbb{CP}^n$

RG

Next example: nonabelian version **((2,2) susy)**

Gauge theory:

U(k) gauge group,
matter: n chiral multiplets in fund' **k**, n > k

Similar analysis:

Nonlinear sigma model on G(k,n)

Consistency check: when k=1, G(k,n) = $\mathbb{CP}^{n-1}$

Now, let's turn to 2d dualities….

Gauge dualities ((2,2) susy)

Baby example:

U(k) gauge theory, n chirals in fundamental, n>k	$\longleftrightarrow$	U(n-k) gauge theory, n chirals in fundamental
RG		RG
G(k,n)	=	G(n-k,n)

More interesting example next….

Gauge dualities ((2,2) susy)

Example in 2d:

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' $\mathbf{k}^*$, A<n

$\xleftrightarrow[\text{Benini-Cremonesi '12}]{\text{Seiberg}}$

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' $\mathbf{k}^*$,
nA neutral chirals M,
superpotential: W = M Φ P

$\downarrow$ RG $\qquad\qquad$ $\downarrow$ RG

NLSM on $\mathrm{Tot}\left(S^A \to \mathrm{G(k,n)}\right)$ $= \mathrm{Tot}\left((Q^*)^A \to \mathrm{G(n\text{-}k,n)}\right)$

$= (\mathbb{C}^{kn} \times \mathbb{C}^{kA}) // GL(k)$

$\downarrow$ RG $\qquad\qquad$ $\downarrow$ RG

. . . $\qquad\qquad$ . . .

Build physics for RHS using $0 \to S \xrightarrow{\Phi} O^n \to Q \to 0$
& discover the upper RHS.

So, 2d analogue of Seiberg duality has geometric description.

Gauge dualities ((2,2) susy)

Dualities between gauge theories are of significant interest in the physics community, as they can be used to extract otherwise inaccessible information.

Strategy: use easy math to make physics predictions.

Our next example will be constructed from

$G(2,4)$ = degree 2 hypersurface in $\mathbb{P}^5$

Abelian/nonabelian dualities ((2,2) susy)

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

$\longleftrightarrow$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

$\Big\downarrow$ RG $\qquad\qquad\qquad\qquad$ $\Big\downarrow$ RG

G(2,4) = degree 2 hypersurface in $\mathbb{P}^5$

$$= \mathbb{C}^{2\cdot 4} // GL(2) \qquad = \{z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}\} \subset \mathbb{C}^6 // \mathbb{C}^\times$$

$\Big\downarrow$ RG

…

The physical duality implied at top relates abelian & nonabelian gauge theories, which in 4d for ex would be surprising.

Abelian/nonabelian dualities ((2,2) susy)

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

$\longleftrightarrow$

U(1) gauge theory,
6 chirals z_{ij} = $-z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

Relation: $z_{ij} = \epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta$

Consistency checks:

Compare symmetries: GL(4) action

$$\phi_i^\alpha \mapsto V_i^{\ j} \phi_j^\alpha \qquad\qquad z_{ij} \mapsto V_i^{\ k} V_j^{\ \ell} z_{k\ell}$$

Chiral rings, anomalies, Higgs moduli space match automatically.

Can also show elliptic genera match, applying computational methods of Benini-Eager-Hori-Tachikawa '13, Gadde-Gukov '13.

Abelian/nonabelian dualities ((2,2) susy)

This little game is entertaining,
but why's it useful ?

Standard physics methods rely on matching global symmetries and corresponding 't Hooft anomalies between prospective gauge duals.

However, generic superpotentials break all symmetries.

Identifying gauge duals as different presentations of the same geometry allows us to construct duals when standard physics methods do not apply.

Abelian/nonabelian dualities ((2,2) susy)

A simple set of examples in which global symmetry broken:

$$G(2,4)[d_1, d_2, \cdots] = \mathbb{P}^5[2, d_1, d_2, \cdots]$$

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**
chirals p_a of charge $-d_a$
under det U(2)
superpotential

$$W = \sum_a p_a f_a(\epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta)$$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
chirals P_a of charge $-d_a$,
superpotential

$$W = P(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}) + \sum_a P_a f_a(z_{ij})$$

$$\epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta = z_{ij}$$

Abelian/nonabelian dualities ((0,2) susy)

Let's build on the previous example

$$G(2,4)[d_1,d_2,\cdots] = \mathbb{P}^5[2,d_1,d_2,\cdots]$$

by extending to heterotic cases: describe space + bundle.

Example:

Bundle $0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$

on the CY G(2,4)[4].

Described by

U(2) gauge theory

4 chirals in fundamental

1 Fermi in (-4,-4) (hypersurface)

8 Fermi's in (1,1) (gauge bundle E)

1 chiral in (-2,-2) (gauge bundle E)

2 chirals in (-3,-3) (gauge bundle E)

plus superpotential

rep' of U(2)

Abelian/nonabelian dualities ((0,2) susy)

U(2) gauge theory	$\longleftrightarrow$	U(1) gauge theory
4 chirals in fundamental		6 chirals charge +1
1 Fermi in (-4,-4) (hypersurface)		2 Fermi's charge -2, -4
8 Fermi's in (1,1) (gauge bundle E)		8 Fermi's charge +1
1 chiral in (-2,-2) (gauge bundle E)		1 chiral charge -2
2 chirals in (-3,-3) (gauge bundle E)		2 chirals charge -3
plus superpotential		plus superpotential
$\Big\downarrow$ RG		$\Big\downarrow$ RG
Bundle	=	Bundle
$0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$		$0 \to E \to \oplus^8 O(1) \to O(2) \oplus^2 O(3) \to 0$
on the CY G(2,4)[4].		on the CY $\mathbb{P}^5[2,4]$

- both satisfy anomaly cancellation
- elliptic genera match

Gadde-Gukov-Putrov triality ('13) ((0,2) susy)

bundle space

$$S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \quad \cdots\overset{\text{phase}}{\cdots}\cdots \quad (S^*)^A \oplus (Q^*)^n \to G(k,2k+A-n)$$

$$\updownarrow =$$

$$(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \quad \cdots\overset{\text{phase}}{\cdots}\cdots \quad (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k,A)$$

$$\updownarrow =$$

$$S^n \oplus (Q^*)^A \to G(A-n+k,2k+A-n) \cdots (S^*)^n \oplus (Q^*)^{2k+A-n} \to G(A-n+k,A)$$

$$\updownarrow =$$

$$(Q^*)^n \oplus S^A \to G(k,2k+A-n) \quad \cdots\overset{\text{phase}}{\cdots}\cdots \quad (Q)^{2k+A-n} \oplus S^A \to G(k,n)$$

For brevity, I've omitted writing out the (0,2) gauge theory.

Utilizes another duality: CFT(X,E) = CFT(X,E*)

Further examples ((2,2) susy)

Start with standard fact:

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Further examples ((2,2) susy)

U(2) gauge theory,
n chirals in fundamental

$\longleftrightarrow$

U(n-2)xU(1) gauge theory,
n chirals X in fundamental of U(n-2),
n chirals P in antifundamental of U(n-2)
(n choose 2) chirals $z_{ij} = - z_{ji}$
each of charge +1 under U(1),
W = tr PAX

$\downarrow$ RG $\qquad\qquad$ $\downarrow$ RG

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

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(using description of Pfaffians of
Hori '11, Jockers et al '12)

In this fashion, straightforward to generate examples.....

Dualities ((0,2) & (2,2) susy)

How do these gauge dualities relate to (0,2) mirrors?

As we've seen, gauge dualities often relate different presentations of the same geometry, whereas (0,2) mirrors exchange different geometries.

Existence of (0,2) mirrors seems to imply that there ought to exist more `exotic' gauge dualities, that present different geometries.

We've just used math to make predictions for physics. Next, we'll turn that around, and use physics to make predictions for math….

Decomposition

In a 2d orbifold or gauge theory,
if a finite subgroup of the gauge group acts trivially on all matter, the theory decomposes as a disjoint union.

(Hellerman et al '06)

Ex: $\mathrm{CFT}([\mathrm{X}/\mathbb{Z}_2]) = \mathrm{CFT}\left(X \coprod X\right)$

On LHS, the $\mathbb{Z}_2$ acts triv'ly on X,
hence there are dim' zero twist fields.
Projection ops are lin' comb's of dim 0 twist fields.

Ex: $\mathrm{CFT}([\mathrm{X}/\mathrm{D}_4])$ where $\mathbb{Z}_2 \subset D_4$ acts trivially on X

$$= \mathrm{CFT}\left([\mathrm{X}/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [\mathrm{X}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

where $D_4 / \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$

This is what's meant by `decomposition'….

Decomposition

Decomposition is also a statement about mathematics.
Dictionary:

	2d Physics	Math
	D-brane	Derived category
	Gauge theory	Stack
	Gauge theory w/ trivially acting subgroup	Gerbe
Conjecture:	Landau-Ginzburg model	Derived scheme
	Universality class of renormalization group flow	Categorical equivalence

(Kontsevich '95, ES '99, Douglas '00)

(Pantev-ES '05)

(Pantev, Calaque, Katzarkov, Toen, Vezzosi, Vaquie, **work in progress**)

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Conjecture:

Decomposition is a statement about physics of strings on gerbes, summarized in the decomposition conjecture….

Decomposition

Decomposition conjecture: (version for banded gerbes)

(Hellerman et al '06)

string on gerbe

$$\mathrm{CFT}(\text{G-gerbe on X}) = \mathrm{CFT}\left(\coprod_{\hat{G}} (X,B) \right)$$

string on disjoint union of spaces

where the B field is determined by the image of

characteristic class

$$H^2(X,Z(G)) \xrightarrow{Z(G)\mapsto U(1)} H^2(X,U(1))$$

flat B field

Consistent with:
- multiloop orbifold partition f'ns
- q.c. ring rel'ns as derived from GLSM's
- D-branes, K theory, sheaves on gerbes

Applications:
- predictions for GW inv'ts, checked by H H Tseng et al '08-'10
- understand GLSM phases, via giving a physical realization of Kuznetsov's homological projective duality for quadrics (Caldararu et al '07, Hori '11, Halverson et al '13…)

Decomposition

$$\mathrm{CFT}(\text{G-gerbe on X}) = \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

Checking this statement in orbifolds involved comparing e.g. multiloop partition functions, state spaces, D-branes, …

In gauge theories, there are further subtleties.

Example:

Ordinary $\mathbb{CP}^n$ model = U(1) gauge theory with n+1 chiral superfields, each of charge +1

Gerby $\mathbb{CP}^n$ model = U(1) gauge theory with n+1 chiral superfields, each of charge +k, k>1

Require physics of charge k > 1 different from charge 1
— but how can multiplying the charges by a factor change anything?

Decomposition

Require physics of charge k > 1 different from charge 1
— but how can multiplying the charges by a factor change anything?

For physics to see gerbes, there must be a difference,
but why isn't this just a convention?
How can physics see this?

Answer: nonperturbative effects

Noncompact worldsheet: distinguish via θ periodicity

Compact worldsheet: define charged field via specific bundle

(Adams-Distler-Plesser, Aspen '04)

Decomposition has been extensively checked for *abelian* gauge theories and orbifolds;
nonabelian gauge theories much more recent….

Decomposition

Extension of decomposition to nonabelian gauge theories:

Since 2d gauge fields don't propagate,
analogous phenomena should happen in nonabelian gauge theories with center-invariant matter.

Proposal: **(ES, '14)**

For G semisimple, with center-inv't matter,
G gauge theories decompose into a sum of theories with variable discrete theta angles:

Ex: $SU(2) = SO(3)_+ + SO(3)_-$

— SO(3)'s have different discrete theta angles

Decomposition

Extension of decomposition to nonabelian gauge theories:

Aside: discrete theta angles

(Gaiotto-Moore-Neitzke '10,
Aharony-Seiberg-Tachikawa '13, Hori '94)

Consider 2d gauge theory, group $G = \tilde{G} / K$

$\tilde{G}$ compact, semisimple, simply-connected

K finite subgroup of center of $\tilde{G}$

The theory has a degree-two K-valued char' class w

For λ any character of K, can add a term to the action

$$\lambda(w)$$

— discrete theta angles, classified by characters

Ex: $SO(3) = SU(2) / \mathbb{Z}_2$ has 2 discrete theta angles

Decomposition

Ex: SU(2) = SO(3)$_+$ + SO(3)$_-$

Let's see this in pure nonsusy 2d QCD.

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$

Decomposition

More general statement of decomposition for 2d nonabelian gauge theories with center-invariant matter:

For G semisimple, K a finite subgp of center of G,

$$G = \sum_{\lambda \in \hat{K}} (G/K)_\lambda$$

indexes discrete theta angles

Other checks include 2d susy partition functions, utilizing Benini-Cremonesi '12, Doroud et al '12; arguments there revolve around cocharacter lattices.

Outline:

Review of quantum sheaf cohomology

Dualities in 2d

- (0,2) mirror symmetry
- Gauge dualities — Seiberg(-like) dualities
  - — corresponding geometry
  - — 2d tricks one can't play in 4d
- Decomposition in 2d nonabelian gauge theories
  Ex: $SU(2) = SO(3)_{+} + SO(3)_{-}$

Brief overview of moduli

Brief overview of moduli

It was known historically that for large-radius het' NLSM's on the (2,2) locus, there were three classes of infinitesimal moduli:

$H^1(X, T^*X)$ Kahler moduli

$H^1(X, TX)$ Complex moduli

$H^1(X, \mathrm{End}\,\mathrm{E})$ Bundle moduli

where, on (2,2) locus, $E = TX$

When the gauge bundle $E \neq TX$,
the correct moduli counting is more complicated….

Brief overview of moduli

For Calabi-Yau (0,2) compactifications off the (2,2) locus, moduli are as follows:

(Anderson-Gray-Lukas-Ovrut, '10)

$H^1(X, T^*X)$ Kahler moduli

$H^1(Q)$ where

$$0 \to \mathrm{End}\,\mathrm{E} \to Q \to TX \to 0 \qquad (F)$$

(Atiyah sequence)

There remained for a long time the question of moduli of non-Kahler compactifications….

Brief overview of moduli

For non-Kahler (0,2) compactifications,
in the **formal** $\alpha' \to 0$ limit,

(Melnikov-ES, '11)

$H^1(S)$ where

$$0 \to T^* X \to S \to Q \to 0 \qquad (H, \; dH = 0)$$

$$0 \to \mathrm{End}\,\mathrm{E} \to Q \to TX \to 0 \qquad (F)$$

Now, we also need α' corrections….

Brief overview of moduli

Through first order in α',
the moduli are *overcounted* by

(Anderson-Gray-ES '14; de la Ossa-Svanes '14)

$H^1(S)$ where

$$0 \to T^*X \to S \to Q \to 0 \qquad (H, \text{Green-Schwarz})$$

$$0 \to \mathrm{End}\,\mathrm{E} \oplus \mathrm{End}\,TX \to Q \to TX \to 0 \qquad (F, R)$$

on manifolds satisfying the $\partial\bar{\partial}$ lemma.

Current state-of-the-art

WIP to find correct counting, & extend to higher orders

Brief overview of moduli

So far I've outlined infinitesimal moduli — marginal operators.

These can be obstructed by eg nonperturbative effects.

Dine-Seiberg-Wen-Witten '86 observed that a single worldsheet instanton can generate a superpotential term obstructing def's off (2,2) locus….

… but then Silverstein-Witten '95, Candelas et al '95, Basu-Sethi '03, Beasley-Witten '03 observed that for polynomial moduli in GLSM's, the contributions of all pertinent worldsheet instantons cancel out.

— those moduli are unobstructed; math not well-understood.

Moduli w/o such a description can still be obstructed, see for example Aspinwall-Plesser '11, Braun-Kreuzer-Ovrut-Scheidegger '07

Summary:

Review of quantum sheaf cohomology

Dualities in 2d

- (0,2) mirror symmetry
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  — corresponding geometry
  — 2d tricks one can't play in 4d
- Decomposition in 2d nonabelian gauge theories
  Ex: $SU(2) = SO(3)_{+} + SO(3)_{-}$

Brief overview of moduli