A-twisted Landau-Ginzburg models, gerbes, and Kuznetsov's homological projective duality

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M. Ando, ES, arXiv: 0905.1285

J. Guffin, ES, arXiv: 0801.3836, 0803.3955

T Pantev, ES, hepth/0502027, 0502044, 0502053

S Hellerman, A Henriques, T Pantev, ES, M Ando, hepth/0606034

A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855

Outline:

* A, B topological twists of Landau-Ginzburg models
on nontrivial spaces

* Stacks in physics: how to build the QFT,
puzzles and problems w/ new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj' to LG:
physical realization of Kuznetsov's homological
projective duality,
LG for K's noncommutative resolutions

A **Landau-Ginzburg model** is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

$$S = \int_\Sigma d^2x \left(g_{i\bar\jmath} \partial\phi^i \overline{\partial}\phi^\jmath + i g_{i\bar\jmath} \psi_+^\jmath D_{\overline{z}} \psi_+^i + i g_{i\bar\jmath} \psi_-^\jmath D_z \psi_-^i + \cdots \right.$$

$$\left. + g^{i\bar\jmath} \partial_i W \partial_\jmath \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar\imath} \psi_-^{\bar\jmath} D_{\bar\imath} \partial_{\bar\jmath} \overline{W} \right.$$

The superpotential $W : X \longrightarrow \mathbf{C}$ is holomorphic, (so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....

For nonlinear sigma models (ie, LG w/ W=0), there are 2 topological twists: the A, B models.

1) A model

$$\psi_+^i \in \Gamma(\phi^*(T^{1,0}X)) \rightarrow \chi^i \qquad \psi_-^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X)) \rightarrow \chi^{\bar{\imath}}$$

$$Q \cdot \phi^i \;=\; \chi^i, \;\; Q \cdot \phi^{\bar{\imath}} \;=\; \chi^{\bar{\imath}}, \;\; Q \cdot \chi \;=\; 0, \;\; Q^2 \;=\; 0$$

Identify $\chi^\mu \sim dx^\mu \qquad Q \sim d$

States $b_{\mu \cdots \nu} \chi^\mu \cdots \chi^\nu \;\leftrightarrow\; H^{\cdot,\cdot}(X)$

2) B model

$$\psi_\pm^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X))$$

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} \qquad \theta_i = g_{i\bar{\jmath}}\left(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}}\right)$$

$$Q \cdot \phi^i = 0, \;\; Q \cdot \phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \;\; Q \cdot \eta^{\bar{\imath}} = 0, \;\; Q \cdot \theta_j = 0, \;\; Q^2 = 0$$

Identify $\eta^{\bar{\imath}} \leftrightarrow d\overline{z}^{\bar{\imath}}$ $\qquad \theta_j \leftrightarrow \frac{\partial}{\partial z^j}$ $\qquad Q \leftrightarrow \overline{\partial}$

States:

$$b_{\bar{\imath}_1 \cdots \bar{\imath}_n}^{j_1 \cdots j_m} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m} \leftrightarrow H^n\left(X, \Lambda^m TX\right)$$

We can also talk about A, B twists of LG models over nontrivial spaces....

LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

$$b(\phi)^{j_1\cdots j_m}_{\bar{\imath}_1\cdots\bar{\imath}_n}\,\eta^{\bar{\imath}_1}\cdots\eta^{\bar{\imath}_n}\theta_{j_1}\cdots\theta_{j_m}$$

where η, θ are linear comb's of ψ

$$Q\cdot\phi^i = 0, \;\; Q\cdot\phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \;\; Q\cdot\eta^{\bar{\imath}} = 0, \;\; \underline{Q\cdot\theta_j = \partial_j W}, \;\; Q^2 = 0$$

Identify $\eta^{\bar{\imath}} \leftrightarrow d\bar{z}^{\bar{\imath}}, \;\; \theta_j \leftrightarrow \dfrac{\partial}{\partial z^j}, \;\; Q \leftrightarrow \bar{\partial}$

so the states are hypercohomology

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

Quick checks:

1) W=0, standard B-twisted NLSM

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto \; H^{\cdot}\left(X, \Lambda^{\cdot} TX\right)$$ ✓

2) X=**C**ⁿ, W = quasihomogeneous polynomial

Seq' above resolves fat point {dW=0}, so

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto \; \mathbf{C}[x_1, \cdots, x_n]/(dW)$$ ✓

LG A model:

Defining the A twist of a LG model is more interesting.

(Fan, Jarvis, Ruan) (Ito; J Guffin, ES)

Producing a TFT from a NLSM involves changing what bundles the ψ couple to, e.g.

$$\psi \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* TX) \mapsto \Gamma(\Sigma, \phi^* TX), \Gamma(\Sigma, K_\Sigma \otimes \phi^* TX)$$

The two inequivalent possibilities are the A, B twists.

To be consistent, the action must remain well-defined after the twist.

True for A, B NLSM's & B LG, but not A LG....

LG A model:

The problem is terms in the action of the form

$$\psi_+^i \psi_-^j D_i \partial_j W$$

If do the standard A NLSM twist,
this becomes a 1-form on Σ,
which can't integrate over Σ .

Fix: modify the A twist.

LG A model:

There are several ways to fix the A twist,
and hence, several different notions of a LG A model.

One way: multiply offending terms in the action
by another 1-form.

Another way: use a different prescription for
modifying bundles.

The second is advantageous for physics, so I'll use it,
but,
disadvantage: not all LG models admit A twist
in this prescription.

To twist, need a U(1) isometry on X w.r.t. which the superpotential is quasi-homogeneous.

Twist by ``R-symmetry + isometry"

Let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) \,=\, \lambda W(\phi_i)$$

then twist: $\psi \,\mapsto\, \Gamma\left(\text{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K_\Sigma}^{-(1/2)Q_L}\right)$

where $$Q_{R,L}(\psi) \,=\, Q(\psi) \,+\, \left\{ \begin{array}{cc} 1 & \psi = \psi^i_+, R \\ 1 & \psi = \psi^i_-, L \\ 0 & \text{else} \end{array} \right.$$

Example: X = **C**^{n}, W quasi-homog' polynomial

Here, to twist, need to make sense of e.g. $K_\Sigma^{1/r}$

where r = 2(degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)

-- but then usually can't make sense of $K_\Sigma^{1/r}$

I'll work with the latter case.

LG A model:

A twistable example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)
with $W = p\pi^* s,\ s \in \Gamma(B, \mathcal{E})$

U(1) action acts as phases on fibers

Turns out that correlation functions in this theory match those in a NLSM on $\{s = 0\} \subset B$.

Correlation functions:

B-twist:

Integrate over X, weight by

$$\exp\left(-|dW|^2 \;+\; \text{fermionic}\right)$$

and then perform transverse Gaussian, to get the standard expression.

A-twist:

Similar: integrate over $\mathcal{M}_X$ and weight as above.

Witten equ'n in A-twist:

BRST: $\delta\psi^i_- = -\alpha\left(\overline{\partial}\phi^i - i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W}\right)$

implies localization on sol'ns of

$$\overline{\partial}\phi^i - i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W} = 0$$ (``Witten equ'n")

On complex Kahler mflds, there are 2 independent BRST operators:

$$\delta\psi^i_- = -\alpha_+\overline{\partial}\phi^i + \alpha_- i g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W}$$

which implies localization on sol'ns of

$$\begin{aligned} \overline{\partial}\phi^i &= 0 \\ g^{i\overline{\jmath}}\partial_{\overline{\jmath}}\overline{W} &= 0 \end{aligned}$$

which is what we're using.

Sol'ns of Witten equ'n:

$$\int_\Sigma \left|\overline{\partial}\phi^i - ig^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}\right|^2 = \int_\Sigma \left(\left|\overline{\partial}\phi^i\right|^2 + \left|\partial_i W\right|^2\right)$$

LHS = 0 iff RHS = 0

hence sol'ns of Witten equ'n
same as the moduli space we're looking at.

LG A model, cont'd

In prototypical cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 \, - \, \chi^p dz^i D_i s \, - \, \text{c.c.} \, - \, F_{i\overline{\jmath}} dz^i d\overline{z}^{\overline{\jmath}} \chi^p \chi^{\overline{p}}\right.}_{\text{Mathai}-\text{Quillen form}}$$

The MQ form rep's a Thom class, so

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \mathrm{Eul}(N_{\{s=0\}/\mathcal{M}}) \\ &= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \end{aligned}$$

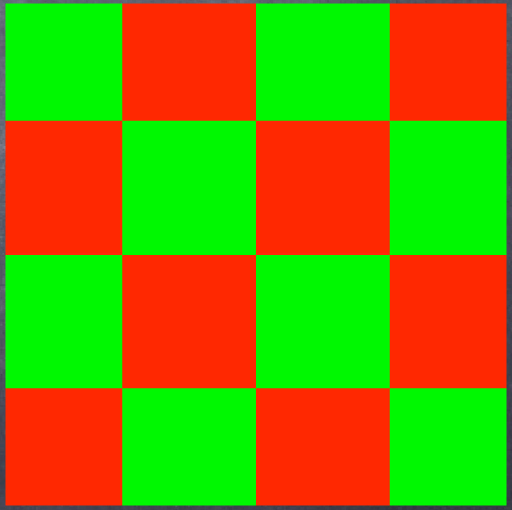
-- same as A twisted NLSM on {s=0}

Not a coincidence, as we shall see shortly....

Renormalization (semi)group flow

Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.



Problem: cannot follow it explicitly.

Renormalization group

Longer distances

Lower energies

Space of physical theories

Furthermore, RG preserves TFT’s.

If two physical theories are related by RG,
then, correlation functions in a top’ twist of one
=
correlation functions in corresponding twist of other.

Example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p s

Renormalization group flow

NLSM on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

This is why correlation functions match.

Another way to associate LG models to NLSM.

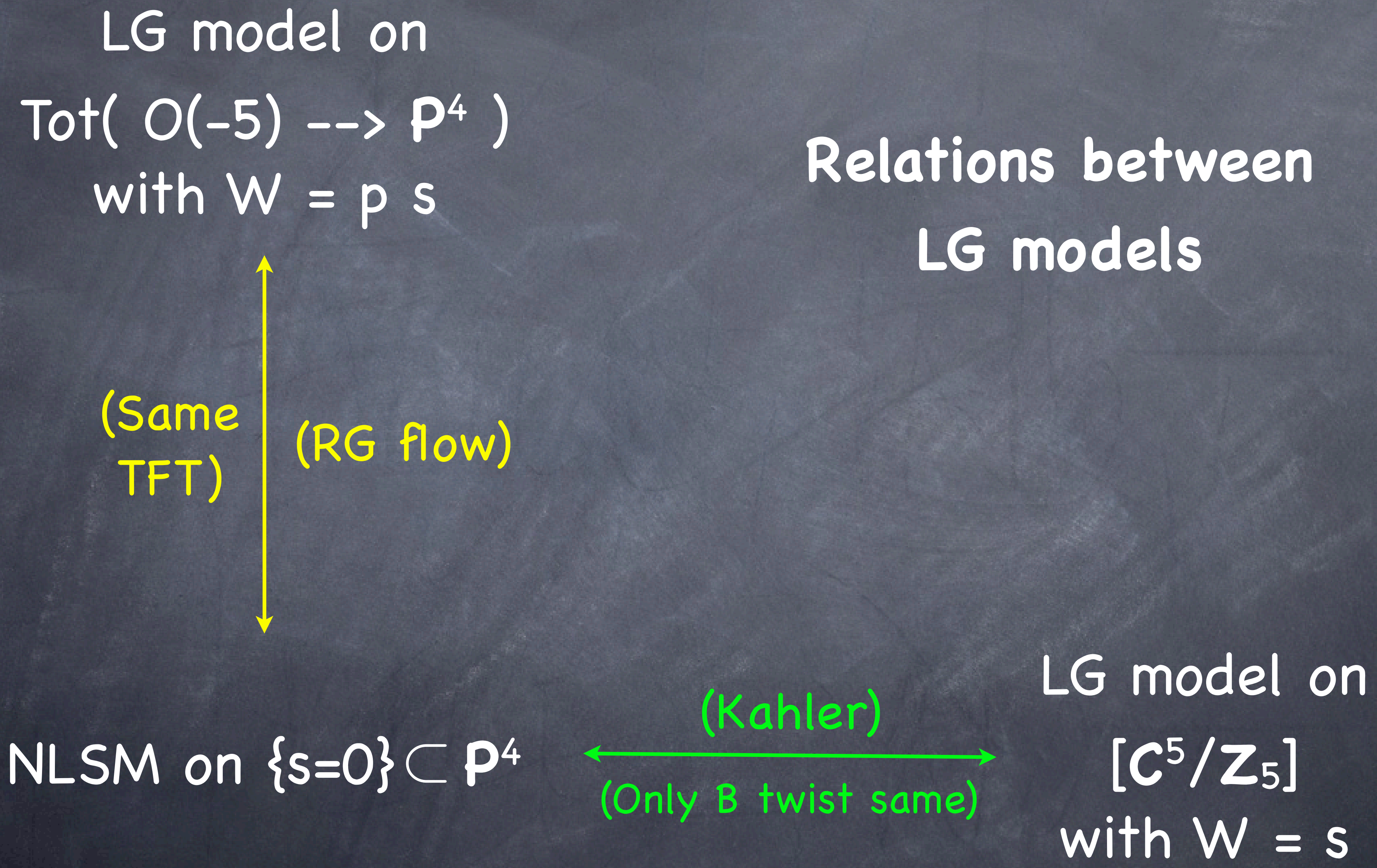
S'pose, for ex, the NLSM has target space
= hypersurface {G=0} in $\mathbf{P}^n$ of degree d

Associate LG model on $[\mathbf{C}^{n+1}/\mathbf{Z}_d]$
with $W = G$

* **Not** related by RG flow

* But, related by Kahler moduli,
so have same B model

Relations between LG models



LG model on

Tot($O(-5) \dashrightarrow \mathbf{P}^4$)

with $W = p\, s$

(Same TFT)

(RG flow)

NLSM on $\{s=0\} \subset \mathbf{P}^4$

(Kahler)

(Only B twist same)

LG model on

$[\mathbf{C}^5/\mathbf{Z}_5]$

with $W = s$

Elliptic genera:

Elliptic genus of LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch} \Bigg(\Lambda_{-1}(TB) \otimes \Lambda_{-1}(\mathcal{E}^\vee) \bigotimes_{n=1,2,3,\cdots} S_{q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\cdots} S_{q^n}((\mathcal{E}^\vee)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\cdots} \Lambda_{-q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\cdots} \Lambda_{q^n}((\mathcal{E}^\vee)^{\mathbf{C}}) \Bigg)$$

matches Witten genus of $\{s = 0\} \subset B$

by virtue of a Thom class computation.

(M Ando, ES, '09)

RG flow interpretation:

In the case of the A-twisted correlation f'ns, we got a Mathai-Quillen rep of a Thom form.

Something analogous happens in elliptic genera: elliptic genera of the LG & NLSM models are related by Thom forms.

Suggests: RG flow interpretation in twisted theories as Thom class.

(possibly from underlying Atiyah-Jeffrey, Baulieu-Singer description)

Next:

* decomposition conjecture for strings on gerbes

* LG duals to gerbes

* application of gerbes to LG's & GLSM's as,
physical realization of Kuznetsov's
homological projective duality

To do this, need to review how stacks appear in physics....

String compactifications on stacks

First, motivation:

-- new string compactifications

-- better understand certain existing string compactifications

Next: how to construct QFT's for strings propagating on stacks?

Stacks

How to make sense of strings on stacks concretely?

Most (smooth, Deligne-Mumford) stacks can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

(G need not be finite; need not act effectively.)

To such a presentation, associate a ``G-gauged sigma model on X."

Problem: such presentations not unique

Stacks

If to [X/G] we associate ``G-gauged sigma model,"
then:

$[\mathbf{C}^2/\mathbf{Z}_2]$ defines a 2d theory with a symmetry called conformal invariance

$[X/\mathbf{C}^\times]$ defines a 2d theory w/o conformal invariance

Same stack, different physics!

Potential presentation-dependence problem:
fix with renormalization group flow
(Can't be checked explicitly, though.)

The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.

Stacks

Other issues: deformation theory
massless spectra

To justify application of stacks to physics,
need to conduct tests of presentation-dependence,
understand issues above.

This was the subject of several papers.

For the rest of today's talk,
I want to focus on special kinds of stacks, namely,
gerbes.

(= quotient by noneffectively-acting group)

Gerbes

Gerbes have add'l problems when viewed from this physical perspective.

Example: The naive massless spectrum calculation contains multiple dimension zero operators, which manifestly violates cluster decomposition, one of the foundational axioms of quantum field theory.

There is a single known loophole: if the target space is disconnected. We **think** that's what's going on....

Decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

Claim

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K \right] \right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G-\mathrm{gerbe\ on\ }X) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}}(X,B)\right)$$

where the B field is determined by the image of

$$H^2(X,Z(G)) \;\overset{Z(G)\to U(1)}{\longrightarrow}\; H^2(X,U(1))$$

Checks:

* For global quotients by finite groups, can compute partition f'ns exactly at arb' genus

* Implies $K_H(X) = \text{twisted}\, K_K(X \times \hat{G})$ which can be checked independently

* Implies known facts about sheaf theory on gerbes

* Implications for Gromov-Witten theory

(Andreini, Jiang, Tseng, 0812.4477, 0905.2258, 0907.2087, and to appear)

In more detail:

global quotients by nonfinite groups

The banded $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$
with characteristic class $-1 \bmod k$
can be described mathematically as the quotient

$$\left[\frac{ \mathbf{C}^{N+1} - \{0\} }{ \mathbf{C}^{\times} } \right]$$

where the $\mathbf{C}^{\times}$ acts as rotations by k times

which physically can be described by a U(1) susy gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbf{P}^N$ model?

The difference lies in nonperturbative effects. (Perturbatively, having nonminimal charges makes no difference.)

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies
$\Phi \in \Gamma(L^{\otimes Q})$

Different bundles => different zero modes
=> different anomalies => different physics

(Noncompact worldsheet - theta angle -- J Distler, R Plesser)

Return to the example $\left[\frac{\mathbf{C}^{N+1} - \{0\}}{\mathbf{C}^{\times}} \right]$

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1} : \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$

$$\mathrm{Here} : \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1} : \quad < X^{N(d+1)-1} > = q^d$$

$$\mathrm{Here} : \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1} : \quad \mathbf{C}[x]/(x^N - q)$$

$$\mathrm{Here} : \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

Quantum cohomology

We can see the decomposition conjecture in the quantum cohomology rings of toric stacks.

Ex: Q.c. ring of a $\mathbf{Z}_k$ gerbe on $\mathbf{P}^N$ is given by

$$\mathbf{C}[x,y]/(y^k - q_2, x^{N+1} - y^n q_1)$$

In this ring, the y's index copies of the quantum cohomology ring of $\mathbf{P}^N$ with variable q's.

The gerbe is banded, so this is exactly what we expect -- copies of $\mathbf{P}^N$, variable B field.

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The LG mirror of $\mathbf{P}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbf{Z}_k$ gerbes over $\mathbf{P}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05;
E Mann, '06)

GLSM's

These are families of QFT's that RG flow to families of CFT's.

Ex:

$$\begin{tikzcd}
{} \arrow[d, "\text{RG}"] & \text{GLSM for } \mathbf{P}^7[2,2,2,2] \arrow[l, "\text{large}", "\text{radius}"'] \arrow[r, "r \ll 0"] & {} \arrow[d, "\text{RG}"'] \\
\begin{array}{c}\text{LG model on} \\ \text{Tot(} \mathcal{O}(-2)^4 \text{ --> } \mathbf{P}^7 \text{)}\end{array} \arrow[rr, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] \arrow[d] & & \begin{array}{c}\text{LG model on} \\ \text{Tot(} \mathcal{O}(-1)^8 \text{ --> } \mathbf{P}^3_{[2,2,2,2]} \text{)}\end{array} \arrow[d] \\
\text{NLSM on } \mathbf{P}^7[2,2,2,2] & & \text{?????}
\end{tikzcd}$$

GLSM's

Let's apply decomposition conjecture.

At r << 0 limit, X = Tot($O(-1)^8$ --> $\mathbf{P}^3_{[2,2,2,2]}$),
have superpotential

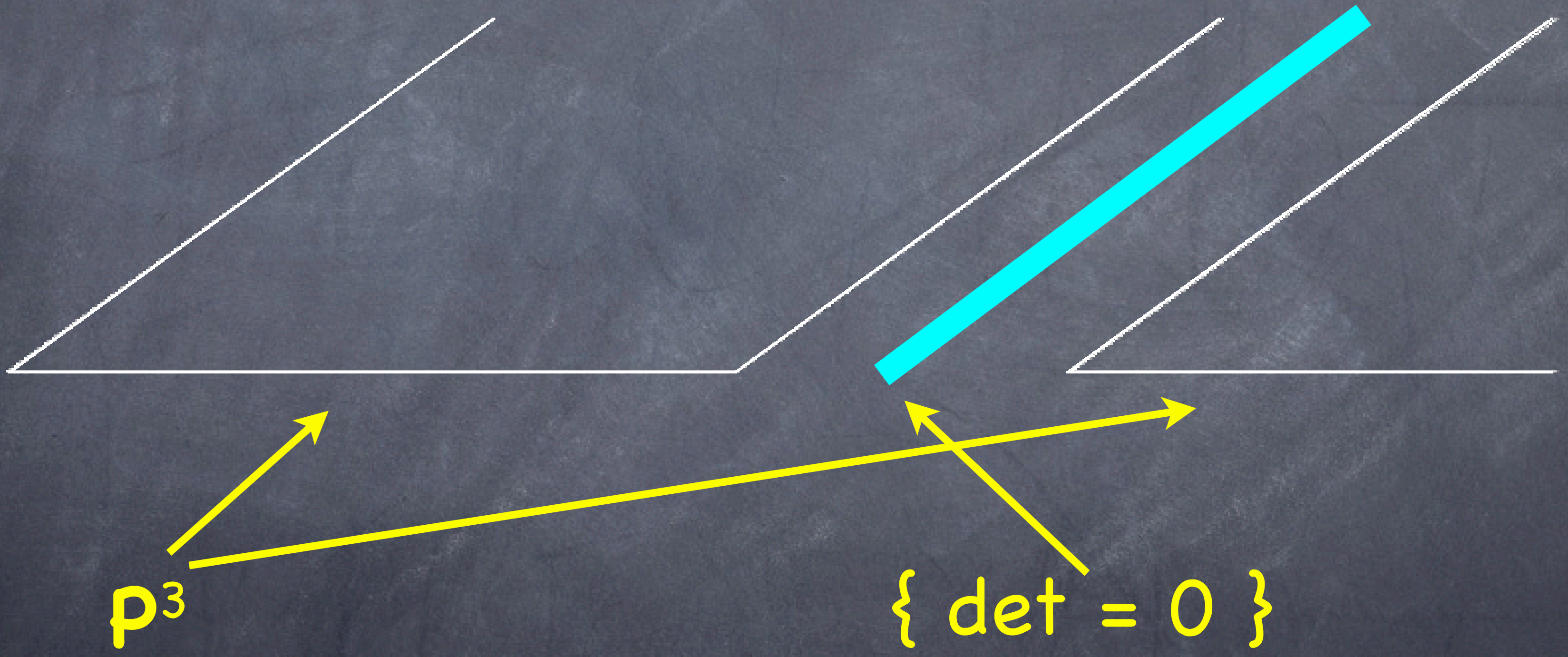
$$\sum_a p_a G_a(\phi) \; = \; \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

* leaves just the p fields, of charge 2

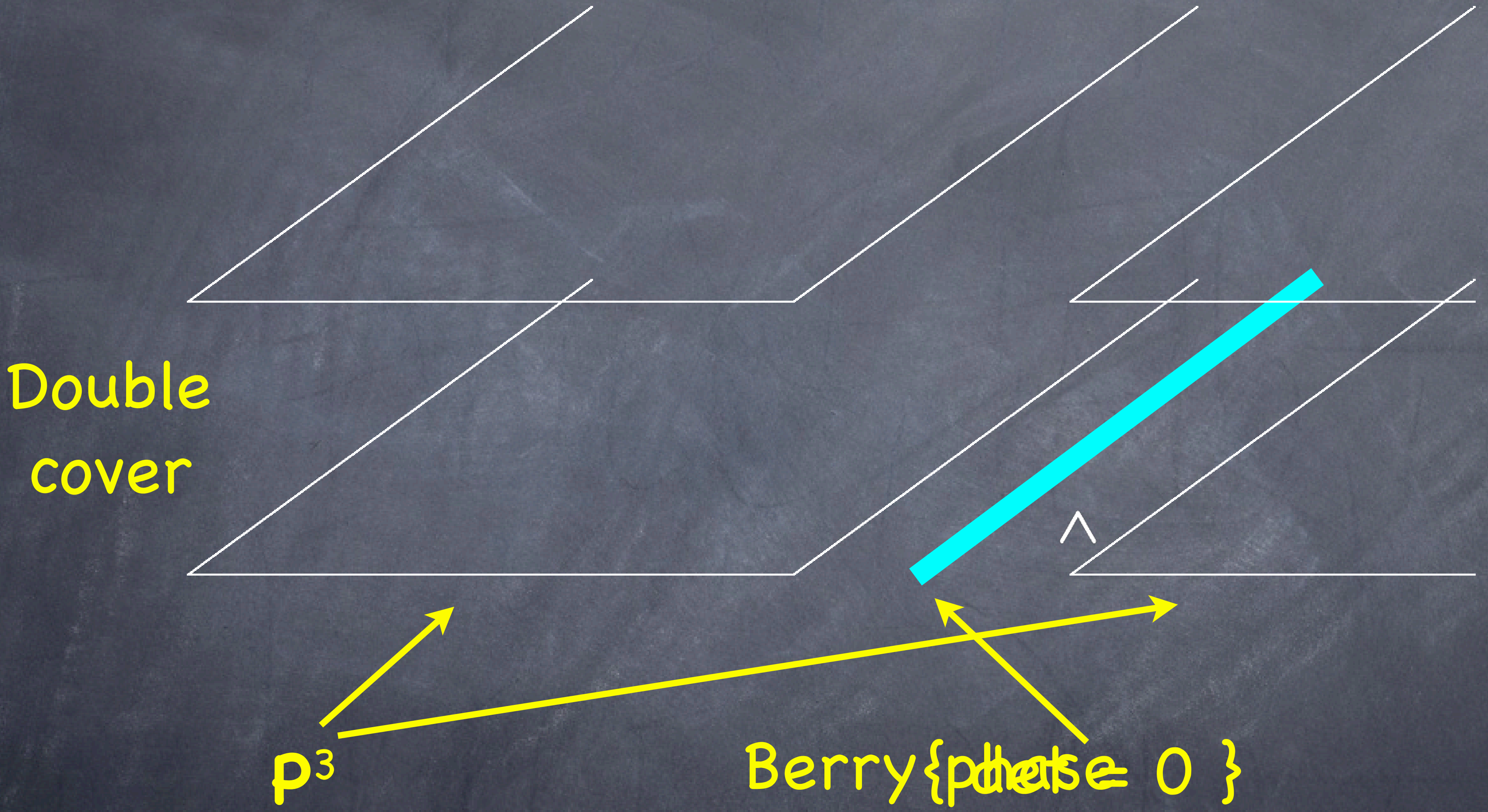
* $\mathbf{Z}_2$ gerbe, hence double cover

The $r \ll 0$ limit:



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{P}^3$ - det....

The r << 0 limit:



Result looks like branched double cover of $\mathbf{P}^3$

So far:

The GLSM seems to realize:

$\mathbf{P}^7[2,2,2,2] \xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^3$

(Clemens' octic double solid)

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence
Unusual physical realization of geometry

Rewrite:

$$\begin{tikzcd}
{} & \text{GLSM for } \mathbf{P}^7[2,2,2,2] \arrow[l, "\text{large}", "\text{radius}"'] \arrow[r, "r \ll 0"] & {} \\
{} \arrow[from=1-1, d, "\text{RG}"] \text{LG model on } \mathrm{Tot}(\,\mathcal{O}(-2)^4 \dashrightarrow \mathbf{P}^7) \arrow[rr, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] & & \text{LG model on } \mathrm{Tot}(\,\mathcal{O}(-1)^8 \dashrightarrow \mathbf{P}^3_{[2,2,2,2]}\,) \arrow[from=1-3, "\text{RG}"] \\
\text{NLSM on } \mathbf{P}^7[2,2,2,2] \arrow[from=2-1, "\text{RG}"] \arrow[rr, leftrightarrow] & & \text{NLSM on branched double cover of } \mathbf{P}^3\text{, branched over deg 8 locus} \arrow[from=2-3, "\text{RG}"]
\end{tikzcd}$$

Puzzle:

the branched double cover will be singular,
but the GLSM is smooth at those singularities.

Solution?....

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Kuznetsov has defined `homological projective duality' that relates $\mathbf{P}^7[2,2,2,2]$ to the noncommutative resolution above.

Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves -- so for example, a Landau-Ginzburg model can be a noncommutative space via matrix factorizations.

Here, K's noncomm' res'n = ($\mathbf{P}^3$,B)
where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the GLSM superpotential.

B ~ structure sheaf; other sheaves ~ B-modules.

Physics?......

Physics:

B-branes in the RG limit theory
= B-branes in the intermediate LG theory.

Claim: matrix factorizations in intermediate LG
= Kuznetsov's B-modules

K has a rigorous proof of this;
B-branes = Kuznetsov's nc res'n sheaves.

Intuition....

Local picture:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$,
gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential)
and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Summary so far:

The GLSM realizes:

$P^7[2,2,2,2] \xleftrightarrow{\text{Kahler}}$ nc res'n of branched double cover of $\mathbf{P}^3$

where RHS realized at r << 0 limit via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Unusual physical realization of geometry

Physical realization of Kuznetsov's homological projective duality

More examples:

CI of
n quadrics in $\mathbf{P}^{2n-1}$

$\xleftrightarrow{\text{Kahler}}$

branched double
cover of $\mathbf{P}^{n-1}$,
branched over deg 2n
locus

Both sides CY

Homologically projective dual

Rewrite with Landau-Ginzburg models:

$$\begin{tikzcd}
\begin{array}{c}\text{LG model on}\\ \text{Tot}(\ O(-2)^k \dashrightarrow \mathbf{P}^n)\end{array} \arrow[r, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] \arrow[d, "\text{RG}"] & \begin{array}{c}\text{LG model on}\\ \text{Tot}(\ O(-1)^{n+1} \dashrightarrow \mathbf{P}^{k-1}_{[2,\ldots,2]}\ )\end{array} \arrow[d, "\text{RG}"] \\
\begin{array}{c}\text{NLSM on}\\ \mathbf{P}^n[2,\ldots,2]\end{array} \arrow[r, leftrightarrow, "\text{Kuznetsov's}", "\text{h.p.d.}"'] & \begin{array}{c}\text{NLSM on n.c. res'n of}\\ \text{branched double cover}\\ \text{of } \mathbf{P}^{k-1},\\ \text{branched over deg n+1 locus}\end{array}
\end{tikzcd}$$

A math conjecture:

Kuznetsov defines his h.p.d. in terms of coherent sheaves. In the physics language

$$\begin{tikzcd}
\begin{array}{c}\text{LG model on}\\ \text{Tot}(\ \mathcal{O}(-2)^k \text{ --> } \mathbf{P}^n)\end{array} \arrow[r, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] & \begin{array}{c}\text{LG model on}\\ \text{Tot}(\ \mathcal{O}(-1)^{n+1} \text{ --> } \mathbf{P}^{k-1}_{[2,\dots,2]}\ )\end{array}
\end{tikzcd}$$

Kuznetsov's h.p.d. becomes a statement about matrix factorizations, analogous to those in Orlov's work.

Math conjecture: Kuznetsov's h.p.d. has an alternative (& hopefully easier) description in terms of matrix factorizations between LG models on birational spaces.

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$$\xleftrightarrow{\text{Kahler}}$$

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

* Related to an example of Vafa-Witten involving discrete torsion

(Caldararu, Borisov)

* Believed to be homologically projective dual

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved

Negatively curved

r flows: $\dashrightarrow$ $\dashrightarrow$ $\dashrightarrow$

More examples:

Hori-Tong 0609032 found closely related phenomena in nonabelian GLSMs:

$G(2,7)[1^7] \longleftrightarrow$ Pfaffian CY

Also: * novel realization of geometry
* nonbirational
* Kuznetsov's h.p.d.

Further nonabelian examples:
Donagi, ES, 0704.1761

So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way

* the geometric phases are not birational

* instead, related by Kuznetsov's homological projective duality

Conjecture: **all** phases of GLSM's are related by Kuznetsov's h.p.d.

Summary:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj' to LG & GLSM's:
physical realization of Kuznetsov's homological projective duality,
GLSM's for K's noncommutative resolutions

Mathematics		Physics
Geometry: Gromov-Witten Donaldson-Thomas quantum cohomology etc	⟺	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⟺	Renormalization group