An introduction to decomposition

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An overview of hep-th/0502027, 0502044, 0502053, 0606034, ... (many ...),
& recently arXiv: 2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423

w/ D. Robbins, T. Vandermeulen

My talk today concerns the application of **decomposition**,
a new notion in quantum field theory (QFT),
to resolution of anomalies as proposed in Wang-Wen-Witten.

Briefly, decomposition is the observation that some QFTs
are secretly equivalent to
sums of other QFTs, known as 'universes.'

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \left(\Pi_i \mathcal{O}_1 \right) \cdots \left(\Pi_i \mathcal{O}_m \right) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i Z_i = \sum_i \sum \exp(-\beta H_i)$$

(on a connected spacetime)

This reflects a (higher-form) symmetry....

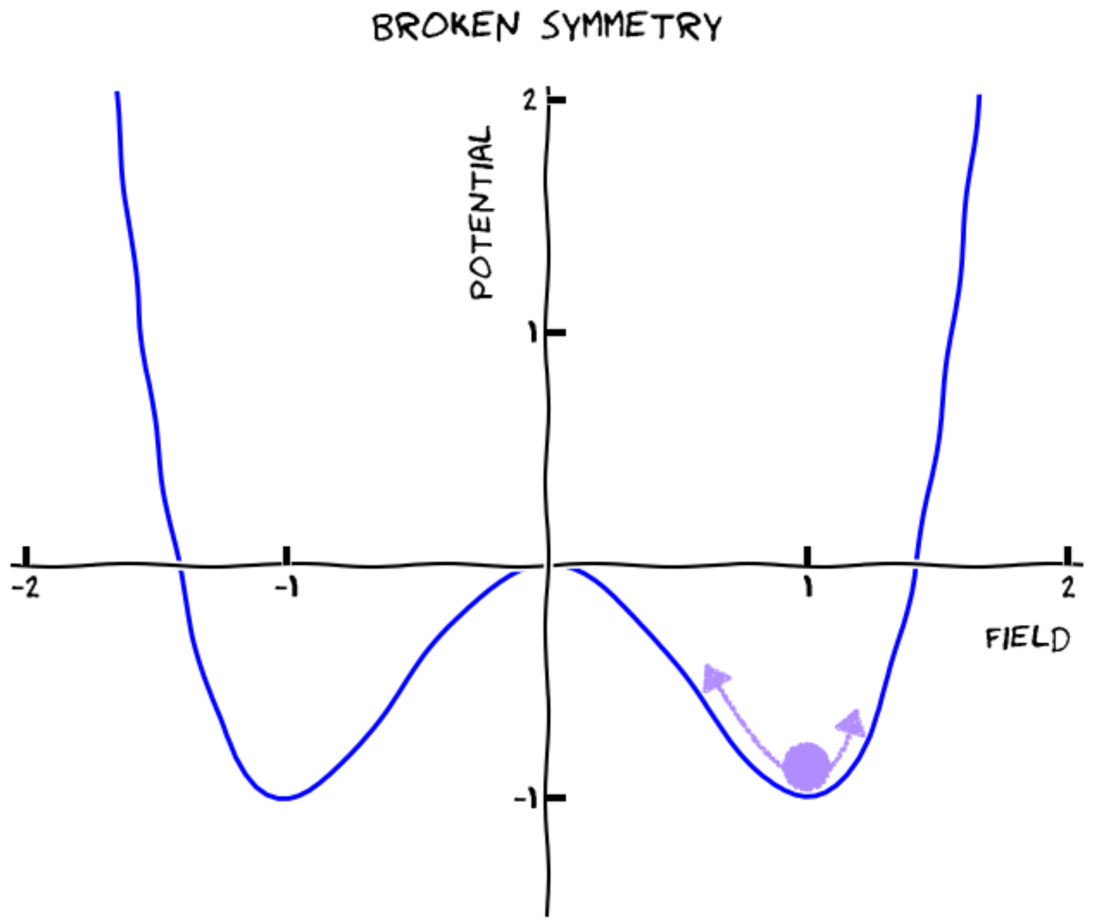
Decomposition $\neq$ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

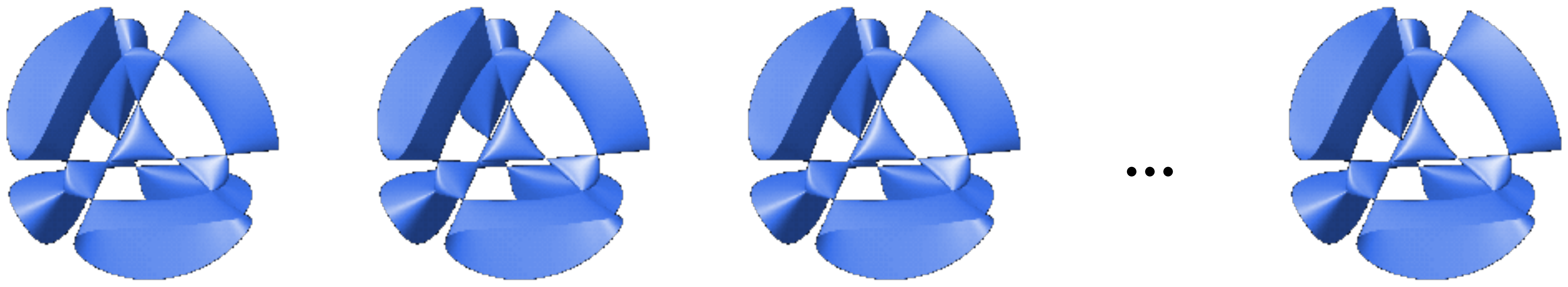


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:



(see e.g. Tanizaki-Unsal 1912.01033)

There are lots of examples of decomposition !

Orbifolds: we'll see many examples later today. (T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

(In these examples, a subgroup of the orbifold group acts trivially.)

Gauge theories:

- 2d G gauge theory w/ center-invt matter = union of $G/Z(G)$ theories w/ discrete theta (ES '14)
- 2d pure G Yang-Mills = sum of invertibles indexed by irreps of G (Nguyen, Tanizaki, Unsal '21)
- 4d gauge theory with restriction on instantons (Tanizaki, Unsal '19)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

Solves tech issue w/ cluster decomposition. (T Pantev, ES '05)

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

Examples: (Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)
- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps

(In fact, is a special case of orbifolds discussed later in this talk.)

Fun features of decomposition:

Multiverse interference effects

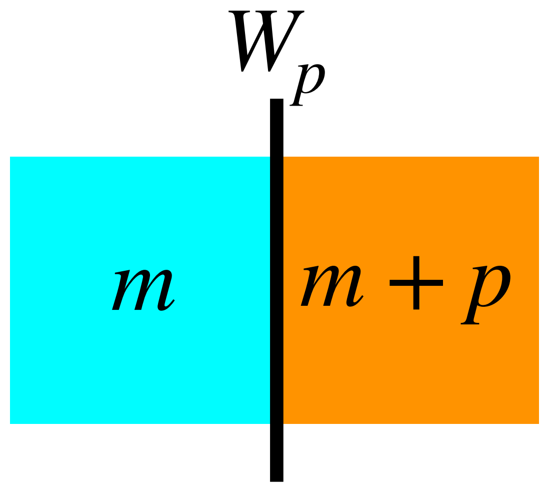
Ex: 2d $SU(2)$ gauge theory w/ center-invariant matter = $SO(3)_+ + SO(3)_-$

Summing over the two universes ($SO(3)$ gauge theories)

cancels out $SO(3)$ bundles which don't arise from $SU(2)$.

Wilson lines = defects between universes

Ex: 2d abelian BF theory at level k



Projectors: $\Pi_m = \frac{1}{k}\sum_{n=0}^{k-1} \xi^{nm}\mathcal{O}_n$ $\quad \xi = \exp(2\pi i/k)$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q O_p \quad \Leftrightarrow \quad \Pi_m W_p = W_p \Pi_{m+p \mod k}$

Wormholes between universes

Ex: U(1) susy gauge theory in 2d: 2 chirals p charge 2, 4 chirals ϕ charge -1, $W = \sum_{ij} \phi_i \phi_j A^{ij}(p)$

Describes double cover of $\mathbb{P}^1$ (sheets are universes), linked over locus where ϕ massless — Euclidean wormhole

What do the examples have in common?
When is one QFT a sum of other QFTs ?

Answer: in d spacetime dimensions,
a theory decomposes when it has a $(d-1)$-form symmetry.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

Decomposition & higher-form symmetries go hand-in-hand.

Today I'm interested in the case $d = 2$,
so get a decomposition if a $(d-1) = 1$-form symmetry is present.

What is a 1-form symmetry?

What is a (linearly realized) one-form symmetry in 2d ?

For this talk, *intuitively*, this will be a `group' that exchanges nonperturbative sectors.

Example: G gauge theory or orbifold in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume K abelian, but decompositions exist more generally.)

Then, at least for K central, nonperturbative sectors are invariant under

$$(G - \text{bundle}) \mapsto (G - \text{bundle}) \otimes (K - \text{bundle})$$

$$A \mapsto A + A'$$

At least when K central, this is the action of the `group' of K-bundles.

(Technically, is a 2-group, only weakly associative.)

That group is denoted BK or $K^{(1)}$

One-form symmetries can also be seen in algebra of topological local operators, where they are often realized *non*linearly (eg 2d TFTs).

What sort of QFTs will I look at today ? ....

The particular QFTs I'm interested in today, which have a decomposition, are (1+1)-dimensional theories with global 1-form symmetries of the following form:

(Pantev, ES '05; Hellerman et al '06)

Symmetry	
1-form	• Gauge theory or orbifold w/ trivially-acting subgroup (<-> non-complete charge spectrum)
$(d-1)$-form	• Theory w/ restriction on instantons
1-form	• Sigma models on gerbes = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$
$(d-1)$-form	• Algebra of topological local operators

Decomposition (into 'universes') often relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ...,Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies (Robbins et al '21)

Today, I'll look at application to anomalies....

Let's switch gears now.

So far, I've given a broad overview of decomposition.

Next, I'm going to discuss a specific application in orbifolds, namely to Wang-Wen-Witten's work on anomaly resolution.

Not only will this be an excellent example of a use of decomposition, but we'll also see explicitly in concrete examples how decomposition works.

My goal for the rest of this talk is to apply decomposition
to an anomaly resolution procedure in orbifolds (Wang-Wen-Witten '17).

Briefly, the idea of WWW is that if a given orbifold $[X/G]$ is ill-defined because of an anomaly (which obstructs the gauging),
then replace G with a larger group Γ whose action is anomaly-free.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

The larger group Γ has a subgroup $K \subset \Gamma$ that acts trivially on X,
and $G = \Gamma / K$.

However, orbifolds with trivially-acting subgroups are standard examples in which decomposition arises (in 1+1 dimensions), so one expects decomposition is relevant here.

(Hellerman et al '06)

Plan for the remainder of the talk:

- Describe decomposition in orbifolds with trivially-acting subgroups,
- Add a new modular invariant phase: "quantum symmetry," in $H^1(G, H^1(K, U(1)))$,
- Review the anomaly-resolution procedure of (Wang-Wen-Witten '17),
- and apply decomposition to that procedure.

What we'll find is that, in (1+1)-dimensions,

$$\mathrm{QFT}(\text{ “}\widetilde{[X/G]}\text{” } = [X/\Gamma]_B\,) \;=\; \mathrm{QFT}(\text{ copies and covers of } [X/(\text{nonanomalous subgp of } G]\,)$$

as a consequence of decomposition.
This gives a simple understanding of why the WWW procedure works,
as well as of the result.

Decomposition in orbifolds in (1+1) dimensions

Let's begin by discussing ordinary orbifolds w/o extra phases.
(We'll need a more complicated version for anomaly resolution,
but let's start here, and build up.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central) (K, Γ, G finite)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

G acts on $\hat{K}$: $\rho(k) \mapsto \rho(hkh^{-1})$ for $h \in \Gamma$ a lift of $g \in G$

$\hat{\omega}$ = phases called "discrete torsion" — see refs for details.

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

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(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

Universes (summands of decomposition)
correspond to orbits of G action on $\hat{K}$.

Projectors: For $R = \oplus_i R_i$, $R_i \in \hat{K}$ related by the action of G, we have

$$\Pi_R \;=\; \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

(Wedderburn's theorem for center of group algebra)

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

If K is in the center of Γ, then the G action on $\hat{K}$ is trivial, and decomposition specializes to

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

— a disjoint union, as many elements as $\hat{K}$

More gen'ly, get both copies and covers of $[X/G]$, as we shall see.

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\hat{\omega}}\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{\mathbb{Z}}_2}{\mathbb{Z}_2 \times \mathbb{Z}_2}\right]_{\hat{\omega}}\right)$$

$$= \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

(b/c $K = \mathbb{Z}_2$ central in $\Gamma = D_4$)

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \; = \; \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \; = \; 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} \; = \; \frac{1}{2}(1 \pm \hat{z})$$

(= specialization of formula given earlier)

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_{+} + \Pi_{-} = 1$$

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

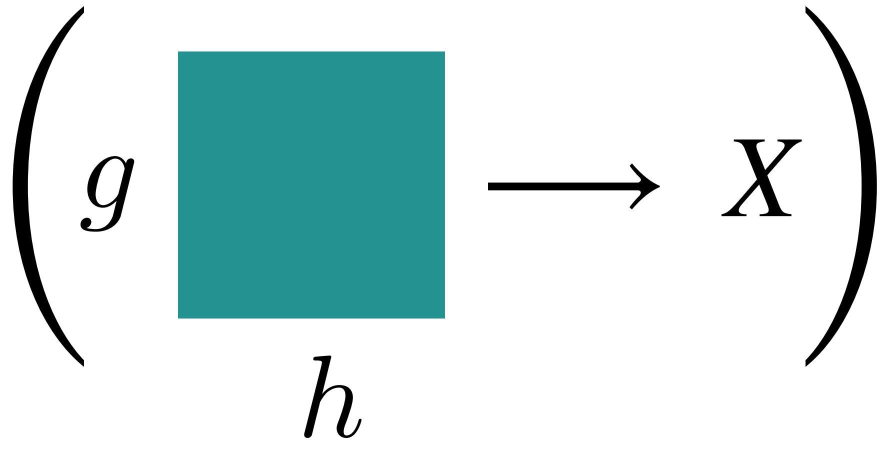
where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h}$$

where $Z_{g,h} =$



("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

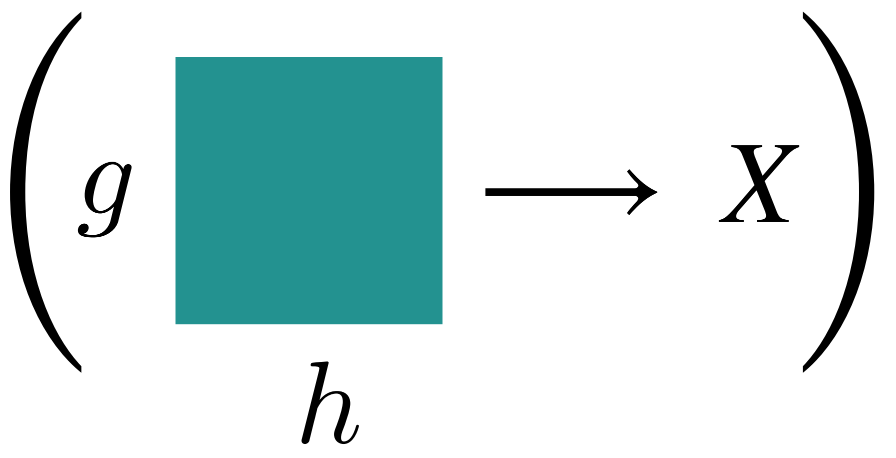
$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

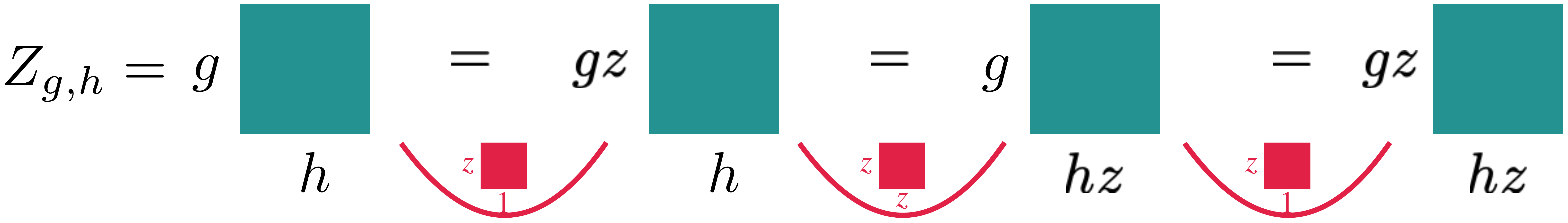
$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{1}{|D_4|} \sum_{g,h \in D_4,\; gh=hg} Z_{g,h}$$

where $Z_{g,h} =$



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{1}{|D_4|} \sum_{gh \in D_4, gh=hg} Z_{g,h}$$ where $Z_{g,h} \;=\; \left(g \underset{h}{\blacksquare} \longrightarrow X \right)$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a} \underset{\overline{b}}{\blacksquare}$ $\quad \overline{a} \underset{\overline{ab}}{\blacksquare}$ $\quad \overline{b} \underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; \left(\text{some twisted sectors}\right)\right)$$

$$= \; 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $\quad Z_{T^2}([X/G]) \;=\; \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' \;=\; \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h)\, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

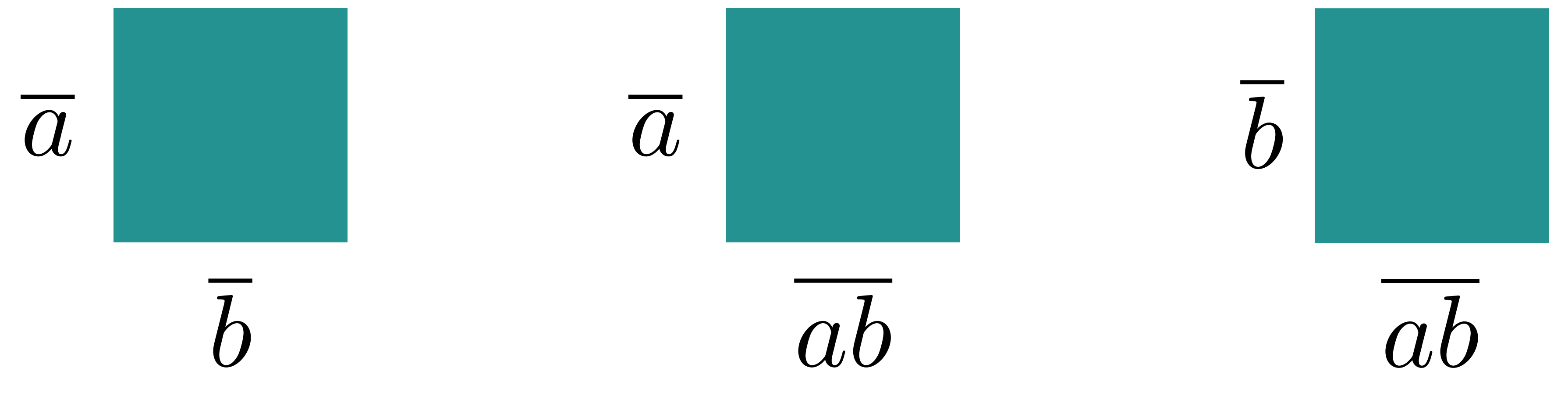
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, discrete torsion $\in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

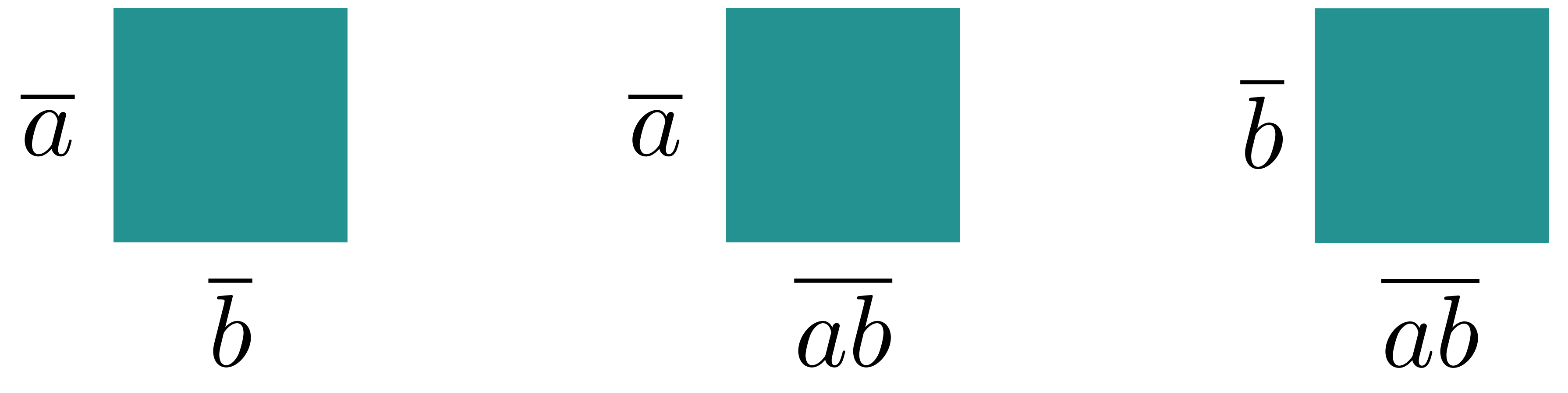
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) = \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \; = \; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \; + \; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\mathrm{QFT}\left([X/D_4]\right) \; = \; \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

Example, cont'd

Massless spectra for $X = T^6$

(T Pantev, ES '05)

Massless spectrum of D_4 orbifold

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad + \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/ d.t.

matching the prediction of decomposition

$$
\mathrm{CFT}\left([X/D_4]\right) \; = \; \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)
$$

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H} =$ eight-element gp of unit quaternions, where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

where $\hat{K} =$ irreps of K

$\hat{\omega} =$ discrete torsion on universes

Here, $G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

so $$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

(Hellerman et al, hep-th/0606034, sect. 5.4)

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

So far I've outlined how decomposition works in orbifolds $[X/\Gamma]$,
with trivially-acting $K \subset \Gamma$,
and no discrete torsion or other phase modifications (in the Γ orbifold).

However, in order to apply this to anomaly resolution,
we're going to need to understand decomposition in orbifolds
modified by (modular-invariant) phases.

Next: decomposition in orbifolds $[X/\Gamma]_\omega$ with discrete torsion $\omega \in H^2(\Gamma, U(1))$....

Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\,\iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) = \mathrm{Hom}(G, \hat{K})$$

Cases:

1) If $\iota^*\omega \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{G}\right]_{\hat{\omega}}\right)$$

2) If $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right)$$

Checked in numerous examples

3) If $\iota^*\omega = 0$ and $\beta(\omega) = 0$, then $\omega = \pi^*\overline{\omega}$ for $\overline{\omega} \in H^2(G, U(1))$ and

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\overline{\omega}+\hat{\omega}}\right)$$

Let's get back on track.

My goal today is to talk about anomaly resolution in 1+1 dimensions.

Decomposition will play a vital role in understanding how the anomalies are resolved.

Recall the idea of www is that given an anomalous (ill-defined) $[X/G]$, replace G by a larger finite group Γ obeying certain properties,

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

and add phases.

Because Γ has a subgroup K that acts trivially, orbifolds $[X/\Gamma]$ will decompose, into copies & covers of $[X/G]$.

However, just getting copies of $[X/G]$ won't help.
We also need to add certain new phases, which I will describe next....

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.
Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

It acts on twisted sector states by phases. Schematically:

$$gz \underset{h}{\blacksquare} \; = \; B(\pi(h), z) \left(g \underset{h}{\blacksquare} \right)$$

where
$z \in K \qquad g, h \in \Gamma$
$B \in H^1(G, H^1(K, U(1)))$

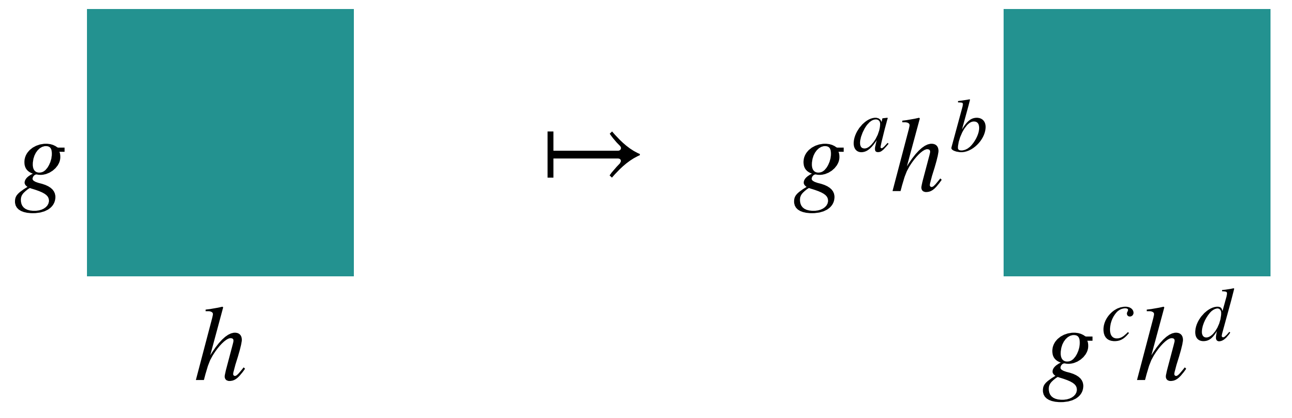
These generalize the old notion of `quantum symmetries' in the orbifolds literature;
those old quantum symmetries were determined by discrete torsion,
but the ones we need for anomaly resolution, aren't....

New modular invariant phases: quantum symmetries

These are modular invariant — analogous to (but different from) discrete torsion.

Work on T^2. Geometrically, this admits `Dehn twists'

Under such a twist,



for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$

Discrete torsion: $\epsilon(g^a h^b, g^c h^d) = \epsilon(g,h)$

Quantum symmetry: $\displaystyle\sum_{k_1,k_2\in K} \epsilon(g^a k_1^a h^b k_2^b, g^c k_1^c h^d k_2^d) = \sum_{k_1,k_2\in K} \epsilon(g k_1, h k_2)$

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

Those quantum symmetries in the image of β are equivalent to discrete torsion:

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

(Hochschild '77)

Specifically, $\beta(\omega) \in H^1(G, H^1(K, U(1)))$ for $\omega \in H^2(\Gamma, U(1))$ s.t. $\omega|_K = 0$.

Example: old-fashioned quantum symmetry in orbifolds

Start with $[X/\mathbb{Z}_n]$. Old story: This admits a $\mathbb{Z}_n$ symmetry that acts on twist fields, with the property that $\mathrm{QFT}([[X/\mathbb{Z}_n]/\mathbb{Z}_n]) \; = \; \mathrm{QFT}([X/\mathbb{Z}_n \times \mathbb{Z}_n]_B) = \mathrm{QFT}(X)$

However, the phases are determined by discrete torsion; $B = \beta(\omega)$

(and rel'n to X is a special case of decomposition)

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.

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(Hochschild '77)

For purposes of resolving anomalies, we need $B \in H^1(G, H^1(K, U(1)))$ such that $d_2 B \neq 0$.

These cases are *not* in im β, so *not* determined by discrete torsion $\omega \in H^2(\Gamma, U(1))$.

They're also of independent interest, beyond anomaly resolution.

How does decomposition work with such phases?....

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

$$\text{where } B \;\in\; H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$$

This is more or less uniquely determined by consistency with previous results.

Recall for discrete torsion $\omega \in \mathrm{Ker}\;\iota^* \subset H^2(\Gamma, U(1))$, with $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right)$$

The result at top needs to include this as a special case, and it does.

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times \widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

Example: $\Gamma = \mathbb{Z}_4, \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Pick nontrivial $B \in H^1(G, H^1(K, U(1))) = H^1(\mathbb{Z}_2, \hat{\mathbb{Z}}_2) = \mathbb{Z}_2.$

$$\mathrm{Ker}\,B = 0, \;\; \mathrm{Coker}\,B = 0$$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\,(X)$

Check in partition function....

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B}\right]_{\hat{\omega}}\right)$$

Example: $\Gamma = \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) = \mathrm{QFT}\,(X)$

Check T^2 partition function:

$$Z_{ij} = (-)^i Z_{i,j-2} = (-)^j Z_{i-2,j}$$

$$Z\left([X/\mathbb{Z}_4]_B\right) = \frac{1}{|\mathbb{Z}_4|}\sum_{i,j=0}^{4} Z_{ij} = \frac{1}{4}\left(Z_{00} + Z_{02} + Z_{20} + Z_{22}\right) = Z_{00} = Z(X)$$

Works!

Decomposition in the presence of a quantum symmetry

If there is also discrete torsion $\omega \in H^2(\Gamma, U(1))$:

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1$$

Assume for simplicity $\iota^* \omega = 0$.

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1)) \right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

Cases:

1) Suppose $\beta(\omega) \neq 0$:

$$\mathrm{QFT}\left([X/\Gamma]_{B,\omega} \right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,(B/\beta(\omega))}}{\mathrm{Ker}\,(B/\beta(\omega))} \right]_{\hat{\omega}} \right)$$

2) Suppose $\omega = \pi^* \overline{\omega}, \;\; \overline{\omega} \in H^2(G, U(1))$:

$$\mathrm{QFT}\left([X/\Gamma]_{B,\omega} \right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\overline{\omega} + \hat{\omega}} \right)$$

All checked in examples;
I'll spare you....

Now, finally, we have the tools to start applying to anomalies.

For the purposes of this talk, anomalies in a finite G gauge theory in $(n+1)$ dimensions will be classified by $H^{n+2}(G, U(1))$.

This arises from considerations of `topological defect lines.'
On the next slide I'll outline how that works in the case $n = 0$.

Then, I'll outline how anomaly resolution in (1+1) dimensions can be understood via decomposition.

Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions
- why are anomalies are associated to group cohomology?

Suppose a (finite) group G acts on the states of a QM system: For any $|\psi\rangle$, get $\rho(g)|\psi\rangle$.

For an honest group action, require $\rho(g)\rho(h) = \rho(gh)$

However, b/c we only care about states up to phases, we might instead have

$\rho(g)\rho(h) = \omega(g,h)\,\rho(gh)$ for some $\omega(g,h) \in U(1)$

Associativity $\Rightarrow$ $\omega(g_2, g_3)\,\omega(g_1, g_2 g_3) = \omega(g_1 g_2, g_3)\,\omega(g_1, g_2)$ (coclosed)

Multiply ρ by phase $\epsilon(g) \Rightarrow$ $\omega(g,h) \mapsto \omega(g,h)\dfrac{\epsilon(gh)}{\epsilon(g)\epsilon(h)}$ (coboundaries)

Thus, the obstructions ω are classified by $H^2(G, U(1))$

Anomaly in 0+1 dims

States are all in ω-projective representations of G.

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (I'll assume central)

where Γ is chosen so that $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

The idea is then to replace $[X/G]$ with $[X/\Gamma]$,
but, need to describe how Γ acts on X.

If K acts triv'ly on X, and we do nothing else,
then we have accomplished nothing:

decomposition $\Rightarrow$ $\quad \mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\hat{K}} \mathrm{QFT}\left([X/G]\right)$ — still anomalous

Fix by adding quantum symmetry....

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (assumed central)

2) Turn on quantum symmetry $B \in H^1(G, H^1(K, U(1)))$
chosen so that $d_2 B = \alpha$. This implies $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

K acts trivially on X, but nontrivially on twisted sector states via B

These two together — extension Γ plus B — resolve anomaly.

Decomposition explains how....

Application to anomaly resolution

Procedure: replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$

where $d_2 B \;=\; \alpha \in H^3(G, U(1))$, the anomaly of the G orbifold.

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\hat{\omega}} \right)$$

— using earlier results for decomp' in orb' w/ quantum symmetry

Note that since $d_2 B = \alpha$, $\quad \alpha|_{\mathrm{Ker}\, B} = 0$

So, $\mathrm{Ker}\, B \subset G$ is automatically anomaly-free!

Summary: $[X/\Gamma]_B =$ copies of orbifold by anomaly-free subgroup.

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 1: Define $\Gamma = D_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	w/o d.t. in D4	w/ d.t. in D4
1	1	—	$[X/G] \coprod [X/G]_{\mathrm{dt}}$	$[X/\langle b \rangle]$
-1	1	—	$[X/\langle b \rangle]$	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
1	-1	$\langle b \rangle$	$[X/\langle a \rangle]$	$[X/\langle ab \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle a \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 2: Define $\Gamma = \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	Result
1	1	—	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
-1	1	$\langle a \rangle, \langle ab \rangle$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 3: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	w/o d.t. in Z2 x Z4	w/ d.t. in Z2 x Z4
1	1	—	$[X/G] \coprod [X/G]$	$[X/G]_{\rm dt} \coprod [X/G]_{\rm dt}$
-1	1	$\langle ab \rangle$	$[X/\langle b \rangle]$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$	$[X/\langle a \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

In the examples so far, we picked a `minimal' resolution Γ.
If we pick larger K, we get copies.

Extension 4: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Results:

$B(a)$	$B(b)$	**d_2(B) (anomaly)**	**Result**
1	1	—	$\coprod_2 \left([X/G] \coprod [X/G]_{\mathrm{dt}} \right)$
-1	1	$\langle a \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$\coprod_2 [X/\langle ab \rangle]$

Get copies of orb's w/ anomaly-free subgroups.

Works!

Summary

Decomposition: `one' QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories
with $n-$form symmetries.

(I've focused on examples in 1+1d,
but examples exist in other dim's too.)

Can be used to understand anomaly-resolution procedure of www:

replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$,
but decomposition implies

$$\text{QFT}\left([X/\Gamma]_B\right) = \text{copies of QFT}\left([X/\text{Ker}\, B \subset G]\right),$$

which is explicitly non-anomalous.

Thank you for your time!