Application of decomposition to anomaly resolution

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w/ D. Robbins, T. Vandermeulen

My talk today concerns the application of **decomposition**,
a new notion in quantum field theory (QFT),
to resolution of anomalies as proposed in Wang-Wen-Witten.

Briefly, decomposition is the observation that some QFTs
are secretly equivalent to
sums of other QFTs, known as 'universes.'

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \left(\Pi_i \mathcal{O}_1 \right) \cdots \left(\Pi_i \mathcal{O}_m \right) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i Z_i = \sum_i \sum \exp(-\beta H_i)$$

(on a connected spacetime)

This reflects a (higher-form) symmetry....

When is one QFT a sum of other QFTs ?

Answer: in d spacetime dimensions, when it has a $(d-1)$-form symmetry.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

Decomposition & higher-form symmetries go hand-in-hand.

Today I'm interested in the case $d = 2$,
so get a decomposition if a $(d-1) = 1$-form symmetry is present.

1-form symmetries arise in
e.g. gauge theories, orbifolds in which a subgroup of the gauge group acts trivially
(<-> incomplete charge lattice).

So, expect 2d theories of that form to decompose.

What is a 1-form symmetry?

What is a one-form symmetry?

For this talk, *intuitively*, this will be a `group' that exchanges nonperturbative sectors.

Example: G gauge theory or orbifold in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume K abelian,
but decompositions exist more generally.)

Then, at least for K central, nonperturbative sectors are invariant under

$$(G - \text{bundle}) \mapsto (G - \text{bundle}) \otimes (K - \text{bundle})$$
$$A \mapsto A + A'$$

At least when K central, this is the action of the `group' of K-bundles.
That group is denoted BK or $K^{(1)}$

(Technically, is a 2-group, only weakly associative.)

One-form symmetries can also be seen in algebra of topological local operators.

What sort of QFTs will I look at today ? ....

The QFTs I'm interested in, which have a decomposition,
are (1+1)-dimensional theories with global 1-form symmetries,
and can be described in several ways, such as

(Pantev, ES '05; Hellerman et al '06)

Symmetry

- 1-form — Gauge theory or orbifold w/ trivially-acting subgroup (<-> non-complete charge spectrum)
- $(d-1)$-form — Theory w/ restriction on instantons
- 1-form — Sigma models on gerbes = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$
- $(d-1)$-form — Algebra of topological local operators

Decomposition (into 'universes') relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Decomposition in (1+1)-d gauge theories

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (Honda et al '21)
- Plus version for (3+1)d theories w/ 3-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to anyone not listed.

Applications include:

- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ..., Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies (Robbins et al '21)

Today, I'll look at application to anomalies....

Decomposition in (1+1)-d gauge theories

My goal today is to apply decomposition to an anomaly resolution procedure in finite gauge theories (Wang-Wen-Witten '17), of which my go-to examples are orbifolds.

Briefly, the idea of WWW is that if a given orbifold $[X/G]$ is ill-defined because of an anomaly (which obstructs the gauging), then replace G with a larger group Γ whose action is anomaly-free.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

The larger group Γ has a subgroup $K \subset \Gamma$ that acts trivially on X, and $G = \Gamma / K$.

However, orbifolds with trivially-acting subgroups are standard examples in which decomposition arises (in 1+1 dimensions), so one expects decomposition is relevant here.

(Hellerman et al '06)

Plan for the remainder of the talk:

- Describe decomposition in orbifolds with trivially-acting subgroups,
- Add a new modular invariant phase: "quantum symmetry," in $H^1(G, H^1(K, U(1)))$,
- Review the anomaly-resolution procedure of (Wang-Wen-Witten '17),
- and apply decomposition to that procedure.

What we'll find is that, in (1+1)-dimensions,

$$\text{QFT}\left(\text{“}\widetilde{[X/G]}\text{”} = [X/\Gamma]_B \right) \; = \; \text{QFT}\left(\text{copies and covers of } [X/(\text{nonanomalous subgp of } G] \right)$$

as a consequence of decomposition.
This gives a simple understanding of why the WWW procedure works,
as well as of the result.

Decomposition in orbifolds in (1+1) dimensions

Let's begin by discussing ordinary orbifolds w/o extra phases.
(We'll need a more complicated version for anomaly resolution,
but let's start here, and build up.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central) (K, Γ, G finite)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

G acts on $\hat{K}$: $\rho(k) \mapsto \rho(hkh^{-1})$ for $h \in \Gamma$ a lift of $g \in G$

$\hat{\omega}$ = phases called "discrete torsion" — see refs for details.

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

Universes (summands of decomposition) correspond to orbits of G action on $\hat{K}$.

Projectors: For $R = \oplus_i R_i$, $R_i \in \hat{K}$ related by the action of G, we have

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

(Wedderburn's theorem for center of group algebra)

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

Decomposition implies

$$\mathrm{QFT}([X/\Gamma]) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

If K is in the center of Γ, then the G action on $\hat{K}$ is trivial, and decomposition specializes to

$$\mathrm{QFT}([X/\Gamma]) \; = \; \mathrm{QFT}\left(\coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

— a disjoint union, as many elements as $\hat{K}$

More gen'ly, get both copies and covers of $[X/G]$, as we shall see.

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\hat{\omega}}\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{\mathbb{Z}}_2}{\mathbb{Z}_2 \times \mathbb{Z}_2}\right]_{\hat{\omega}}\right)$$

$$= \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

(b/c $K = \mathbb{Z}_2$ central in $\Gamma = D_4$)

Let's check this explicitly....

Example, cont'd

$$\text{QFT}\left([X/D_4]\right) \; = \; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \; = \; 1.$

Using that relation, we form projection operators:

$$\Pi_\pm \; = \; \frac{1}{2}(1 \pm \hat{z})$$

(= specialization of formula given earlier)

$$\Pi_\pm^2 = \Pi_\pm \qquad \Pi_\pm \Pi_\mp = 0 \qquad \Pi_+ + \Pi_- = 1$$

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h}$$

where $Z_{g,h} = \Big(g$  $\longrightarrow X \Big)$

h

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

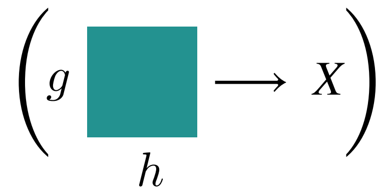
$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

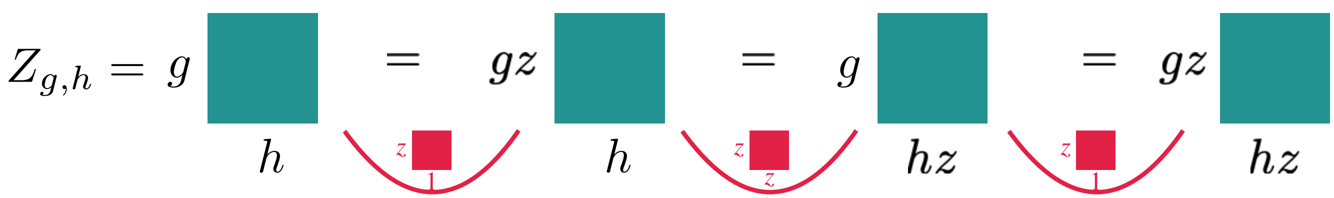
$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$

where $Z_{g,h}$ =



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\} \quad \text{where } \overline{a} = \{a, az\} \text{ etc}$$

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{gh \in D_4, gh=hg} Z_{g,h}$$

where $Z_{g,h} = \Big(g$  $\longrightarrow X \Big)$, with h below the square

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a}$  $\overline{b}$, $\overline{a}$  $\overline{ab}$, $\overline{b}$  $\overline{ab}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h) Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

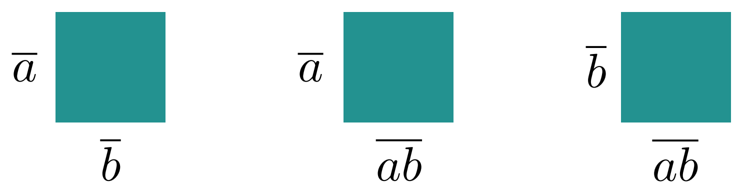
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - (\text{some twisted sectors})\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - (\text{some twisted sectors})\right)$$

In a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold, discrete torsion $\in H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

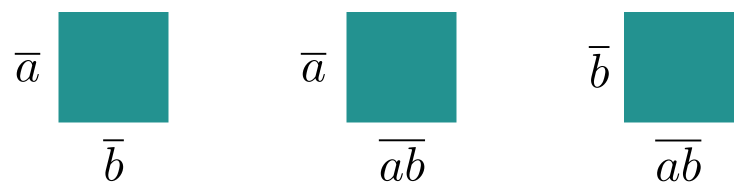
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

$$= \; 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \; = \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \; + \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \; = \; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2 has the form predicted by decomposition.
The same is also true of partition functions at higher genus — just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts, which mostly I'll suppress in this talk.

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions, where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

where $\hat{K}$ = irreps of K
$\hat{\omega}$ = discrete torsion on universes

Here, $G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

so $$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

(Hellerman et al, hep-th/0606034, sect. 5.4)

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

So far I've outlined how decomposition works in orbifolds $[X/\Gamma]$,
with trivially-acting $K \subset \Gamma$,
and no discrete torsion or other phase modifications (in the Γ orbifold).

However, in order to apply this to anomaly resolution,
we're going to need to understand decomposition in orbifolds
modified by (modular-invariant) phases.

Next: decomposition in orbifolds $[X/\Gamma]_\omega$ with discrete torsion $\omega \in H^2(\Gamma, U(1))$....

Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) = \mathrm{Hom}(G, \hat{K})$$

Cases:

1) If $\iota^*\omega \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{G}\right]_{\hat{\omega}}\right)$$

2) If $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right)$$

3) If $\iota^*\omega = 0$ and $\beta(\omega) = 0$, then $\omega = \pi^*\overline{\omega}$ for $\overline{\omega} \in H^2(G, U(1))$ and

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\overline{\omega}+\hat{\omega}}\right)$$

Checked in numerous examples

Let's get back on track.

My goal today is to talk about anomaly resolution in 1+1 dimensions.

Decomposition will play a vital role in understanding how the anomalies are resolved.

Recall the idea of www is that given an anomalous (ill-defined) $[X/G]$,
replace G by a larger finite group Γ obeying certain properties,

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

and add phases.

Because Γ has a subgroup K that acts trivially,
orbifolds $[X/\Gamma]$ will decompose,
into copies & covers of $[X/G]$.

However, just getting copies of $[X/G]$ won't help.
We also need to add certain new phases, which I will describe next....

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

It acts on twisted sector states by phases. Schematically:

$$gz \, \underset{h}{\blacksquare} \; = \; B(\pi(h), z) \left(g \, \underset{h}{\blacksquare} \right)$$

where
$z \in K \qquad g, h \in \Gamma$
$B \in H^1(G, H^1(K, U(1)))$

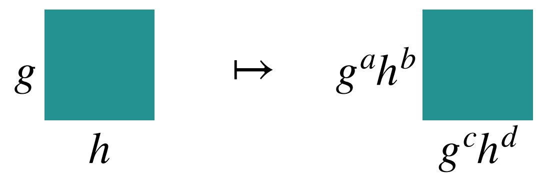
These generalize the old notion of `quantum symmetries' in the orbifolds literature; those old quantum symmetries were determined by discrete torsion, but the ones we need for anomaly resolution, aren't....

New modular invariant phases: quantum symmetries

These are modular invariant — analogous to (but different from) discrete torsion.

Work on T^2. Geometrically, this admits `Dehn twists'

Under such a twist,



for $\left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL(2,\mathbb{Z})$

Discrete torsion: $\epsilon(g^a h^b, g^c h^d) \;=\; \epsilon(g,h)$

Quantum symmetry: $\sum_{k_1,k_2 \in K} \epsilon(g^a k_1^a h^b k_2^b, g^c k_1^c h^d k_2^d) \;=\; \sum_{k_1,k_2 \in K} \epsilon(g k_1, h k_2)$

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

Those quantum symmetries in the image of β are equivalent to discrete torsion:

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

(Hochschild '77)

Specifically, $\beta(\omega) \in H^1(G, H^1(K, U(1)))$ for $\omega \in H^2(\Gamma, U(1))$ s.t. $\omega|_K = 0$.

Example: old-fashioned quantum symmetry in orbifolds

Start with $[X/\mathbb{Z}_n]$. Old story: This admits a $\mathbb{Z}_n$ symmetry that acts on twist fields, with the property that $\mathrm{QFT}([[X/\mathbb{Z}_n]/\mathbb{Z}_n]) \; = \; \mathrm{QFT}([X/\mathbb{Z}_n \times \mathbb{Z}_n]_B) = \mathrm{QFT}(X)$

However, the phases are determined by discrete torsion; $B = \beta(\omega)$

(and rel'n to X is a special case of decomposition)

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.
Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

Those quantum symmetries in the image of β are equivalent to discrete torsion:

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

(Hochschild '77)

For purposes of resolving anomalies,
we need $B \in H^1(G, H^1(K, U(1)))$ such that $d_2 B \neq 0$.

These cases are *not* in im β, so *not* determined by discrete torsion $\omega \in H^2(\Gamma, U(1))$.

They're also of independent interest, beyond anomaly resolution.

How does decomposition work with such phases?....

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B}\right]_{\hat{\omega}}\right)$$

$$\text{where } B \;\in\; H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$$

This is more or less uniquely determined by consistency with previous results.

Recall for discrete torsion $\omega \in \mathrm{Ker}\ \iota^* \subset H^2(\Gamma, U(1))$, with $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times \widehat{\mathrm{Coker}\, \beta(\omega)}}{\mathrm{Ker}\, \beta(\omega)}\right]_{\hat{\omega}}\right)$$

The result at top needs to include this as a special case, and it does.

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\hat{\omega}} \right)$$

Example: $\Gamma = \mathbb{Z}_4, \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Pick nontrivial $B \in H^1(G, H^1(K, U(1))) = H^1(\mathbb{Z}_2, \hat{\mathbb{Z}}_2) = \mathbb{Z}_2.$

$$\mathrm{Ker}\, B = 0, \;\; \mathrm{Coker}\, B = 0$$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\,(X)$

Check in partition function....

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

Example: $\Gamma = \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\,(X)$

Check T^2 partition function:

$$Z_{ij} \;=\; (-)^i Z_{i,j-2} \;=\; (-)^j Z_{i-2,j}$$

$$Z\left([X/\mathbb{Z}_4]_B\right) \;=\; \frac{1}{|\mathbb{Z}_4|}\sum_{i,j=0}^{4} Z_{ij} \;=\; \frac{1}{4}\left(Z_{00}+Z_{02}+Z_{20}+Z_{22}\right) \;=\; Z_{00} \;=\; Z(X)$$

Works!

Decomposition in the presence of a quantum symmetry

If there is also discrete torsion $\omega \in H^2(\Gamma, U(1))$:

$$1 \longrightarrow K \stackrel{\iota}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$$

Assume for simplicity $\iota^* \omega = 0$.

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \stackrel{\beta}{\longrightarrow} H^1(G, H^1(K, U(1))) \stackrel{d_2}{\longrightarrow} H^3(G, U(1))$$

Cases:

1) Suppose $\beta(\omega) \neq 0$:

$$\mathrm{QFT}\left([X/\Gamma]_{B,\omega}\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,(B/\beta(\omega))}}{\mathrm{Ker}\,(B/\beta(\omega))} \right]_{\hat{\omega}} \right)$$

2) Suppose $\omega = \pi^* \overline{\omega}$, $\overline{\omega} \in H^2(G, U(1))$:

$$\mathrm{QFT}\left([X/\Gamma]_{B,\omega}\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\overline{\omega} + \hat{\omega}} \right)$$

All checked in examples;
I'll spare you....

Now, finally, we have the tools to start applying to anomalies.

For the purposes of this talk, anomalies in a finite G gauge theory in $(n+1)$ dimensions will be classified by $H^{n+2}(G, U(1))$.

This arises from considerations of `topological defect lines.' On the next slide I'll outline how that works in the case $n = 0$.

Then, I'll outline how anomaly resolution in (1+1) dimensions can be understood via decomposition.

Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions
- why are anomalies are associated to group cohomology?

Suppose a (finite) group G acts on the states of a QM system: For any $|\psi\rangle$, get $\rho(g)|\psi\rangle$.

For an honest group action, require $\rho(g)\rho(h) = \rho(gh)$

However, b/c we only care about states up to phases, we might instead have

$\rho(g)\rho(h) = \omega(g,h)\,\rho(gh)$ for some $\omega(g,h) \in U(1)$

Associativity $\Rightarrow$ $\omega(g_2,g_3)\,\omega(g_1,g_2g_3) = \omega(g_1g_2,g_3)\,\omega(g_1,g_2)$ (coclosed)

Multiply ρ by phase $\epsilon(g) \Rightarrow$ $\omega(g,h) \mapsto \omega(g,h)\dfrac{\epsilon(gh)}{\epsilon(g)\epsilon(h)}$ (coboundaries)

Thus, the obstructions ω are classified by $H^2(G,U(1))$

Anomaly in 0+1 dims

States are all in ω-projective representations of G.

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (I'll assume central)

where Γ is chosen so that $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

The idea is then to replace $[X/G]$ with $[X/\Gamma]$,
but, need to describe how Γ acts on X.

If K acts triv'ly on X, and we do nothing else,
then we have accomplished nothing:

decomposition $\Rightarrow$ $\text{QFT}([X/\Gamma]) = \coprod_{\hat{K}} \text{QFT}([X/G])$ — still anomalous

Fix by adding quantum symmetry....

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \stackrel{\pi}{\longrightarrow} 1$ (assumed central)

2) Turn on quantum symmetry $B \in H^1(G, H^1(K, U(1)))$
chosen so that $d_2 B = \alpha$. This implies $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

K acts trivially on X, but nontrivially on twisted sector states via B

These two together — extension Γ plus B — resolve anomaly.

Decomposition explains how....

Application to anomaly resolution

Procedure: replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$

where $d_2 B \; = \; \alpha \in H^3(G, U(1))$, the anomaly of the G orbifold.

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\hat{\omega}} \right)$$

— using earlier results for decomp' in orb' w/ quantum symmetry

Note that since $d_2 B = \alpha$, $\quad \alpha|_{\mathrm{Ker}\, B} = 0$

So, $\mathrm{Ker}\, B \subset G$ is automatically anomaly-free!

Summary: $[X/\Gamma]_B =$ copies of orbifold by anomaly-free subgroup.

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

Anomaly $\alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$

Extension 1: Define $\Gamma = D_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	w/o d.t. in D4	w/ d.t. in D4
1	1	–	$[X/G] \coprod [X/G]_{\rm dt}$	$[X/\langle b \rangle]$
-1	1	–	$[X/\langle b \rangle]$	$[X/G] \coprod [X/G]_{\rm dt}$
1	-1	$\langle b \rangle$	$[X/\langle a \rangle]$	$[X/\langle ab \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle a \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G],\ G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

Anomaly $\alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$

Extension 2: Define $\Gamma = \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	Result
1	1	–	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
-1	1	$\langle a \rangle, \langle ab \rangle$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

Anomaly $\alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$

Extension 3: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	w/o d.t. in Z2 x Z4	w/ d.t. in Z2 x Z4
1	1	–	$[X/G] \coprod [X/G]$	$[X/G]_{\text{dt}} \coprod [X/G]_{\text{dt}}$
-1	1	$\langle ab \rangle$	$[X/\langle b \rangle]$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$	$[X/\langle a \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

Anomaly $\alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$

In the examples so far, we picked a `minimal' resolution Γ.
If we pick larger K, we get copies.

Extension 4: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{H}$, $1 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Results:

$B(a)$	$B(b)$	**d_2(B) (anomaly)**	**Result**
1	1	—	$\coprod_2 \left([X/G] \coprod [X/G]_{\rm dt} \right)$
-1	1	$\langle a \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$\coprod_2 [X/\langle ab \rangle]$

Get copies of orb's w/ anomaly-free subgroups.

Works!

Summary

Decomposition: `one' QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories
with $n-$form symmetries.

(I've focused on examples in 1+1d,
but examples exist in other dim's too.)

Can be used to understand anomaly-resolution procedure of www:

replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$,
but decomposition implies
$\text{QFT}\left([X/\Gamma]_B\right) = \text{copies of QFT}\left([X/\text{Ker}\, B \subset G]\right)$,
which is explicitly non-anomalous.

Thank you for your time !

Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions

So far, have obstruction to honest action of G encoded in anomaly $\omega \in H^2(G, U(1))$

Fix: extend G to larger group Γ for which states are in an honest representation.

1) Pick extension Γ, $\quad 1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$

such that $\quad \pi^* \omega = 0 \in H^2(\Gamma, U(1))$

2) Describe action of Γ, by picking $A \in H^0(G, H^1(K, U(1)))$

such that $\quad A(s_1 s_2 s_{12}^{-1}) = \omega(g_1, g_2) \quad$ for $s_i = s(g_i)$, $\; s : G \to \Gamma \;$ a section.

Then, define $\quad \tilde{\rho}(s(g)k) \; \equiv \; A(k)\rho(g)$

and one can show that $\tilde{\rho}$ defines an honest representation of Γ.

Anomaly fixed!

Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions

Fix: extend G to larger group Γ for which states are in an honest representation.

1) Pick extension Γ, $1 \longrightarrow K \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$

such that $\pi^* \omega = 0 \in H^2(\Gamma, U(1))$

2) Describe action of Γ, by picking $A \in H^0(G, H^1(K, U(1)))$

such that $A(s_1 s_2 s_{12}^{-1}) = \omega(g_1, g_2)$ for $s_i = s(g_i)$, $s : G \to \Gamma$ a section.

That was just QM, but the same pattern applies in higher dimensions.
In 1+1 dimensions, we'll see how decomposition gives a very
explicit understanding of how anomaly resolution works.

Future directions

Boundaries in orbifolds with quantum symmetries

We saw earlier that in orbifolds $[X/\Gamma]$ with triv'ly acting $K \subset \Gamma$,
the boundaries are naturally associated to universes of decomposition:

the boundary carries a (possibly projective) action of Γ,
so restrict to K,
that action descends to a (possibly projective) representation of K,
which tells us which universe(s) the boundary is associated to.

That works fine in cases in which $[X/\Gamma]$ has discrete torsion,
just projectivize. But what about quantum symmetries?

Specifically, quantum symmetries B with $d_2 B \neq 0$?

Boundaries in orbifolds with quantum symmetries

Specifically, quantum symmetries B with $d_2 B \neq 0$?

In this case, the associativity of the Γ action is broken, albeit weakly — the action is `homotopy associative.'

In principle, this structure should be understood formally in terms of a groupoid quotient.

WIP w/ Tony Pantev to give a careful description.