A-twisted Landau-Ginzburg models, gerbes, and Kuznetsov's homological projective duality

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J. Guffin, ES, arXiv: 0801.3836, 0803.3955
T Pantev, ES, hepth/0502027, 0502044, 0502053
S Hellerman, A Henriques, T Pantev, ES, M Ando, hepth/0606034
A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855

Outline:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj' to LG & GLSM's: physical realization of Kuznetsov's homological projective duality,
GLSM's for K's noncommutative resolutions

* Heterotic LG models

A **Landau-Ginzburg model** is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

$$S = \int_{\Sigma} d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\jmath} D_z \psi_-^i + \cdots \right.$$

$$+ g^{i\bar{\jmath}} \partial_i W \partial_{\jmath} \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar{\imath}} \psi_-^{\bar{\jmath}} D_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W}$$

The superpotential $W : X \longrightarrow \mathbf{C}$ is holomorphic, (so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....

LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

$$b(\phi)^{j_1 \cdots j_m}_{\overline{\imath}_1 \cdots \overline{\imath}_n} \eta^{\overline{\imath}_1} \cdots \eta^{\overline{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where η, θ are linear comb's of ψ

$$Q \cdot \phi^i = 0, \;\; Q \cdot \phi^{\overline{\imath}} = \eta^{\overline{\imath}}, \;\; Q \cdot \eta^{\overline{\imath}} = 0, \;\; Q \cdot \theta_j = \partial_j W, \;\; Q^2 = 0$$

Identify $\eta^{\overline{\imath}} \leftrightarrow d\overline{z}^{\overline{\imath}}, \;\; \theta_j \leftrightarrow \frac{\partial}{\partial z^j}, \;\; Q \leftrightarrow \overline{\partial}$

so the states are hypercohomology

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

Quick checks:

1) W=0, standard B-twisted NLSM

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto \; H^{\cdot}\left(X, \Lambda^{\cdot} TX\right)$$ ✓

2) X=$\mathbf{C}^n$, W = quasihomogeneous polynomial

Seq' above resolves fat point {dW=0}, so

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto \; \mathbf{C}[x_1, \cdots, x_n]/(dW)$$ ✓

LG A model:

Defining the A twist of a LG model is more interesting.

(Fan, Jarvis, Ruan) (Ito; J Guffin, ES)

Producing a TFT from a NLSM involves changing what bundles the ψ couple to, e.g.

$$\psi \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* TX) \mapsto \Gamma(\Sigma, \phi^* TX), \Gamma(\Sigma, K_\Sigma \otimes \phi^* TX)$$

The two inequivalent possibilities are the A, B twists.

To be consistent, the action must remain well-defined after the twist.

True for A, B NLSM's & B LG, but not A LG....

LG A model:

The problem is terms in the action of the form

$$\psi_+^i \psi_-^j D_i \partial_j W$$

If do the standard A NLSM twist,
this becomes a 1-form on Σ,
which can't integrate over Σ .

Fix: modify the A twist.

LG A model:

There are several ways to fix the A twist,
and hence, several different notions of a LG A model.

One way: multiply offending terms in the action
by another 1-form.

Another way: use a different prescription for
modifying bundles.

The second is advantageous for physics, so I'll use it,
but,
disadvantage: not all LG models admit A twist
in this prescription.

To twist, need a U(1) isometry on X w.r.t. which the superpotential is quasi-homogeneous.

Twist by ``R-symmetry + isometry”

Let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) \;=\; \lambda W(\phi_i)$$

then twist: $\psi \;\mapsto\; \Gamma\left(\text{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K_\Sigma}^{-(1/2)Q_L}\right)$

where $$Q_{R,L}(\psi) \;=\; Q(\psi) \;+\; \begin{cases} 1 & \psi = \psi_+^i, R \\ 1 & \psi = \psi_-^i, L \\ 0 & \text{else} \end{cases}$$

Example: X = $\mathbf{C}^n$, W quasi-homog' polynomial

Here, to twist, need to make sense of e.g. $K_{\Sigma}^{1/r}$

where r = 2(degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)
-- but then usually can't make sense of $K_{\Sigma}^{1/r}$

I'll work with the latter case.

LG A model:

A twistable example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)
with $W = p\pi^* s,\ s \in \Gamma(B, \mathcal{E})$

Accessible states are Q-closed (mod Q-exact) prod's:

$$b(\phi)_{\overline{\imath}_1 \cdots \overline{\imath}_n j_1 \cdots j_m} \psi_-^{\overline{\imath}_1} \cdots \psi_-^{\overline{\imath}_n} \psi_+^{j_1} \cdots \psi_+^{j_m}$$

where

$$\phi \sim \{s = 0\} \subset B \qquad \psi \sim TB|_{\{s=0\}}$$

$$Q \cdot \phi^i = \psi_+^i,\ \ Q \cdot \phi^{\overline{\imath}} = \psi_-^{\overline{\imath}},\ \ Q \cdot \psi_+^i = Q \cdot \psi_-^{\overline{\imath}} = 0,\ \ Q^2 = 0$$

Identify $\psi_+^i \leftrightarrow dz^i,\ \ \psi_-^{\overline{\imath}} \leftrightarrow d\overline{z}^{\overline{\imath}},\ \ Q \leftrightarrow d$

so the states are elements of $H^{m,n}(B)|_{\{s=0\}}$

Correlation functions:

B-twist:

Integrate over X, weight by

$$\exp\left(-|dW|^2 \ + \ \mathrm{fermionic}\right)$$

and then perform transverse Gaussian, to get the standard expression.

A-twist:

Similar: integrate over $\mathcal{M}_X$ and weight as above.

Witten equ'n in A-twist:

BRST: $\delta \psi_-^i \, = \, -\alpha \left(\overline{\partial} \phi^i \, - \, i g^{i \overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} \right)$

implies localization on sol'ns of

$$\overline{\partial} \phi^i \, - \, i g^{i \overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} \, = \, 0$$ (``Witten equ'n")

On complex Kahler mflds, there are 2 independent BRST operators:

$$\delta \psi_-^i \, = \, -\alpha_+ \overline{\partial} \phi^i + \alpha_- \, i g^{i \overline{\jmath}} \partial_{\overline{\jmath}} \overline{W}$$

which implies localization on sol'ns of

$$\begin{aligned} \overline{\partial} \phi^i &= 0 \\ g^{i \overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} &= 0 \end{aligned}$$

which is what we're using.

Sol'ns of Witten equ'n:

$$\int_\Sigma \left|\overline{\partial}\phi^i - i g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \overline{W}\right|^2 = \int_\Sigma \left(\left|\overline{\partial}\phi^i\right|^2 + \left|\partial_i W\right|^2 \right)$$

LHS = 0 iff RHS = 0

hence sol'ns of Witten equ'n
same as the moduli space we're looking at.

LG A model, cont'd

In prototypical cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 \,-\, \chi^p dz^i D_i s \,-\, \mathrm{c.c.} \,-\, F_{i\overline{\jmath}} dz^i d\overline{z}^{\overline{\jmath}} \chi^p \chi^{\overline{p}}\right.}_{\mathrm{Mathai-Quillen\ form}}$$

The MQ form rep's a Thom class, so

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \mathrm{Eul}(N_{\{s=0\}/\mathcal{M}}) \\ &= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \end{aligned}$$

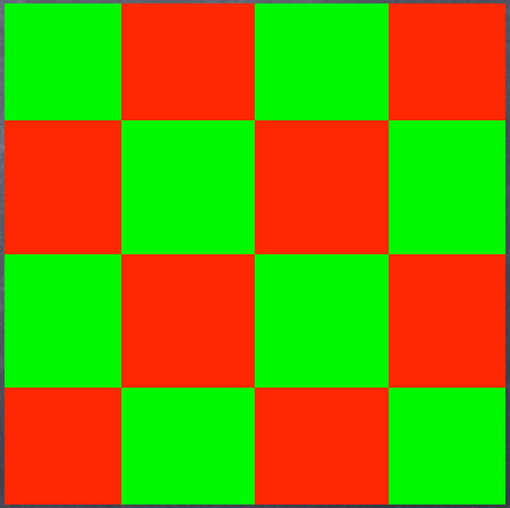
-- same as A twisted NLSM on {s=0}

Not a coincidence, as we shall see shortly....

Renormalization (semi)group flow

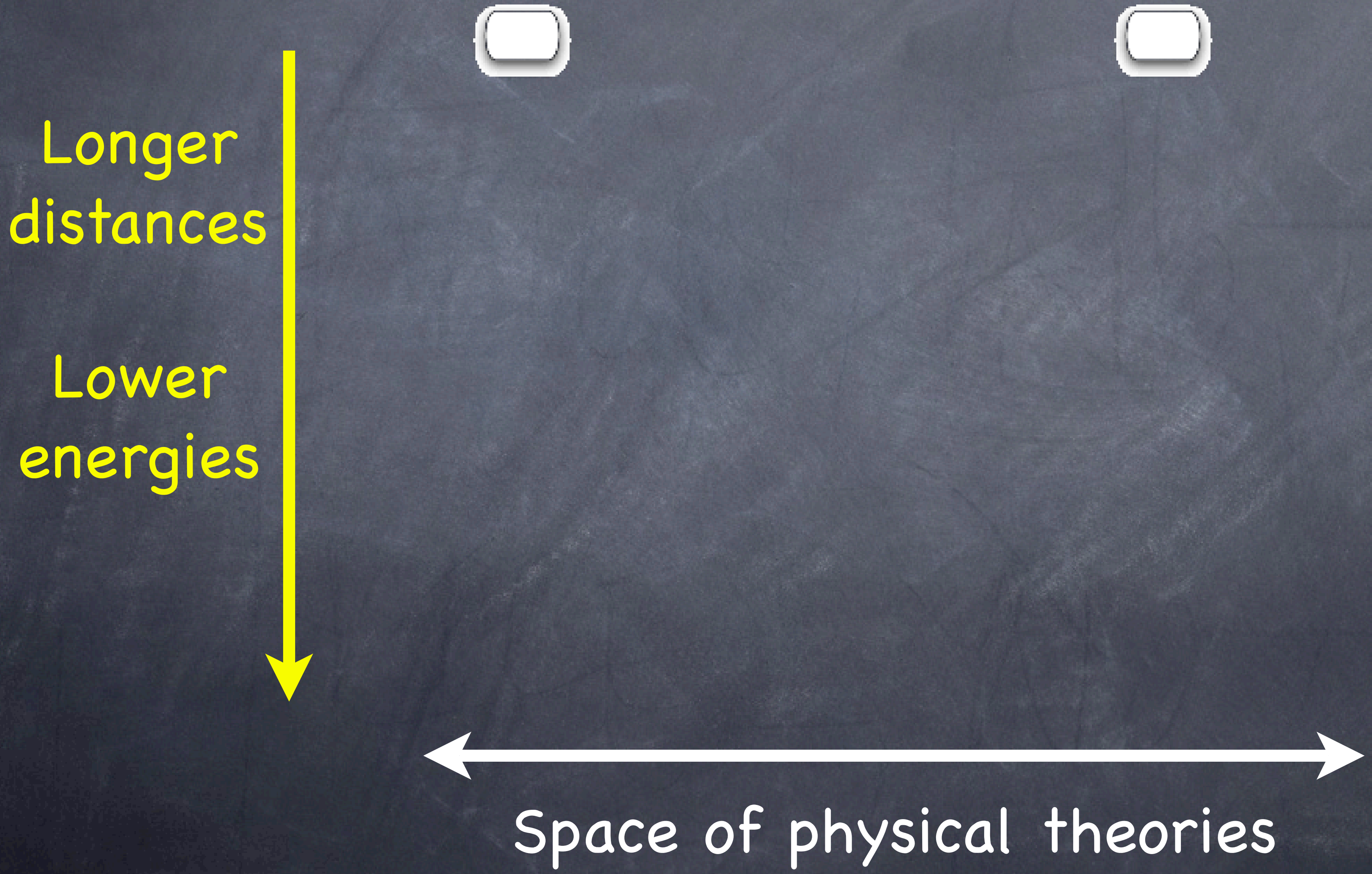
Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.



Problem: cannot follow it explicitly.

Renormalization group



Longer distances

Lower energies

Space of physical theories

Furthermore, RG preserves TFT’s.

If two physical theories are related by RG,
then, correlation functions in a top’ twist of one
=
correlation functions in corresponding twist of other.

Example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p s

↓ Renormalization group flow

NLSM on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

This is why correlation functions match.

Another way to associate LG models to NLSM.

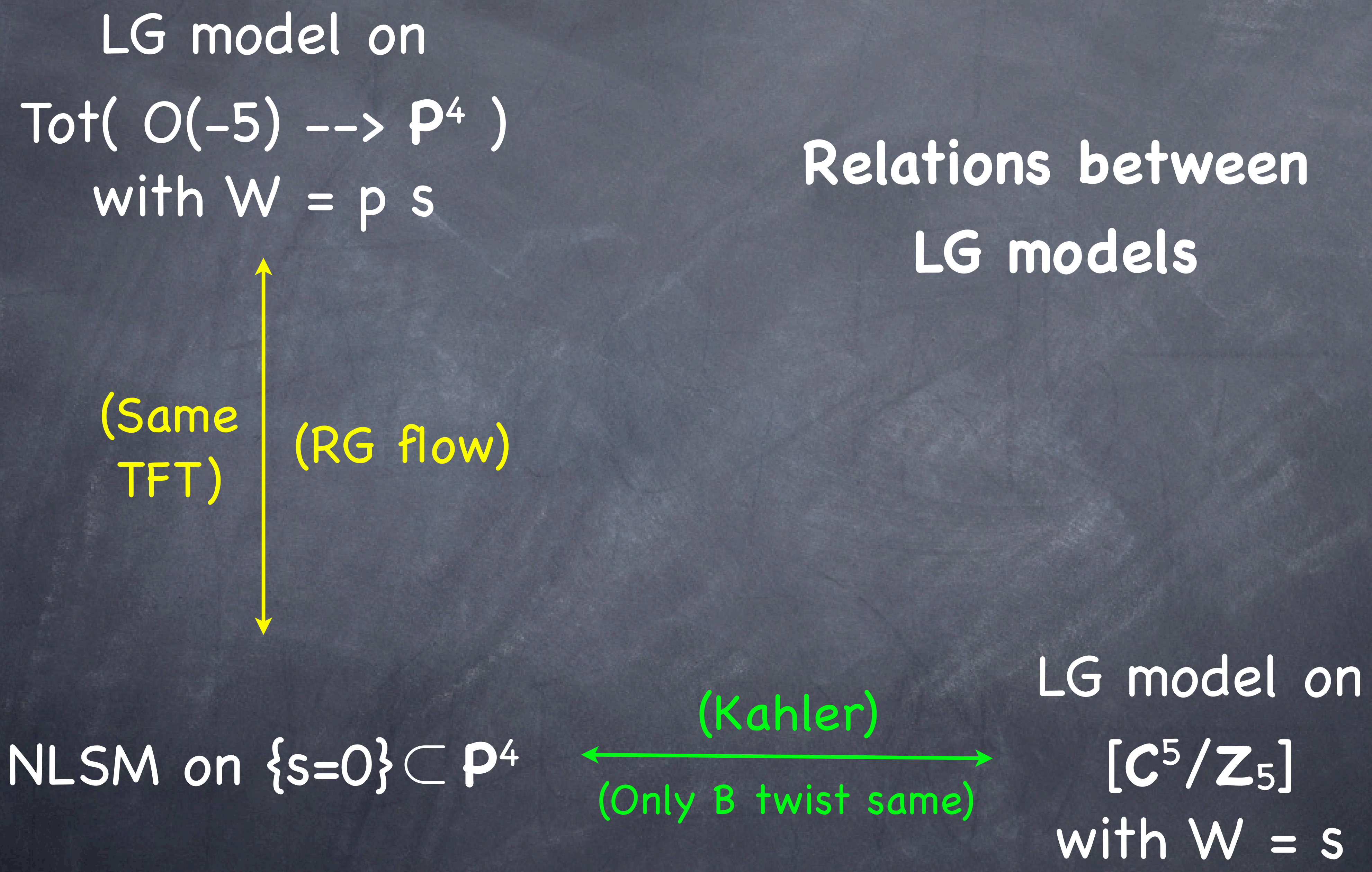
S'pose, for ex, the NLSM has target space
= hypersurface {G=0} in $\mathbf{P}^n$ of degree d

Associate LG model on $[\mathbf{C}^{n+1}/\mathbf{Z}_d]$
with W = G

* **Not** related by RG flow

* But, related by Kahler moduli,
so have same B model

Relations between LG models



LG model on
Tot(O(-5) --> $\mathbf{P}^4$)
with W = p s

(Same TFT)

(RG flow)

NLSM on $\{s=0\} \subset \mathbf{P}^4$

(Kahler)

(Only B twist same)

LG model on
$[\mathbf{C}^5/\mathbf{Z}_5]$
with W = s

RG flow interpretation:

In the case of the A-twisted correlation f'ns,
we got a Mathai-Quillen rep of a Thom form.

Something analogous happens in elliptic genera:
elliptic genera of the LG & NLSM models
are related by Thom forms.

Suggests: RG flow interpretation in twisted theories
as Thom class.

Possible mirror symmetry application:

Part of what we've done is to replace NLSM's with LG models that are `upstairs' in RG flow.

Then, for example, one could imagine rephrasing mirror symmetry as a duality between the `upstairs' LG models.

-- P. Clarke, 0803.0447

Next:

* decomposition conjecture for strings on gerbes

* LG duals to gerbes

* application of gerbes to LG's & GLSM's as,
physical realization of Kuznetsov's
homological projective duality

To do this, need to review how stacks appear in physics....

String compactifications on stacks

First, motivation:

-- new string compactifications

-- better understand certain existing string compactifications

Next: how to construct QFT's for strings propagating on stacks?

Stacks

How to make sense of strings on stacks concretely?

Most (smooth, Deligne-Mumford) stacks can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

(G need not be finite; need not act effectively.)

To such a presentation, associate a ``G-gauged sigma model on X."

Problem: such presentations not unique

Stacks

If to [X/G] we associate ``G-gauged sigma model,”
then:

$[\mathbf{C}^2/\mathbf{Z}_2]$ defines a 2d theory with a symmetry called conformal invariance

$[X/\mathbf{C}^\times]$ defines a 2d theory w/o conformal invariance

Same stack, different physics!

Potential presentation-dependence problem:
fix with renormalization group flow
(Can’t be checked explicitly, though.)

The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.

Stacks

Other issues: deformation theory
massless spectra

To justify application of stacks to physics,
need to conduct tests of presentation-dependence,
understand issues above.

This was the subject of several papers.

For the rest of today's talk,
I want to focus on special kinds of stacks, namely,
gerbes.

(= quotient by noneffectively-acting group)

Gerbes

Gerbes have add’l problems when viewed from this physical perspective.

Example: The naive massless spectrum calculation contains multiple dimension zero operators, which manifestly violates cluster decomposition, one of the foundational axioms of quantum field theory.

There is a single known loophole: if the target space is disconnected. We **think** that’s what’s going on....

Decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

Claim

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on\ } X) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \xrightarrow{Z(G) \to U(1)} H^2(X, U(1))$$

Banded Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

One of the effective orbifolds has vanishing discrete torsion, the other has nonvanishing discrete torsion.

Checks: can show partition functions match:

$$Z([X/D_4]) = Z\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

Another quick check-- compare massless spectra:

Spectrum for $[T^6/D_4]$:

$$\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}$$

and for each $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Sum matches. ✓

Nonbanded example:

Consider $[X/\mathbf{H}]$ where $\mathbf{H}$ is the eight-element group of quaternions, and a $\mathbf{Z}_4$ acts trivially.

$$1 \longrightarrow < i > (\cong \mathbf{Z}_4) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}([X/\mathbf{H}]) = \mathrm{CFT}\left([X/\mathbf{Z}_2] \coprod [X/\mathbf{Z}_2] \coprod X \right)$$

Straightforward to show that this is true at the level of partition functions, as before.

Another class of examples:
global quotients by nonfinite groups

The banded $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$
with characteristic class $-1 \bmod k$
can be described mathematically as the quotient

$$\left[\frac{ \mathbf{C}^{N+1} - \{0\} }{ \mathbf{C}^{\times} } \right]$$

where the $\mathbf{C}^{\times}$ acts as rotations by k times

which physically can be described by a U(1) susy gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbf{P}^N$ model?

The difference lies in nonperturbative effects.
(Perturbatively, having nonminimal charges makes no difference.)

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies
$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

(Noncompact worldsheet - theta angle -- J Distler, R Plesser)

Return to the example $\left[\frac{\mathbf{C}^{N+1} - \{0\}}{\mathbf{C}^{\times}} \right]$

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1} : \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$

$$\mathrm{Here:} \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1} : \quad < X^{N(d+1)-1} > = q^d$$

$$\mathrm{Here:} \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1} : \quad \mathbf{C}[x]/(x^N - q)$$

$$\mathrm{Here:} \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

K theory implications

This equivalence of CFT's implies a statement about K theory (thanks to D-branes).

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

If G acts trivially on X

then the ordinary H-equivariant K theory of X

is the same as

twisted K-equivariant K theory of $X \times \hat{G}$

* Can be derived just within K theory

* Provides a check of the decomposition conjecture

D-branes and sheaves

D-branes in the topological B model can be described with sheaves and, more gen'ly, derived categories.

This also is consistent with the decomp' conjecture:

Math fact:

A sheaf on a banded G-gerbe

is the same thing as

a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

which is consistent with the way D-branes should behave according to the conjecture.

D-branes and sheaves

Similarly, massless states between D-branes should be counted by Ext groups between the corresponding sheaves.

Math fact:

Sheaves on a banded G-gerbe decompose according to irrep' of G,
and sheaves associated to distinct irreps have vanishing Ext groups between them.

Consistent w/ idea that sheaves associated to distinct reps should describe D-branes on different components of a disconnected space.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K\right]$

Banded **Z**$_k$ gerbes:
E Andreini, Y Jiang, H-H Tseng, 0812.4477

Quantum cohomology

Some old results of Morrison-Plesser (q.c. from gauge theory) generalize from toric varieties to toric stacks.

Let the toric stack be described in the form

$$\left[\frac{\mathbf{C}^N \; - \; E}{(\mathbf{C}^\times)^n} \right]$$

E some exceptional set

Q_i^a the weight of the ith vector under ath $\mathbf{C}^\times$

then Batyrev's conjecture becomes

$\mathbf{C}[\sigma_1, \cdots, \sigma_n]$ modulo the relations

$$\prod_{i=1}^{N} \left(\sum_{b=1}^{n} Q_i^b \sigma_b \right)^{Q_i^a} = q_a$$

(ES, T Pantev, '05)

Quantum cohomology

Ex: Quantum cohomology ring of $\mathbf{P}^N$ is

$$\mathbf{C}[x]/(x^{N+1} - q)$$

Quantum cohomology ring of $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$ with characteristic class -n mod k is

$$\mathbf{C}[x,y]/(y^k - q_2, x^{N+1} - y^n q_1)$$

Aside: these calculations give us a check of the massless spectrum -- in physics, can derive q.c. ring w/o knowing massless spectrum.

Quantum cohomology

We can see the decomposition conjecture in the quantum cohomology rings of toric stacks.

Ex: Q.c. ring of a $\mathbf{Z}_k$ gerbe on $\mathbf{P}^N$ is given by

$$\mathbf{C}[x,y]/(y^k - q_2, x^{N+1} - y^n q_1)$$

In this ring, the y's index copies of the quantum cohomology ring of $\mathbf{P}^N$ with variable q's.

The gerbe is banded, so this is exactly what we expect -- copies of $\mathbf{P}^N$, variable B field.

Quantum cohomology

More generally, a gerbe structure is indicated from this quotient description whenever $\mathbf{C}^\times$ charges are nonminimal.

In such a case, from our generalization of Batyrev's conjecture, at least one rel'n will have the form

$$p^k = q$$

where p is a rel'n in q.c. of toric variety, and k is the nonminimal part.

Can rewrite this in same form as for gerbe on $\mathbf{P}^N$, and in this fashion can see our decomp' conj' in our gen'l of Batyrev's q.c.

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The LG mirror of $\mathbf{P}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbf{Z}_k$ gerbes over $\mathbf{P}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05;
E Mann, '06)

GLSM's

Decomposition conjecture can be applied to GLSM's.

Example: $\mathbf{P}^7[2,2,2,2]$

At the Landau-Ginzburg point, have superpotential

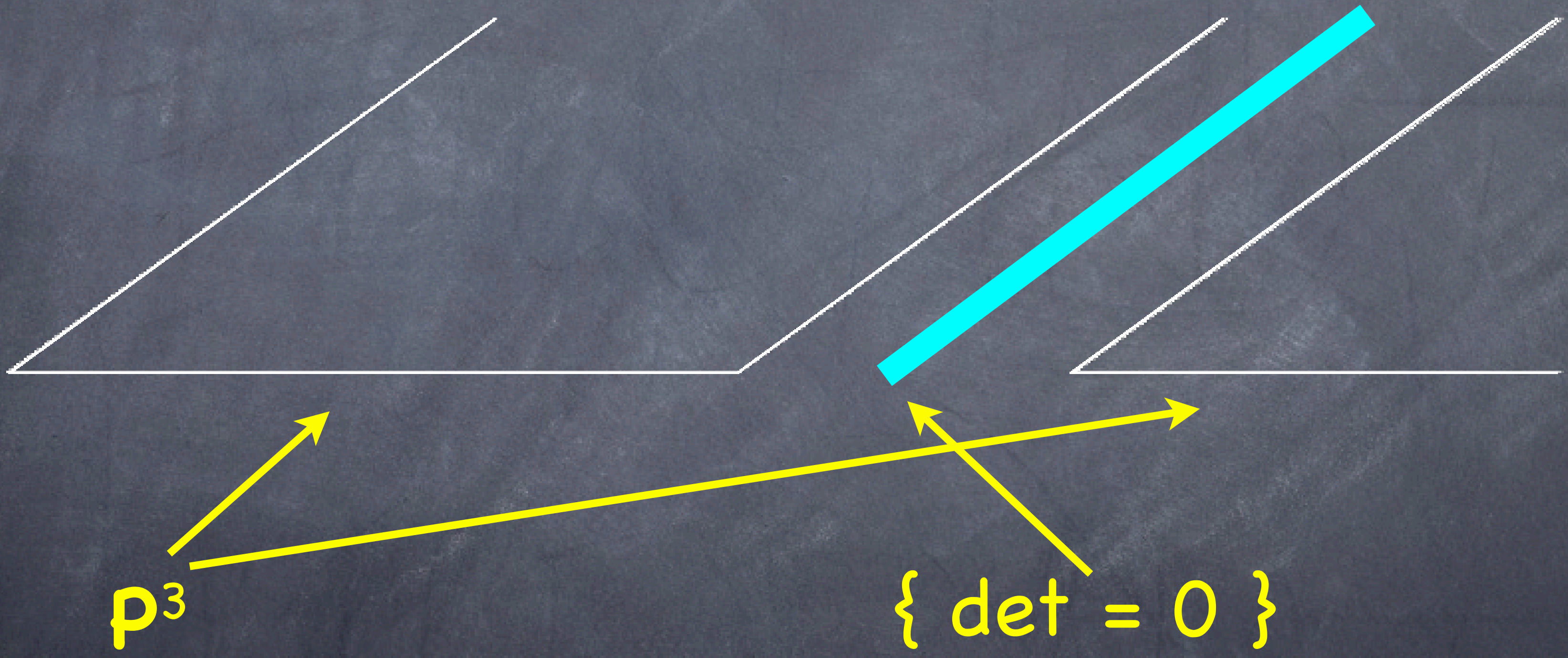
$$\sum_a p_a G_a(\phi) \, = \, \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

* leaves just the p fields, of charge -2

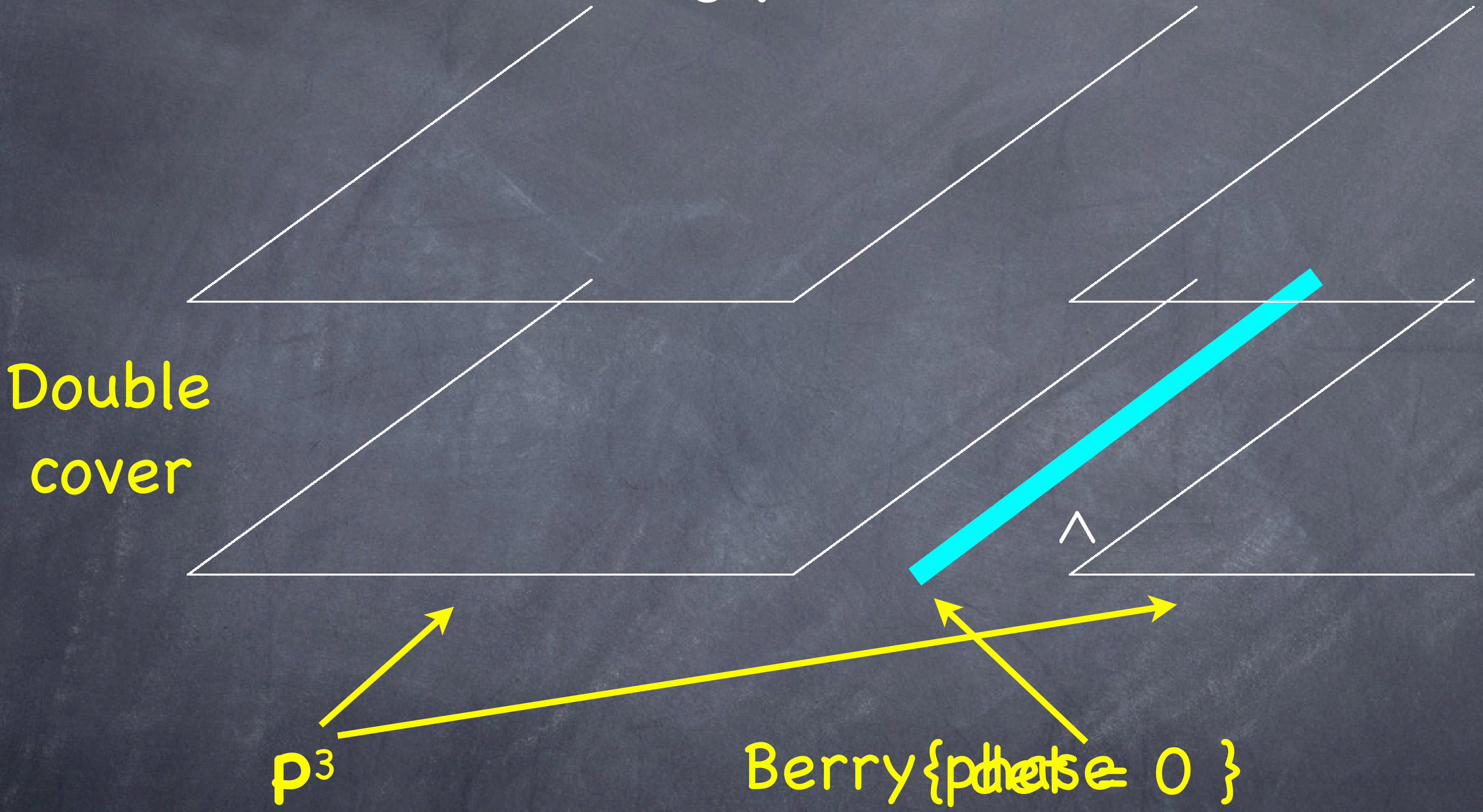
* $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg point:



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{P}^3$ - det....

The Landau-Ginzburg point:



Result: branched double cover of $\mathbf{P}^3$

So far:

The GLSM realizes:

$$P^7[2,2,2,2] \xleftrightarrow{\text{Kahler}} \text{branched double cover of } \mathbf{P}^3$$

(Clemens' octic double solid)

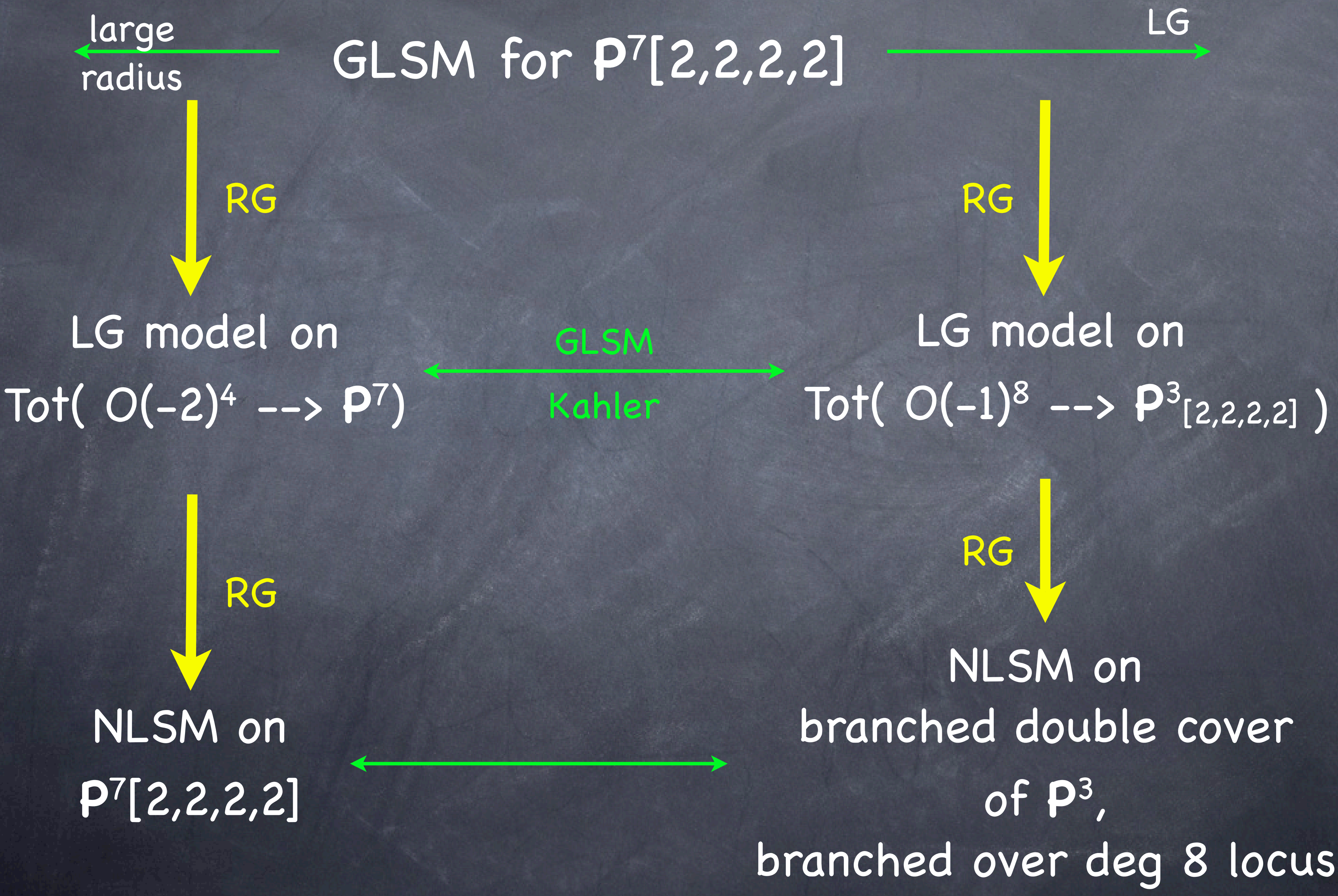
where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Unusual physical realization of geometry

Rewrite with Landau-Ginzburg models:



large radius

GLSM for $\mathbf{P}^7[2,2,2,2]$

LG

RG

RG

LG model on

Tot($O(-2)^4$ --> $\mathbf{P}^7$)

GLSM

Kahler

LG model on

Tot($O(-1)^8$ --> $\mathbf{P}^3_{[2,2,2,2]}$)

RG

RG

NLSM on

$\mathbf{P}^7[2,2,2,2]$

NLSM on

branched double cover

of $\mathbf{P}^3$,

branched over deg 8 locus

Puzzle:

the branched double cover will be singular,
but the GLSM is smooth at those singularities.

Solution?....

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Kuznetsov has defined `homological projective duality' that relates $\mathbf{P}^7[2,2,2,2]$ to the noncommutative resolution above.

Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves -- so for example, a Landau-Ginzburg model can be a noncommutative space via matrix factorizations.

Here, K's noncomm' res'n = ($\mathbf{P}^3$,B)
where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the GLSM superpotential.

B ~ structure sheaf; other sheaves ~ B-modules.

Physics?......

Physics:

B-branes in the RG limit theory
= B-branes in the intermediate LG theory.

Claim: matrix factorizations in intermediate LG
= Kuznetsov's B-modules

K has a rigorous proof of this;
B-branes = Kuznetsov's nc res'n sheaves.

Intuition....

Local picture:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Summary so far:

The GLSM realizes:

$$\mathbf{P}^7[2,2,2,2] \xleftrightarrow{\text{Kahler}} \begin{array}{c}\text{nc res'n of} \\ \text{branched double cover} \\ \text{of } \mathbf{P}^3\end{array}$$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Unusual physical realization of geometry

Physical realization of Kuznetsov's homological projective duality

More examples:

CI of
n quadrics in $\mathbf{P}^{2n-1}$

$\xleftrightarrow{\text{Kahler}}$

branched double
cover of $\mathbf{P}^{n-1}$,
branched over deg 2n
locus

Both sides CY

Homologically projective dual

Rewrite with Landau-Ginzburg models:

$$\begin{tikzcd}
\begin{array}{c}\text{LG model on}\\ \text{Tot}(\ \mathcal{O}(-2)^{k} \dashrightarrow \mathbf{P}^{n})\end{array} \arrow[rr, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] \arrow[d, "\text{RG}"] & & \begin{array}{c}\text{LG model on}\\ \text{Tot}(\ \mathcal{O}(-1)^{n+1} \dashrightarrow \mathbf{P}^{k-1}_{[2,\ldots,2]}\ )\end{array} \arrow[d, "\text{RG}"'] \\
\begin{array}{c}\text{NLSM on}\\ \mathbf{P}^{n}[2,\ldots,2]\end{array} \arrow[rr, leftrightarrow, "\text{Kuznetsov's}", "\text{h.p.d.}"'] & & \begin{array}{c}\text{NLSM on n.c. res'n of}\\ \text{branched double cover}\\ \text{of } \mathbf{P}^{k-1},\\ \text{branched over deg n+1 locus}\end{array}
\end{tikzcd}$$

A math conjecture:

Kuznetsov defines his h.p.d. in terms of coherent sheaves. In the physics language

$$\begin{tikzcd} \begin{array}{c}\text{LG model on}\\ \text{Tot}(\,O(-2)^k \text{ --> } \mathbf{P}^n)\end{array} \arrow[r, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] & \begin{array}{c}\text{LG model on}\\ \text{Tot}(\,O(-1)^{n+1} \text{ --> } \mathbf{P}^{k-1}_{[2,\ldots,2]}\,)\end{array} \end{tikzcd}$$

Kuznetsov's h.p.d. becomes a statement about matrix factorizations, analogous to those in Orlov's work.

Math conjecture: Kuznetsov's h.p.d. has an alternative (& hopefully easier) description in terms of matrix factorizations between LG models on birational spaces.

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$$\xleftrightarrow{\text{Kahler}}$$

branched double cover of $\mathbf{P}^1 \mathrm{x} \mathbf{P}^1 \mathrm{x} \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

* Related to an example of Vafa-Witten involving
discrete torsion

(Caldararu, Borisov)

* Believed to be homologically projective dual

A non-CY example:

CI 2 quadrics
in $\mathbf{P}^{2g+1}$

$\xleftrightarrow{\text{Kahler}}$

branched double
cover of $\mathbf{P}^1$,
over deg 2g+2
(= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively
curved

Negatively
curved

r flows: ⇢ ⇢ ⇢

More examples:

Hori-Tong 0609032 found closely related phenomena in nonabelian GLSMs:

$G(2,7)[1^7] \longleftrightarrow$ Pfaffian CY

Also:
* novel realization of geometry
* nonbirational
* Kuznetsov's h.p.d.

Further nonabelian examples:
Donagi, ES, 0704.1761

So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way

* the geometric phases are not birational

* instead, related by Kuznetsov's homological projective duality

Conjecture: **all** phases of GLSM's are related by Kuznetsov's h.p.d.

Another direction:

Heterotic Landau-Ginzburg models

We'll begin with heterotic nonlinear sigma models....

Heterotic nonlinear sigma models:

Let X be a complex manifold,

$\mathcal{E} \longrightarrow X$ a holomorphic vector bundle

such that $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

Action:

$$S = \int_\Sigma d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i h_{a\overline{b}} \lambda_-^{\overline{b}} D_z \lambda_-^a + \cdots \right.$$

ϕ, ψ_+ as before $\qquad \lambda_-^a \in \Gamma\left(\mathcal{E} \otimes \sqrt{K_\Sigma} \right)$

Reduces to ordinary NLSM when $\mathcal{E} = TX$

Heterotic Landau-Ginzburg model:

$$\begin{aligned} S \,=\, \int_\Sigma d^2x \Big(& g_{i\bar\jmath}\partial\phi^i\overline{\partial}\phi^{\jmath} \,+\, i g_{i\bar\jmath}\psi_+^{\jmath} D_{\bar z}\psi_+^i \,+\, i h_{a\bar b}\lambda_-^{\bar b} D_z\lambda_-^a \,+\, \cdots \\ & + \, h^{a\bar b} F_a \overline{F}_{\bar b} \,+\, \psi_+^i\lambda_-^a D_i F_a \,+\, \text{c.c.} \\ & + h_{a\bar b} E^a \overline{E}^{\bar b} \,+\, \psi_+^i \lambda_-^{\bar a} D_i E^b h_{\bar a b} \,+\, \text{c.c.} \Big) \end{aligned}$$

Has two superpotential-like pieces of data

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^\vee)$$

such that $\displaystyle \sum_a E^a F_a \,=\, 0$

Heterotic LG models are related to heterotic NLSM's via renormalization group flow.

Example:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$

with $\mathcal{E}' = \pi^* \mathcal{F}_2$ & $F_a \equiv 0, \quad E^a \neq 0$

↓

Renormalization group

A heterotic NLSM on B

with $\mathcal{E} = \mathrm{coker}\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Summary:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj' to LG & GLSM's:
physical realization of Kuznetsov's homological projective duality,
GLSM's for K's noncommutative resolutions

* Heterotic LG models

Mathematics		Physics
Geometry: Gromov-Witten Donaldson-Thomas quantum cohomology etc	⟺	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⟺	Renormalization group