Chiral rings in 2d (0,2) theories

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Over the last half dozen years, there's been a *tremendous* amount of progress in perturbative string compactifications.

A few of my favorite examples:

- nonpert' realizations of geometry (Pfaffians, double covers)
  (Hori-Tong '06, Caldararu et al '07,…)
- perturbative GLSM's for Pfaffians (Hori '11, Jockers et al '12,…)
- non-birational GLSM phases - physical realization of homological projective duality (Hori-Tong '06, Caldararu et al '07, Ballard et al '12; Kuznetsov '05-'06,…)
- examples of closed strings on noncommutative res'ns
  (Caldararu et al '07, Addington et al '12, ES '13)
- localization techniques: new GW & elliptic genus computations, role of Gamma classes, …
  (Benini-Cremonesi '12, Doroud et al '12; Jockers et al '12, Halverson et al '13, Hori-Romo '13, Benini et al '13, ….)
- heterotic strings: nonpert' corrections, 2d dualities, non-Kahler moduli (many)

Far too much to cover in one talk! I'll focus on just one….

Today I'll restrict to

- heterotic strings: nonpert' corrections, 2d dualities, non-Kahler moduli

I'm going to outline progress we've made in solving some of the outstanding problems in heterotic string compactifications, through the lens of chiral rings:

- worldsheet instanton corrections: quantum sheaf cohomology

- dualities

  (0,2) mirror symmetry

  Seiberg-like gauge dualities

These heterotic issues generalize ordinary worldsheet instanton corrections & ordinary mirror symmetry.

Outline:

- Chiral states in 2d (0,2) NLSM's
- Product structures in chiral rings in A/2, B/2 twists:

  quantum sheaf cohomology

- Nonabelian GLSMs:
  - Dualities in 2d and their geometry
  - Gadde-Gukov-Putrov triality

(Progress in (0,2) mirrors & non-Kahler moduli left for another time.)

Review: chiral rings in 2d (2,2) NLSM's

(Lerche-Vafa-Warner '89)

Consists of states annihilated by
1 of left-moving & 1 of right-moving supercharges.

4 distinct possibilities, labelled (c,c), (a,c), (c,a), (a,a)

In a NLSM on a complex Kahler manifold X,
all correspond to cohomology of X.

Play a fundamental role in e.g. massless spectra of string compactifications, and are protected against quantum corrections.

More explicitly…

Review: chiral rings in 2d (2,2) NLSM's

In a (R,R) sector, in a NLSM on a space X,
states have the schematic form

$$b^{j_1 \cdots j_p}_{\bar{\imath}_1 \cdots \bar{\imath}_q}(\phi) \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_q} \psi_{-,j_1} \cdots \psi_{-,j_p} |0\rangle$$

$\psi_\pm$ worldsheet fermions, $\sim \; TX$

$$Q \;=\; Q_+ + Q_- \;\leftrightarrow\; d$$

Q-cohomology classes, counted by $H^{p,q}(X)$

Sit in a topologically protected subsector.

What's heterotic analogue?

What's heterotic analogue?

A heterotic worldsheet only has (0,2) susy
instead of (2,2) susy,
so the heterotic analogue will involve states
annihilated by one supercharge instead of two.

For a (0,2) NLSM, on space X with bundle $\mathcal{E}$,
we'll again look at (R,R) sector states….

For 2d (0,2) NLSM's on Calabi-Yau's (CY's), Distler-Greene ('88) worked out the analogue:

In a (R,R) sector, zero-energy Q_+-closed states of form

$$b^{a_1 \cdots a_p}_{\overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,a_1} \cdots \lambda_{-,a_p} |0\rangle$$

close to large radius.

ψ_+, λ_- worldsheet fermions, $\sim \; TX, \mathcal{E}$

$$Q_+ \leftrightarrow \overline{\partial}$$

States counted by Q_+-cohomology = $H^q(X, \wedge^p \mathcal{E}^*)$

= $H^{p,q}(X)$ when $\mathcal{E} \cong TX$ ((2,2) locus)

Assumed $K_X, \det \mathcal{E}$ trivial

Q_+-cohomology no longer in a topological subsector, but should be protected from perturbative corrections.

So, for large-radius CY, should be reliable.

Consider a more general 2d (0,2) NLSM near large-radius:

(Guo, Jia, ES, 2015)

$K_X, \, \det \mathcal{E}$ need not be trivial

The zero-energy Q_+-closed states again of the form

$$b^{a_1 \cdots a_p}_{\overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,a_1} \cdots \lambda_{-,a_p} |0\rangle$$

but now $|0\rangle \sim (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}$

for the Fock vacuum $\psi_+^i |0\rangle = 0 = \lambda_{-,\overline{a}} |0\rangle$

States counted by

$$H^q \left(X, (\wedge^p \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right)$$

Choice of square root encodes eg target space spin structure.

Different Fock vacua choices give equivalent results….

If instead we'd worked with a Fock vacuum defined by

$$\psi_+^i |0\rangle' \;=\; 0 \;=\; \lambda_{-,a} |0\rangle'$$

then this one related to last one by

$$|0\rangle' \;=\; \left(\prod_a \lambda_{-,\overline{a}} \right) |0\rangle \qquad \begin{array}{l} |0\rangle \;\sim\; (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \\ |0\rangle' \;\sim\; (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \end{array}$$

and states of the form

$$b_{\overline{\imath}_1 \cdots \overline{\imath}_q}^{\overline{a}_1 \cdots \overline{a}_p}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,\overline{a}_1} \cdots \lambda_{-,\overline{a}_p} |0\rangle'$$

Counted by $H^q \left(X, (\wedge^p \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right)$

$$= \; H^q \left(X, (\wedge^{r-p} \mathcal{E}^*) \otimes (\det \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right)$$

$$= \; H^q \left(X, (\wedge^{r-p} \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right)$$

(matching previous counting)

States:

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right)$$

Special case: (2,2) locus

$$\mathcal{E} \;=\; TX$$

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, \Omega_X^\bullet\right) \;=\; H^{\bullet,\bullet}(X)$$

as expected

On a Calabi-Yau, or if $K_X^{\otimes 2} \;\cong\; \mathcal{O}_X$

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, \wedge^\bullet TX\right)$$

Other tests:

- Invariance under $\mathcal{E} \leftrightarrow \mathcal{E}^*$ (a duality of (0,2) worldsheets)

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right) = H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right)$$
$$= H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E}^*)^{-1/2} \otimes K_X^{+1/2}\right)$$

— manifest

- Should be implicit in elliptic genera

Leading term is proportional to

$$\int \hat{A}(TX) \wedge \mathrm{ch}\left((\det \mathcal{E})^{+1/2} \wedge_{-1} (\mathcal{E}^*)\right)$$
$$= \int \mathrm{td}(TX) \wedge \mathrm{ch}\left(\wedge_{-1}(\mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right)$$
$$= \sum_i (-)^i \chi\left((\wedge^i \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right)$$

— matches

Sometimes we can perform a (pseudo-) topological twist.

These NLSM's have two anomalous global U(1)'s:

- a right-moving $U(1)_R$
- a canonical left-moving U(1), rotating the phase of all left fermions, which becomes $U(1)_R$ on (2,2) locus

If $\det \mathcal{E}^{\pm 1} \cong K_X$, then a nonanomalous U(1) exists along which we can twist right & left moving fermions.

Possible twists….

A/2 model: Exists when $(\det \mathcal{E})^{-1} \cong K_X$

(on (2,2) locus, always possible; reduces to A model)

States: $H^{\bullet}\left(X, \wedge^{\bullet}\mathcal{E}^*\right)$

B/2 model: Exists when $\det \mathcal{E} \cong K_X$

(on (2,2) locus, requires $K_X^{\otimes 2} \cong \mathcal{O}_X$; reduces to B model)

States: $H^{\bullet}\left(X, \wedge^{\bullet}\mathcal{E}\right)$

Exchanging $\mathcal{E} \leftrightarrow \mathcal{E}^*$ swaps the A/2, B/2 models.

(Physically, just a complex conjugation of left movers.)

Product structures

OPE rings in the A/2, B/2 models
= "quantum sheaf cohomology"

Physical relevance?

On the (2,2) locus, in a perturbative heterotic compactification on a CY 3-fold, say,

A model correlation f'ns & GW inv'ts encoded in $\overline{\mathbf{27}}^3$ couplings

B model correlation f'ns encoded in $\mathbf{27}^3$ couplings

Off the (2,2) locus, Gromov-Witten inv'ts no longer relevant.

Mathematical GW computational tricks no longer apply.

No known analogue of periods, Picard-Fuchs equations.

New methods needed… and a few have been developed.

(A Adams, J Distler, R Donagi, J Guffin, S Katz, J McOrist, I Melnikov, R Plesser, ES, ….)

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

Ex: ordinary quantum cohomology of $\mathbb{P}^n$

$$\mathbb{C}[x]/(x^{n+1}-q)$$

Compare: quantum sheaf cohomology of $\mathbb{P}^n \times \mathbb{P}^n$

with bundle

$$0 \to O \oplus O \xrightarrow{*} O(1,0)^{n+1} \oplus O(0,1)^{n+1} \to E \to 0$$

where

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$x, \tilde{x}$ homog' coord's on $\mathbb{P}^n$'s

is given by $\mathbb{C}[x,y]/(\det(Ax+By)-q_1, \det(Cx+Dy)-q_2)$

Check: When E=T, this becomes $\mathbb{C}[x,y]/(x^{n+1}-q_1, \mathrm{y}^{n+1}-q_2)$

(as expected: q.s.c. should reduce to ordinary q.c. for E=T)

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

When does that OPE ring close into itself?

(2,2) susy **not** required.

For a SCFT, can use combination of

- worldsheet conformal invariance
- right-moving N=2 algebra

to argue closure on patches on moduli space.

(Adams-Distler-Ernebjerg, '05)

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

A model:

Operators: $b_{i_1 \cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \cdots \chi^{i_1} \cdots \chi^{i_p} \leftrightarrow H^{p,q}(X)$

A/2 model:

Operators: $b_{\bar{\imath}_1 \cdots \bar{\imath}_q a_1 \cdots a_p} \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_q} \lambda_-^{a_1} \cdots \lambda_-^{a_p} \leftrightarrow H^q(X, \wedge^p E^*)$

On the (2,2) locus, A/2 reduces to A.

For operators, follows from

$$H^q(X, \wedge^p T^* X) = H^{p,q}(X)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Schematically:

A model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n = \int_X (\text{top-form})$$

$$\omega_i \in H^{p_i, q_i}(X)$$

A/2 model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n$$

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\text{top}}(X, \wedge^{\text{top}} E^*) = \mathrm{H}^{\text{top}}(X, K_X)$

using the anomaly constraint $\det E^* \cong K_X$

Again, a top form, so get a number.

Review of quantum sheaf cohomology

To make this more clear, let's consider an

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \to H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $\cancelto{0}{H^0(Z \otimes W)} \to H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \to \cancelto{0}{H^1(Z \otimes W)}$

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$

0

Since Z is a sum of $O(-1,0)$'s, $O(0,-1)$'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \underset{\delta}{\overset{\sim}{\to}} H^1(S_1) \underset{\delta}{\to} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\text{top}} E^{*} \cong K_{X} \\ \mathrm{ch}_{2}(\mathrm{E}) = \mathrm{ch}_{2}(\mathrm{TX}) \end{array}\right\} \overset{\text{GRR}}{\Longrightarrow} \quad \wedge^{\text{top}} F^{*} \cong K_{M}$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet}\mathrm{W}/\left(Q^{a+1},\tilde{Q}^{b+1}\right)$$

where $\quad Q=\det(A\psi+B\tilde{\psi}), \quad \tilde{Q}=\det(C\psi+D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O\rangle_{a,b}=q^{a'-a}\tilde{q}^{b'-b}\langle OQ^{a'-a}\tilde{Q}^{b'-b}\rangle_{a',b'}$$

for some constants $q,\tilde{q}$ => OPE's $Q=q,\ \tilde{Q}=\tilde{q}$

— quantum sheaf cohomology rel'ns

Review of quantum sheaf cohomology

General result: (Donagi, Guffin, Katz, ES, '11)

For any toric variety, and any def' E of its tangent bundle,

$$0 \rightarrow W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \rightarrow \mathrm{E} \rightarrow 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} = \mathrm{q}_a$$

where the M's are matrices of chiral operators built from *.

Review of quantum sheaf cohomology

So far, I've outlined mathematical computations of quantum sheaf cohomology, but GLSM-based methods also exist:

- Quantum cohomology ((2,2)): Morrison-Plesser '94
- Quantum sheaf cohomology ((0,2)): McOrist-Melnikov '07, '08

Briefly, for (0,2) case:

One computes quantum corrections to effective action of form

$$L_{\text{eff}} = \int d\theta^{+} \sum_{a} \Upsilon_a \log\left[\prod\nolimits_{\alpha} (\det M_{(\alpha)})^{Q_\alpha^a} / q_a \right]$$

from which one derives $\displaystyle \prod_{\alpha} \left(\det M_{(\alpha)} \right)^{Q_\alpha^a} = q_a$

— these are q.s.c. rel'ns — match math' computations

Review of quantum sheaf cohomology

State of the art: computations on toric varieties

To do: compact CY's

Intermediate step: Grassmannians (work in progress)

Briefly, what we need are better computational methods.

Conventional GW tricks seem to revolve around idea that A model is independent of complex structure, not necessarily true for A/2.

- McOrist-Melnikov '08 have argued an analogue for A/2
- Despite attempts to check (Garavuso-ES '13), still not well-understood

So far, I've (secretly) been talking about abelian GLSM's.

Next, let's turn to nonabelian GLSMs:

- Dualities in 2d and their geometry
- Gadde-Gukov-Putrov triality

Dualities in 2d and their geometry

Gauge theory (Seiberg) dualities —
in which two different-looking theories RG flow to the same —
are very interesting, and esp. in 4d have a long history.

Recently, there's been a lot of interest in, and a number of proposals for, 2d gauge theory dualities,
in both (2,2) and (0,2) susy.

However, most of those dualities seem to have a simple geometric understanding, as we'll outline and utilize.

(Jia, ES, Wu, '14)

In 2d theories, dualities often have a purely geometric understanding.

Trivial example:

U(k) gauge theory, n chiral multiplets		U(n-k) gauge theory, n chiral multiplets
$\downarrow$ RG		$\downarrow$ RG
NLSM on G(k,n)	=	NLSM on G(n-k,n)

But G(k,n) = G(n-k,n),
so IR limits equivalent.

Can check chiral rings, elliptic genera, etc.

In less trivial examples, we apply similar tricks to systematize understanding, & to make predictions.

Another example, in 2d, (2,2) susy:

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' **k***, A<n

$\overset{\text{Seiberg}}{\longleftrightarrow}$ Benini-Cremonesi '12

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' **k***,
nA neutral chirals M,
superpotential: W = M Φ P

RG ↓ RG ↓

NLSM on $\mathrm{Tot}\left(S^A \to G(k,n)\right)$ $= \mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k,n)\right)$

$= (\mathbb{C}^{kn} \times \mathbb{C}^{kA}) // GL(k)$

RG ↓ RG ↓

... ...

Build physics for RHS using $0 \to S \xrightarrow{\Phi} O^n \to Q \to 0$
& discover the upper RHS.

So, 2d analogue of Seiberg duality has geometric description.

Another example, in 2d, (2,2) susy:

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' $\mathbf{k}^*$, A<n

Seiberg $\longleftrightarrow$ dual

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' $\mathbf{k}^*$,
nA neutral chirals M,
superpotential: $W = M \Phi P$

To be fair, I've glossed over something….

To play this game in (2,2), I want the geometry to be either Fano or CY, to avoid `discrete Coulomb vacua.'

If the geometry is, say, negatively curved, then the correct intermediate scale description has extra `dust',
and the correct mathematical application is more complicated.

I'll suppress this level of detail in what follows.

(Jia, ES, Wu, '14)

A prediction, in 2d, (2,2) susy:

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

$\longleftrightarrow$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

RG $\downarrow$ RG $\downarrow$

G(2,4) = degree 2 hypersurface in $\mathbb{P}^5$

$$= \mathbb{C}^{2\cdot 4} // GL(2) \qquad = \{z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}\} \subset \mathbb{C}^6 // \mathbb{C}^\times$$

RG $\downarrow$

…

The physical duality implied at top relates abelian & nonabelian gauge theories, which in 4d for ex would be surprising.

Another prediction **((0,2) susy)**

U(2) gauge theory	$\longleftrightarrow$	U(1) gauge theory
4 chirals in fundamental		6 chirals charge +1
1 Fermi in (-4,-4) (hypersurface)		2 Fermi's charge -2, -4
8 Fermi's in (1,1) (gauge bundle E)		8 Fermi's charge +1
1 chiral in (-2,-2) (gauge bundle E)		1 chiral charge -2
2 chirals in (-3,-3) (gauge bundle E)		2 chirals charge -3
plus superpotential		plus superpotential

$\Big\downarrow$ RG $\qquad\qquad$ $\Big\downarrow$ RG

Bundle

$$0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$$

on the CY G(2,4)[4].

=

Bundle

$$0 \to E \to \oplus^8 O(1) \to O(2) \oplus^2 O(3) \to 0$$

on the CY $\mathbb{P}^5[2,4]$

- both satisfy anomaly cancellation
- elliptic genera match

Further predictions ((2,2) susy)

U(2) gauge theory,
n chirals in fundamental

$\longleftrightarrow$

U(n-2)xU(1) gauge theory,
n chirals X in fundamental of U(n-2),
n chirals P in antifundamental of U(n-2)
(n choose 2) chirals $z_{ij} = - z_{ji}$
each of charge +1 under U(1),
W = tr PAX

RG

RG

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(using description of Pfaffians of
Hori '11, Jockers et al '12)

In this fashion, straightforward to generate examples;
let's move on.....

Triality ((0,2) susy)

(Gadde-Gukov-Putrov ’13-’14)

GGP proposed that *triples* of (0,2) GLSM’s might flow to the same IR fixed point.

In terms of lower-energy NLSM’s, the theories are

Gauge bundle $\longrightarrow$ Target space

$$S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2 \longrightarrow G(k,n)$$

$$S^{2k+A-n} \oplus (Q^*)^n \oplus (\det S^*)^2 \longrightarrow G(n-k,A)$$

$$S^n \oplus (Q^*)^A \oplus (\det S^*)^2 \longrightarrow G(A-n+k, 2k+A-n)$$

related by permuting 3 of flavor symmetries.

Susy unbroken iff geometric description above valid.

However, triality is **not** merely a geometric equivalence….

Gadde-Gukov-Putrov triality ('13) **((0,2) susy)**

bundle → space

$$S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \quad \overset{\text{phase}}{\cdots\cdots} \quad (S^*)^A \oplus (Q^*)^n \to G(k,2k+A-n)$$

$$\updownarrow =$$

$$(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \quad \overset{\text{phase}}{\cdots\cdots} \quad (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k,A)$$

$$\updownarrow =$$

$$S^n \oplus (Q^*)^A \to G(A-n+k,2k+A-n) \cdots (S^*)^n \oplus (Q^*)^{2k+A-n} \to G(A-n+k,A)$$

$$\updownarrow =$$

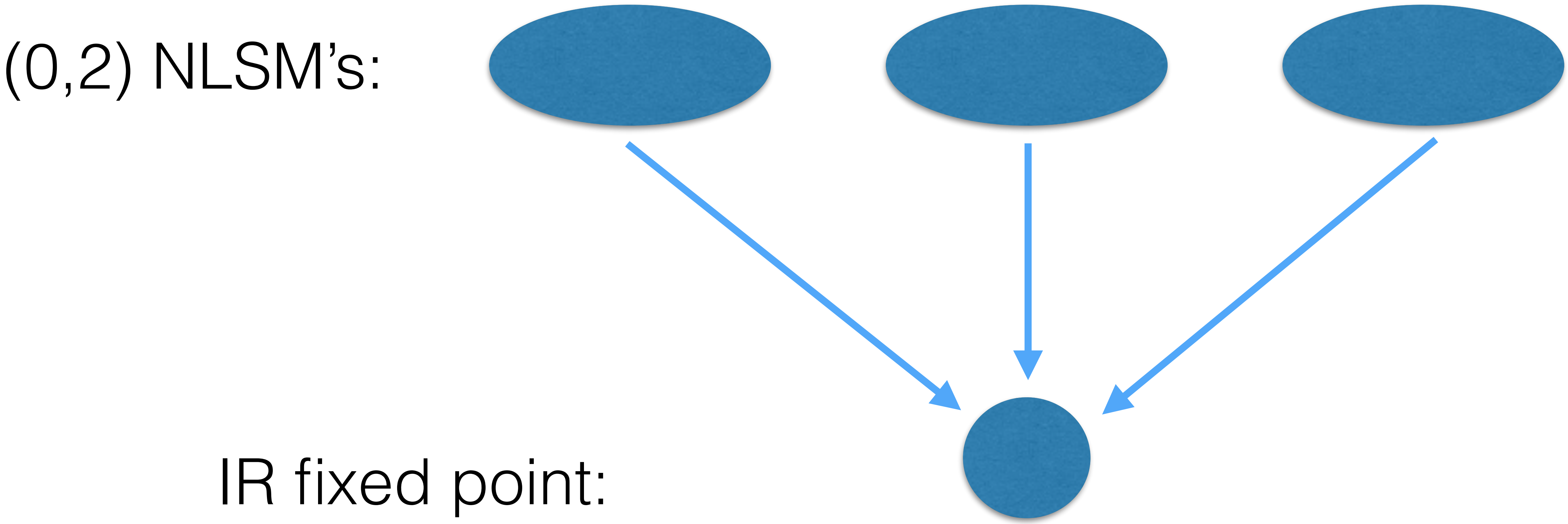
$$(Q^*)^n \oplus S^A \to G(k,2k+A-n) \quad \overset{\text{phase}}{\cdots\cdots} \quad (Q)^{2k+A-n} \oplus S^A \to G(k,n)$$

For brevity, I've omitted writing out the (0,2) gauge theory.

Utilizes another duality: NLSM(X,E) = NLSM(X,E*)

Though related, these spaces & bundles not all the same.

Triality predicts



IR SCFT = (left-moving Kac-Moody) $\otimes$ (rt-moving Kazama-Suzuki)

UV global $\quad SU(n) \times SU(A) \times SU(2k+A-n) \times SU(2)$

(present in GLSM & each NLSM)

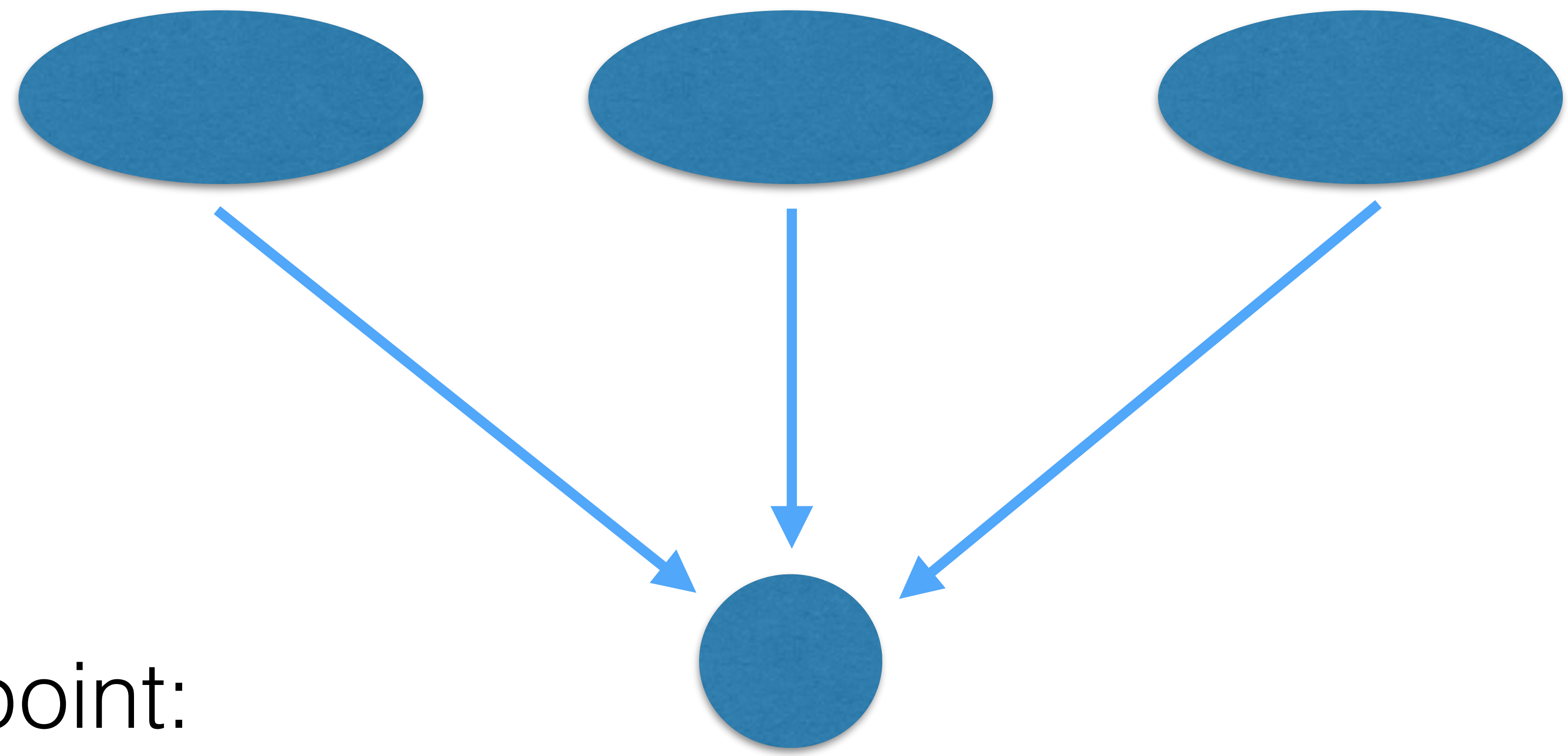
enhanced in IR to affine

$$SU(n)_{k+A-n} \times SU(A)_k \times SU(2k+A-n)_{n-k} \times SU(2)_1$$

Chiral states should live in integrable reps of affine algebras.

Let’s study triality, using chiral rings.

(0,2) NLSM’s:



IR fixed point:

Plan: Compute chiral states in each theory and compare.

Community expectation: (0,2) chiral rings should match.

Alas, not quite so simple….

Subtleties in comparing chiral states:

- Q*-cohomology in large-radius (0,2) NLSM invariant under perturbative corrections, but, here RG flow goes to strong coupling — states might enter/leave.

We'll see exactly that —
not all states will match between different presentations,
but, states that don't match, shouldn't be in IR either.

In fact, this is generic behavior expected in QFT for non-protected states.
What's surprising here is that it happens in (0,2) chiral rings — not widely expected in the (0,2) community — and triality provides clean examples demonstrating this behavior.

Subtleties in comparing chiral states:

- Chiral ring computations in 2d KS models not under good control; Lie algebra cohomology is part of answer.

We'll focus on comparing states across UV presentations, then, merely outline in general terms how form of Lie algebra cohomology is appropriate.

Example 1:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^2$$

$r \ll 0:$

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^1$$

$$U \;=\; \mathbb{C}^3, \quad V \;=\; \mathbb{C}^2, \quad W \;=\; \mathbb{C}^2, \quad \tilde{V} \;=\; \mathbb{C}^3$$

Let’s compare states in these two phases
(= 2 of 3 triality-related geometries)….

Example 1:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^2$$

Compute states:

$$H^\bullet(\mathbb{P}^2, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_{\mathbb{P}^2}^{+1/2})$$

Global symmetries:

$SU(U) \times SU(V) \times SU(W)$ manifest — acts on bundle

$SU(\tilde{V})$ also present:

Compute sheaf cohomology with Bott-Borel-Weil, which gives sheaf cohomology as reps of $U(\tilde{V})$.

These computations are an application of Bott-Borel-Weil, so, brief overview:

For a bundle $\mathcal{E}_\xi$ on G/P defined by rep' ξ of P,
$H^\bullet(G/P, \mathcal{E}_\xi)$ is naturally a rep' of G.

For Grassmannians,
compute $H^\bullet\left(G(k,n), K_{(a_1,\cdots,a_k)}S^* \otimes K_{(b_1,\cdots,b_{n-k})}Q^*\right)$:

$(a_1, \cdots, a_k)$ rep' of U(k) $\qquad a_1 \geq a_2 \geq \cdots \geq a_k$

$(b_1, \cdots, b_{n-k})$ rep' of U(n-k) $\qquad b_1 \geq b_2 \geq \cdots \geq b_{n-k}$

'Mutate' $(a_1, \cdots, a_k, b_1, \cdots, b_{n-k})$ to $(c_1, \cdots, c_n)$ rep of U(n)

$$H^\bullet\left(G(k,n), K_{(a_1,\cdots,a_k)}S^* \otimes K_{(b_1,\cdots,b_{n-k})}Q^*\right) = K_{(c_1,\cdots,c_n)}V^*$$

for $\bullet$ = number of mutations, & zero in other degrees.

Bott-Borel-Weil, cont'd

Ex: $H^\bullet(G(k,\tilde{V}^*), U \otimes S^*)$

$$= U \otimes H^\bullet(G(k,\tilde{V}^*), K_{(1,0,\cdots,0)} S^* \otimes K_{(0,0,\cdots,0)} Q^*)$$

$$= U \otimes K_{(1,0,\cdots,0)} \tilde{V}\, \delta^{\bullet,0} = U \otimes \tilde{V}\, \delta^{\bullet,0}$$

Constraints on results:

- Invariance under Serre duality

$$H^{\bullet}(X,\mathcal{E}) \;=\; H^{\mathrm{dim}-\bullet}(X,\mathcal{E}^* \otimes K_X)^*$$

Should map state spectrum into itself, dualizing representation.

- Integrability of representations

GGP triality predicts that states should live in `integrable' rep's.

$SU(n)_k$: integrable reps have Young tableaux of width $\leq$ k

Let's look at some states in the first example….

Examples of states shared between two phases:

SU(3)xSU(2)xSU(2)xSU(3)	U(1)^3
(1,1,1,1)	(+3,0,-3)
(1,1,2,3)	(+2,-1/2,-3/2)
(1,1,1,3*)	(+1,+2,-3)
(3,2,1,1)	(+2,0,-2)
(3,1,2,1)	(-2,-3/2,-1/2)
(1,2,2,3*)	(+1,+1/2,-3/2)
(3,1,1,3)	(+1,+1,-2)
...	...
(3*,1,1,3*)	(-1,-1,+2)
(1,2,2,3)	(-1,-1/2,+3/2)
(3*,1,2,1)	(-2,+3/2,+1/2)
(3*,2,1,1)	(-2,0,+2)
(1,1,1,3)	(-1,-2,+3)
(1,1,2,3*)	(-2,+1/2,+3/2)
(1,1,1,1)	(-3,0,+3)

Serre duals

Integrable reps of $SU(3)_1 \times SU(2)_2 \times SU(2)_1 \times SU(3)_1$

Non-shared states in $r \gg 0$ phase:

wedge	coh' degree	SU(3)xSU(2)xSU(2)xSU(3)	U(1)³
2	0	**(1,1,1,6)**	(+1,-1,0)
3	0	**(1,2,1,8)**	(0,0,0)
4	0	**(1,1,1,6*)**	(-1,+1,0)
5	2	**(1,1,1,6)**	(+1,-1,0)
6	2	**(1,2,1,8)**	(0,0,0)
7	2	**(1,1,1,6*)**	(-1,+1,0)

Serre duals

Matching rep'

Non-integrable rep' of $SU(3)_1 \times SU(2)_2 \times SU(2)_1 \times SU(3)_1$

- All states come in Serre dual pairs
- Rep's are non-integrable — should not survive to IR
- States come in pairs with matching rep's (so cancel out of elliptic genera)

Example 2:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^1$$

$r \ll 0:$

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^3$$

$$U \;=\; \mathbb{C}^4, \quad V \;=\; \mathbb{C}^4, \quad W \;=\; \mathbb{C}^2, \quad \tilde{V} \;=\; \mathbb{C}^2$$

Let's compare states in these two phases
(= 2 of 3 triality-related geometries)….

Examples of states shared between two phases:

SU(4)xSU(4)xSU(2)xSU(2)	U(1)^3
(1,1,1,3)	(+4,0,-4)
(4,1,1,2)	(+4,-1,-3)
(1,4,1,2)	(-3,+1,-4)
(1,1,2,4)	(+2,0,-2)
(6,1,1,1)	(+4,-2,-2)
(4,4,1,1)	(+3,0,-3)
(4,1,2,3)	(+2,-1,-1)
…	…
(4*,1,2,3)	(-2,+1,+1)
(4*,4*,1,1)	(-3,0,+3)
(6,1,1,1)	(-4,+2,+2)
(1,1,2,4)	(-2,0,+2)
(1,4*,1,2)	(-3,-1,+4)
(4*,1,1,2)	(-4,+1,+3)
(1,1,1,3)	(-4,0,+4)

Serre duals

Integrable reps of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

Non-shared states in $r \ll 0$ phase:

wedge	coh' degree	SU(4)xSU(4)xSU(2)xSU(2)	U(1)³
4	0	**(1,10,1,1)**	(+2,-2,0)
4	3	**(1,10,1,1)**	(+2,-2,0)
5	0	**(1,20,1,2)**	(+1,-1,0)
…	…	**…**	…
7	3	**(1,20,1,2)**	(-1,+1,0)
8	0	**(1,10*,1,1)**	(-2,+2,0)
8	3	**(1,10*,1,1)**	(-2,+2,0)

Serre duals

Matching rep'

Matching rep'

Non-integrable rep' of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

- All states come in Serre dual pairs
- Rep's are non-integrable — should not survive to IR
- States come in pairs with matching rep's (so cancel out of elliptic genera)

So far, I've compared chiral states in two phases of one GLSM, corresponding to 2 of the 3 geometries related by triality.

We can perform the same analysis in phases of other GLSM's, describing geometries related by triality to the two above.

We find the same results:

- There is a set of states shared between all geometries related by triality, falling in integrable representations
- There are non-shared states, in non-integrable representations, and which cancel out of elliptic genera.

So far, only discussed one example of a triple, but the same pattern appears in other examples….

Example 2:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^1$$

$r \ll 0:$

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^3$$

$$U \;=\; \mathbb{C}^4, \quad V \;=\; \mathbb{C}^4, \quad W \;=\; \mathbb{C}^2, \quad \tilde{V} \;=\; \mathbb{C}^2$$

Let's compare states in these two phases
(= 2 of 3 triality-related geometries).

We'll find closely analogous results.

Examples of states shared between two phases:

SU(4)xSU(4)xSU(2)xSU(2)	U(1)³
(1,1,1,3)	(+4,0,-4)
(4,1,1,2)	(+4,-1,-3)
(1,4,1,2)	(-3,+1,-4)
(1,1,2,4)	(+2,0,-2)
(6,1,1,1)	(+4,-2,-2)
(4,4,1,1)	(+3,0,-3)
(4,1,2,3)	(+2,-1,-1)
...	...
(4*,1,2,3)	(-2,+1,+1)
(4*,4*,1,1)	(-3,0,+3)
(6,1,1,1)	(-4,+2,+2)
(1,1,2,4)	(-2,0,+2)
(1,4*,1,2)	(-3,-1,+4)
(4*,1,1,2)	(-4,+1,+3)
(1,1,1,3)	(-4,0,+4)

Serre duals

Integrable reps of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

Non-shared states in $r \ll 0$ phase:

wedge	coh' degree	SU(4)xSU(4)xSU(2)xSU(2)	U(1)³
4	0	**(1,10,1,1)**	(+2,-2,0)
4	3	**(1,10,1,1)**	(+2,-2,0)
5	0	**(1,20,1,2)**	(+1,-1,0)
…	…	**…**	…
7	3	**(1,20,1,2)**	(-1,+1,0)
8	0	**(1,10*,1,1)**	(-2,+2,0)
8	3	**(1,10*,1,1)**	(-2,+2,0)

Serre duals

Matching rep'

Matching rep'

Non-integrable rep' of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

- All states come in Serre dual pairs
- Rep's are non-integrable — should not survive to IR
- States come in pairs with matching rep's (so cancel out of elliptic genera)

So far in example 2,
we've compared states between 2 of 3 triality geometries.

If we compare other pairs of the 3,
we get analogous results.

We've seen that the between geometries that should flow to same fixed point, the chiral states don't all match,
but,
the ones that don't, also have nonintegrable reps,
and make no net contribution to refined elliptic genera.

We believe that they get a mass and disappear from RG flow.

The fact that the remaining states are both

- shared between phases, and
- in integrable reps of proposed IR symmetry algebras,

serves as a check of triality.

In hindsight, we should’ve expected this.

These (0,2) chiral rings do not have the same topological protection as (2,2) chiral rings,
hence,
in general (pairs of) states should be able to
enter & leave RG flow.

However, within the (0,2) community, we’ve implicitly assumed that (0,2) chiral rings were somehow protected,
and GGP’s triality provides clear counterexamples.

How in principle might these UV sheaf cohomology groups relate, in general, to the IR states?

In IR, expect states ~ Lie algebra cohomology.
[*roughly* — correspondence incomplete] (W Lerche, private communication)

How is that related?

We won't pursue this in detail, but,
want to observe that another flavor of BBW provides the missing link:

sheaf cohomology $$H^\bullet(G/P, \mathcal{E}_\xi)_\lambda \;=\; H^\bullet(\mathfrak{n}, V_\lambda)_\xi$$ Lie algebra cohomology

λ a representation of G

ξ a representation of P

$$\mathfrak{p} \;=\; (\mathrm{Levi}) \;+\; \mathfrak{n}$$

Example 3:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^1$$

$r \ll 0:$

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^1$$

$$U \;=\; \mathbb{C}^2, \quad V \;=\; \mathbb{C}^2, \quad W \;=\; \mathbb{C}^2, \quad \tilde{V} \;=\; \mathbb{C}^2$$

We can compare states as before —
the geometries are identical,
but the global symmetries have rotated.

Results follow same pattern:
matching states in integrable reps,
non-matching states cancel out of indices.

Example 3, cont'd

Something new here happens in IR:
($SU(2)_1)^4$ believed to be enhanced to $(E_6)_1$

(GGP '13)

One can show that the matching chiral states fill out
the $\mathbf{27}$, $\overline{\mathbf{27}}$ of E_6:

$$\begin{aligned}\mathbf{27} \;=\; &(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \;+\; (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \;+\; (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) \;+\; (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\ &+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) \;+\; (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \;+\; 3(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\end{aligned}$$

consistent with GGP's predictions.

Math conjecture:

The shared states, the sheaf cohomology that survives to IR, should define some sort of `stable sheaf cohomology.'

Stable under `physics homotopy' = RG flow

Conversely, in 2d physical theories with a continuous global symmetry, there is a weak test for a nontrivial IR limit:

isotypic components of certain indices in nonintegrable representations should vanish.

Summary:

- Chiral states in 2d (0,2) NLSM's
- Product structures in chiral rings in A/2, B/2 twists:

  quantum sheaf cohomology

- Nonabelian GLSMs:
  - Dualities in 2d and their geometry
  - Gadde-Gukov-Putrov triality

Thank you for your time!