Heterotic Mirror Symmetry

Eric Sharpe

Physics Dep't, Virginia Tech

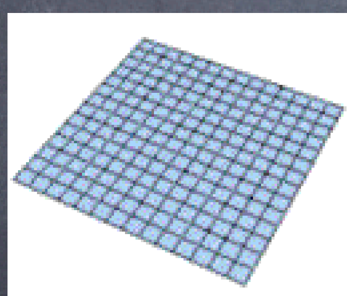
Drexel University Workshop on Topology and Physics

September 8-9, 2008

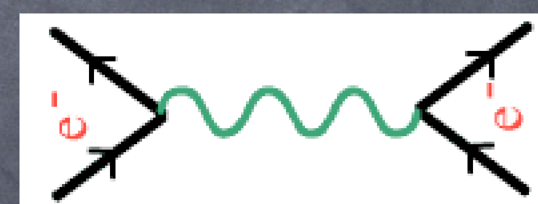
This will be a talk about string theory, so lemme motivate it....

Twentieth-century physics saw two foundational advances:

General relativity
(special relativity)



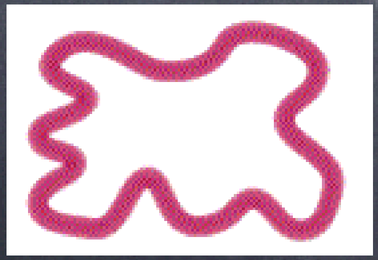
Quantum field theory
(quantum mechanics)

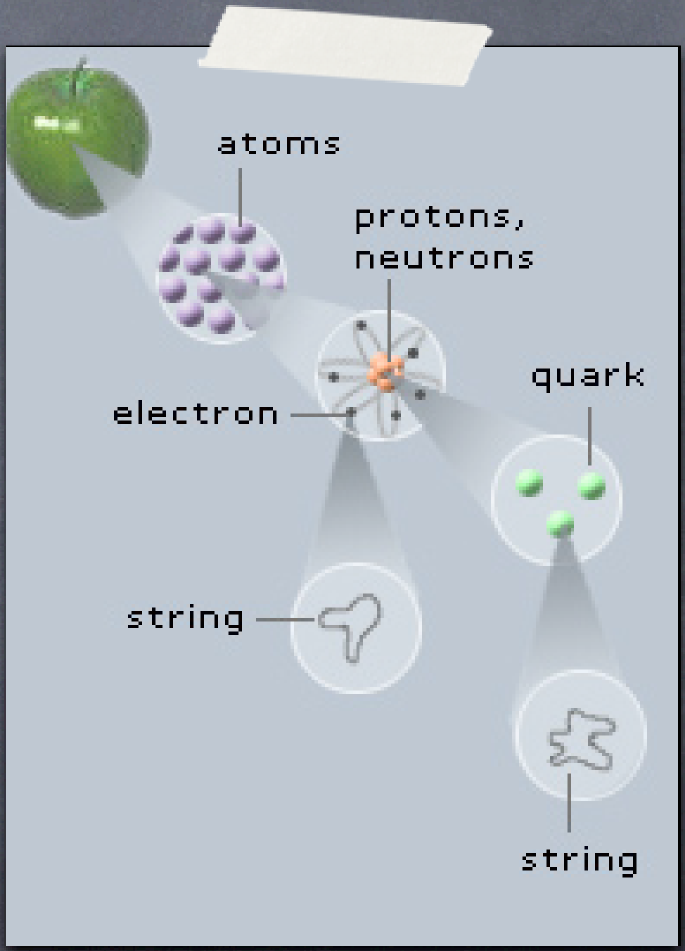


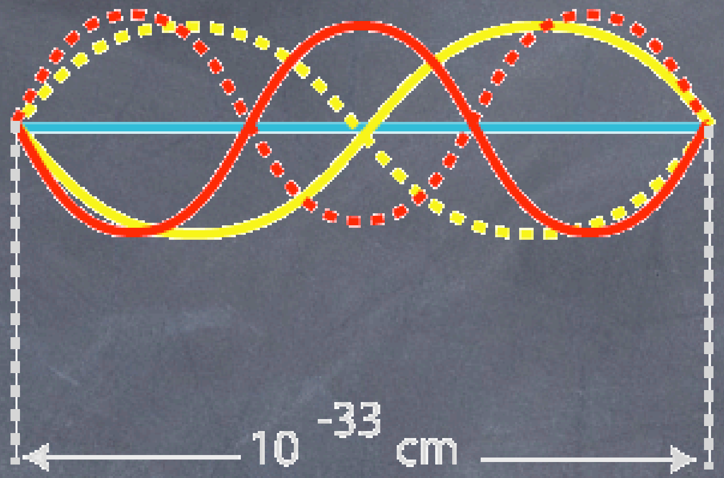
Problem: They contradict each other!

String theory...

... is a physical theory that reconciles GR & QFT, by replacing elementary particles by strings.





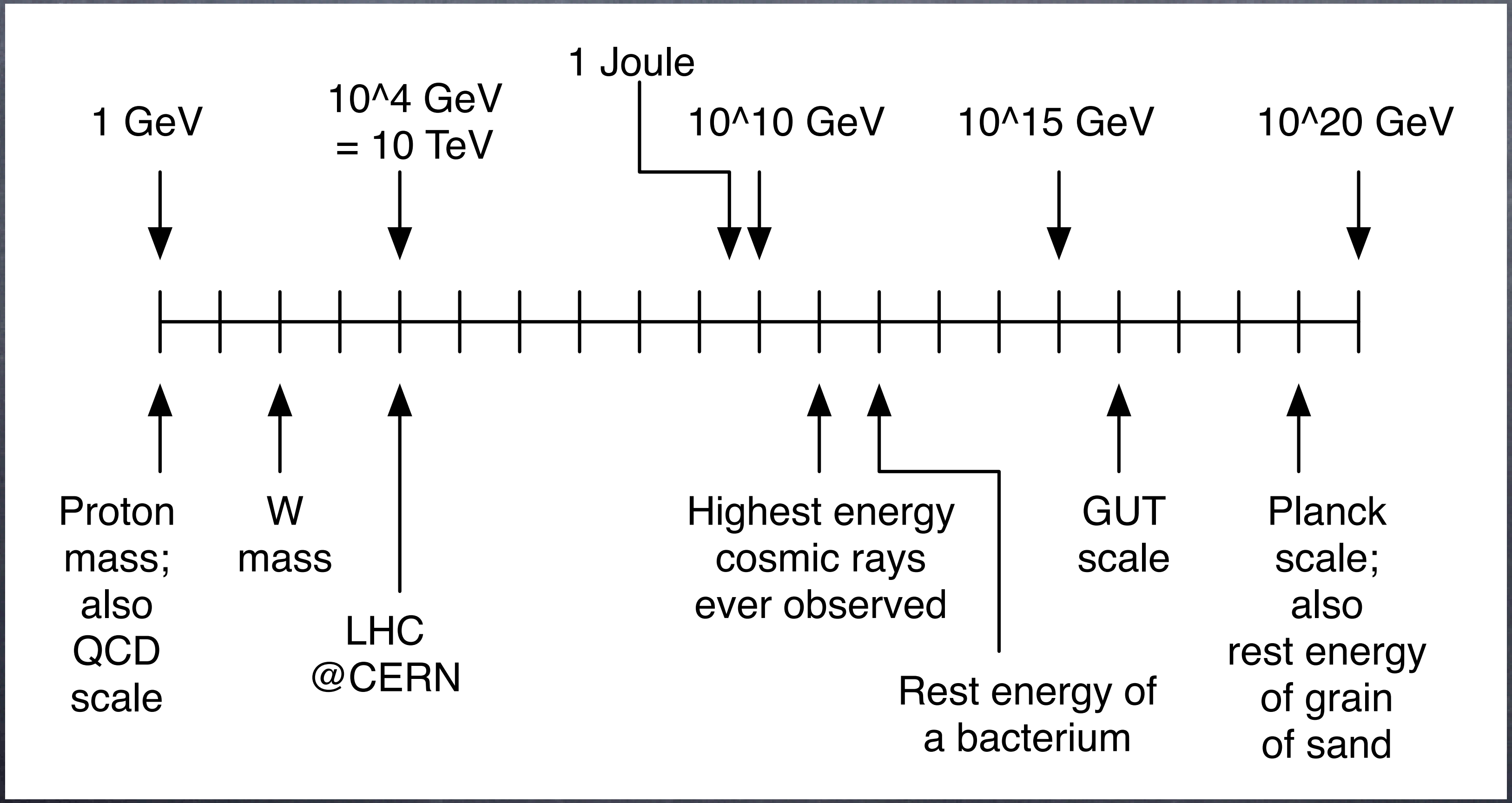


The typical sizes of the strings are very small -- of order the Planck length. To everyday observers, the string appears to be a pointlike object.

From dim'l analysis,

Planck energy = $(h\ c^5 / G)^{1/2} \sim 10^{19}$ GeV

How big is that?



String theory predicts the universe is actually ten-dimensional.

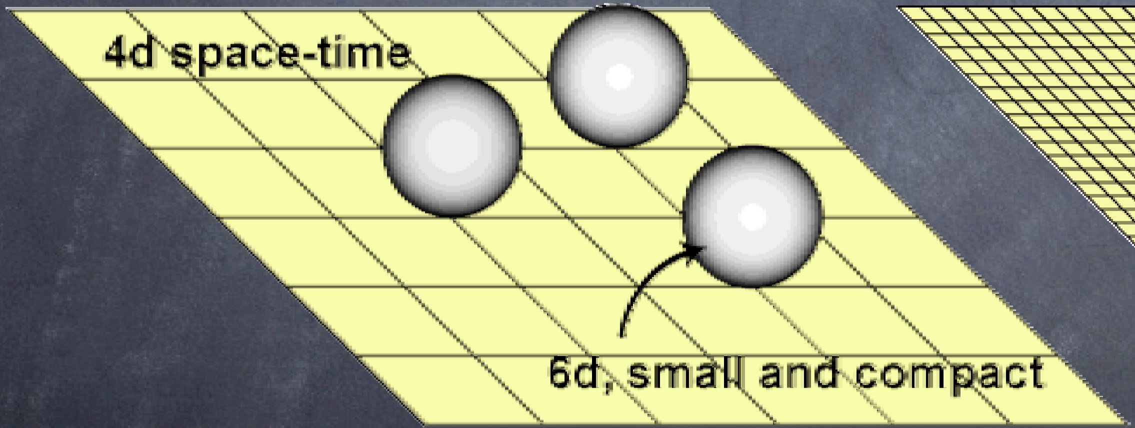
But, we only see 3 space dims + 1 time dim.

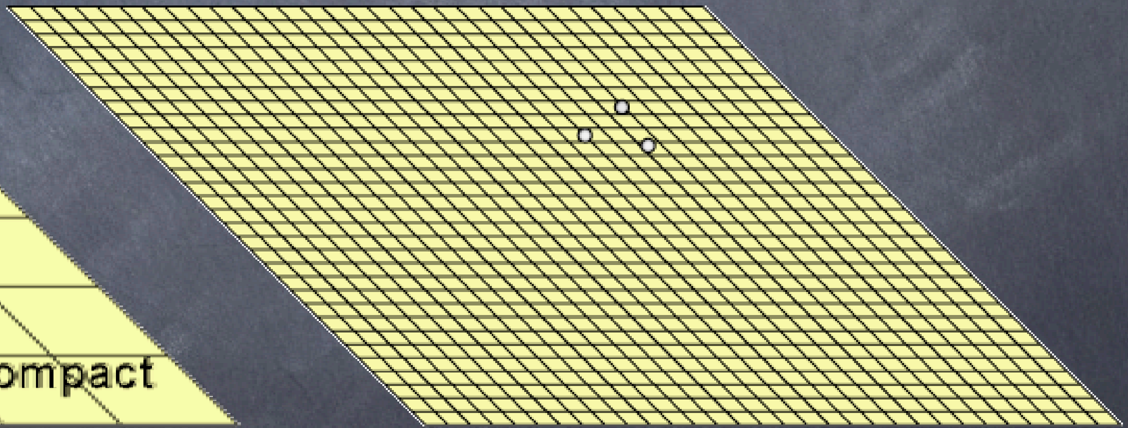
The other 6 dims are believed to be rolled up on a `small' compact space.

10D spacetime = $\mathbf{R}^4$ x (6-manifold)

Compactification scenario

Assume 10d spacetime has form $\mathbf{R}^4 \times M$ where M is some (small) (compact) 6d space





So long as you work at wavelengths much larger than the size of the compact space, spacetime looks like $\mathbf{R}^4$.

Properties of the `internal' 6 dim space determine features of the resulting 4 dim universe.

Ex: light 4 dim particles counted by, additive part of de Rham cohomology ring of 6-mfld

Ex: couplings between those particles determined by product structure of cohomology ring of 6-mfld

& more

In short, learn about physics by studying mathematical structure of the 6-manifold.

I’ve just told you why math is interesting to physicists,
but the reverse has also turned out to be true:

Thinking about the resulting physics has led to new mathematics, which is what I’ll outline today.

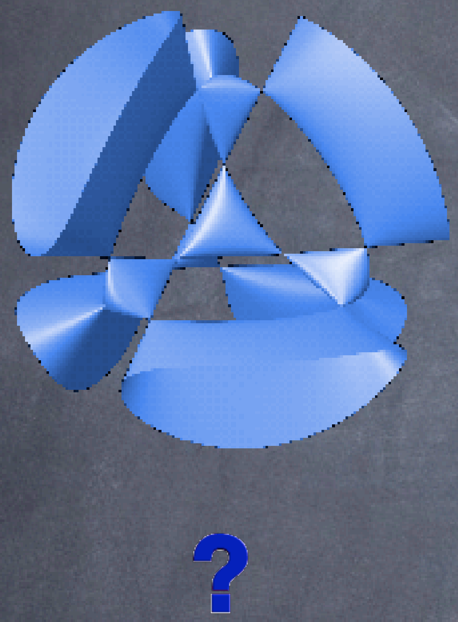
Outline:

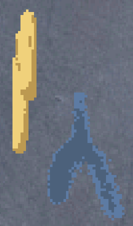
- Review of ordinary mirror symmetry (Greene, Plesser, Morrison, Aspinwall, Candelas, de la Ossa, Berglund, Hubsch, Vafa, Hori, Givental, Yau, ...)

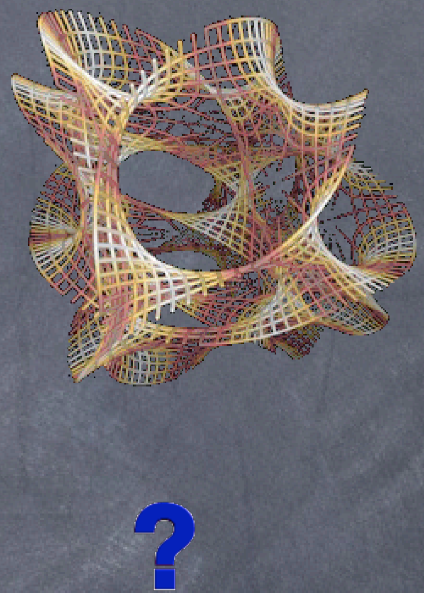
- Heterotic mirror symmetry (Blumenhagen, Sethi, Adams, Basu, ES, Guffin, Clarke, ...)

- Landau-Ginzburg models & the renormalization group

Mirror symmetry







Sometimes strings can't distinguish two spaces....
.... such spaces are called mirrors

This turns out to have fun math applications....

Mirror symmetry

What sorts of spaces can be mirror?

Usually we mean, complex Kahler manifolds with holomorphically-trivializable canonical bundle.

(= special Ricci-flat Riemannian mflds)

These are ``Calabi-Yau" manifolds.

Exs: T^2, quartic hypersurface in $\mathbf{P}^3$,
quintic hypersurface in $\mathbf{P}^4$

Mirror symmetry

When two Calabi-Yau mflds M, W are mirror,
they turn out to be very closely related.
(but topologically distinct)

Ex: dim M = dim W

After all, if strings are unable to distinguish one from the other, then the compactified theory should be the same
-- in particular, the dimension of the compactified theory had better not change

Mirror symmetry

Since the spectrum of light 4 dim particles is determined by de Rham cohomology, we can conclude that

$$\sum \dim H^*_{dR}(M) = \sum \dim H^*_{dR}(W)$$

where $H^n_{dR}(M)$ = (closed deg n diff' forms)/(exact)

Mirror symmetry

A refinement of the last statement exists.

On a cpx Kahler mfld, we can decompose the space of deg n diff' forms

$$b_{i_1 \cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

into (p,q) forms

$$c_{a_1 \cdots a_p \overline{a}_1 \cdots \overline{a}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\overline{z}^{\overline{a}_1} \wedge \cdots \wedge d\overline{z}^{\overline{a}_q}$$

Mirror symmetry

For M a cpx mfld, we can define a group $H^{p,q}(M)$ consisting of the (p,q) differential forms on M (closed mod exact), and for M a cpx Kahler mfld,

$$\dim H^n(M) = \sum_{p+q=n} \dim H^{p,q}(M)$$

Mirror symmetry

The reason I'm mentioning all this is that one of the basic properties of mirror symmetry is that it exchanges (p,q) differential forms with (n-p,q) differential forms

(n = cpx dim)

$$\dim H^{p,q}(M) = \dim H^{n-p,q}(W)$$

Mirror symmetry

The dimensions of the spaces of (p,q) forms can be organized more neatly into ``Hodge diamonds."

For ex, for cpx dim 2, this is

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

Mirror symmetry acts as a rotation about diagonal

Example: T^2

T^2 is self-mirror topologically;
cpx, Kahler structures interchanged

$h^{0,1}$ ↗ (cpx) $h^{1,1}$ ↖ (Kahler)

Hodge diamond:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

Note this symmetry is
specific to genus 1;
for genus g:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Example: Quartics in $\mathbf{P}^3$

(known as K3 mflds)

K3 is self-mirror topologically;

cpx, Kahler structures interchanged

$h^{1,1}$ ↗ ↖ $h^{1,1}$

Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Kummer surface

$$\left(x^2 + y^2 + z^2 - aw^2\right)^2 - \left(\frac{3a-1}{3-a}\right) pqts = 0$$

$$p = w - z - \sqrt{2}x$$

$$q = w - z + \sqrt{2}x$$

$$t = w + z + \sqrt{2}y$$

$$s = w + z - \sqrt{2}y$$

$$a = 1.5$$

Example: Quintic in $\mathbb{P}^4$

The quintic (deg 5) hypersurface in $\mathbb{P}^4$
is mirror to
(res'n of) a deg 5 hypersurface in $\mathbb{P}^4/(\mathbb{Z}_5)^3$

Quintic

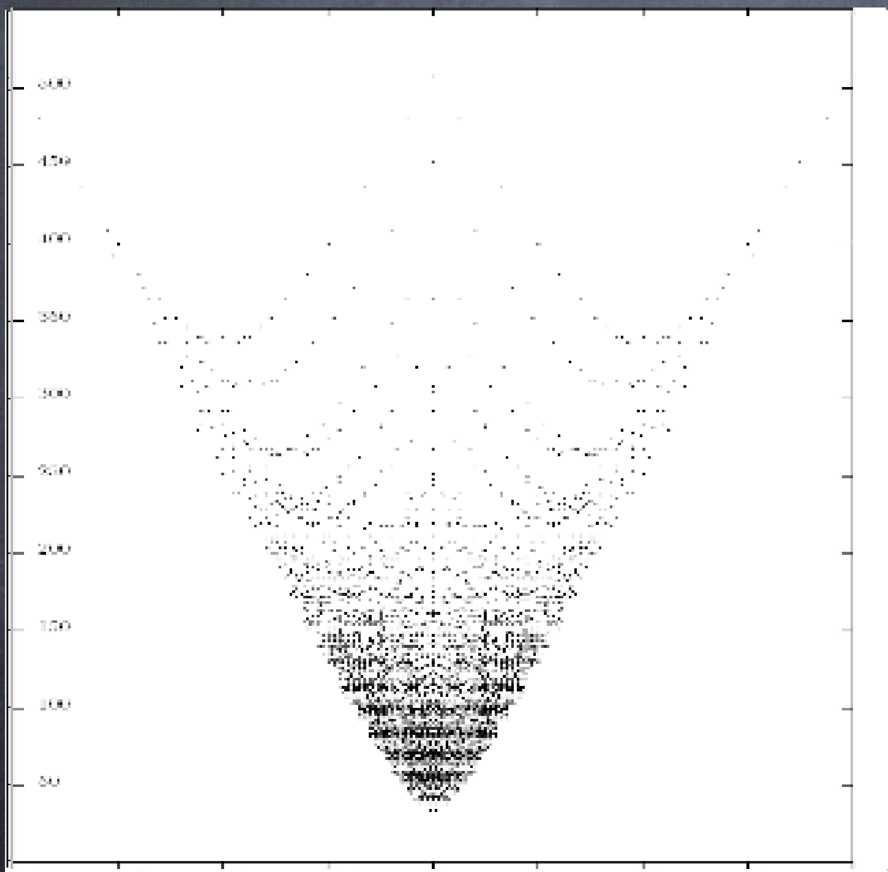
$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Mirror

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

How many Calabi-Yau's ?

Hundreds of thousands of families



Shown are 3-folds:

Vertical axis: $h^{1,1} + h^{2,1}$

Horizontal axis: $2(h^{1,1} - h^{2,1})$

= 2 (# Kahler - # cpx defs)

Mirror symm'

==> symm' across vert' axis

-- numerical evidence for mirror symmetry

(Klemm, Schimmrigk, NPB 411 ('94) 559-583)

How to find mirrors?

One of the original methods:
"Greene-Plesser orbifold construction"

$$Q_5 \subset \mathbf{P}^4 \quad \xleftrightarrow{\text{mirror}} \quad \widetilde{Q_5/\mathbf{Z}_5^3}$$

but only applicable in relatively special cases

How to find mirrors?

Batyrev's construction:

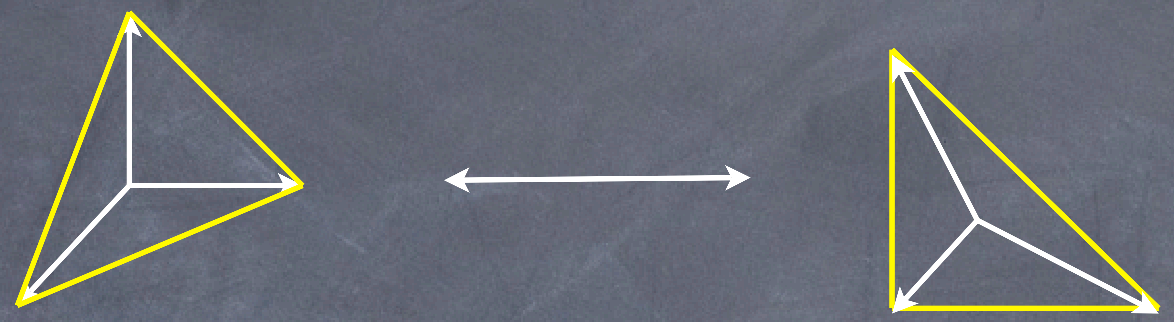
For a hypersurface in a toric variety,
mirror symmetry exchanges

polytope of ambient toric variety	$\longleftrightarrow$	dual polytope, for ambient t.v. of mirror

Example of Batyrev's construction:

T^2 as deg 3 hypersurface in $\mathbf{P}^2$

$\mathbf{P}^2$:



$= \mathbf{P}^2/\mathbf{Z}_3$

$$P^0 = \{y \,|\, \langle x, y\rangle \geq -1 \, \forall x \in P\}$$

Result:

deg 3 hypersurface in $\mathbf{P}^2$

mirror to

$\mathbf{Z}_3$ quotient of deg 3 hypersurface

(Greene-Plesser)

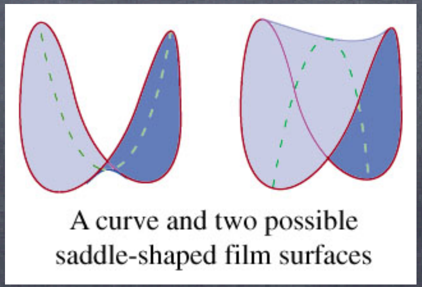
Enumerative geometry

Mirror symmetry exchanges:

classical computations on M

↕

sums over minimal area (holomorphic) curves on W



-- In other words, mirror symmetry makes predictions for mathematics

Deg k	n_k
1	2875
2	609250
3	317206375

Shown: numbers of rat'l curves in the quintic in $\mathbf{P}^4$, of fixed degree.

These three degrees were the state-of-the-art before mirror symmetry (deg 2 in '86, deg 3 in '91)

Then, after mirror symmetry, the list expanded...

Deg k	n_k
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
...	...

So, physics makes (lots of!) math predictions.

Understanding the nature of these calculations, and turning them into rigorous mathematics was an industry in the algebraic geometry community for several years.

``Gromov-Witten"

``Gopakumar-Vafa"

``Donaldson-Thomas"

Conversely:

The predictions that string theory makes for enumerative geometry gives physicists a kind of experimental test:

by checking whether its predictions are true, we learn whether string theory is self-consistent.

Why holomorphic curves?

To explain this, I need to describe a tiny bit of physics.

When I speak of strings propagating on spaces, what I'm secretly thinking of are 2d ``quantum field theories."

In a quantum field theory, one calculates `correlation functions,' closely analogous to correlation functions in statistics:

$$< fg >= \sum_{events} \mathrm{prob}(event)\, f(event)\, g(event)$$

In string theory, we also calculate correlation functions:

$$< fg >= \int [D\phi] \exp(iS(\phi)) f(\phi) g(\phi)$$

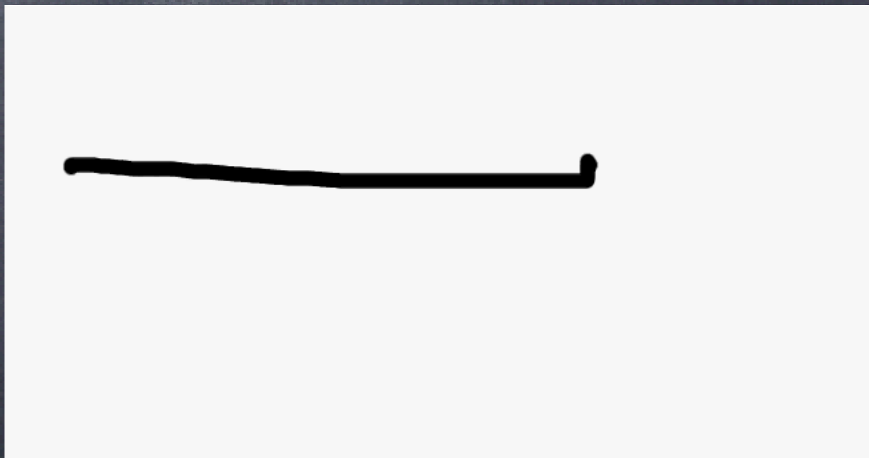
where the ϕ are maps from a Riemann surface into the space

Cultural aside

The real reason I'm talking about sums over maps is b/c this is the stringy version of quantum mechanics.

One description of ordinary quantum mechanics is as a sum over maps from the `worldline' of a particle into the space.

That sum over maps is weighted by phases; dominant contribution from classical paths.



In the string sum over maps, minimal-area curves in the target space play a special role.

These can be described by holomorphic maps.

For certain special correlation functions, the (ill-defined) path integral reduces to an integral over a moduli space of holomorphic maps:

$$< fgh >= \sum_d \int_{\mathcal{M}_d} \exp(-d(\text{Area})) fgh$$

(= A model TFT correlation f'ns)

The fact that an (ill-defined) sum over all maps reduces to something that looks nearly well-defined is a consequence of a physical property of the theory called ``supersymmetry,” as a result of which most fluctuations cancel each other out, leaving only contributions from zero-energy (minimal-area) curves.

Technical complications:

To make sense of expressions involving integrals over moduli spaces,
the moduli spaces need to be compact.

Problem is, they're not.

Ex: Space of deg 1 hol' maps $\mathbf{P}^1 \to \mathbf{P}^1$ is SL(2,**C**)

So, part of the story here involves compactifying moduli spaces.

Ex: SL(2,**C**) -> $\mathbf{CP}^3$ as, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [a, b, c, d]$

We can calculate correlation f'n for cpx Kahler mflds that aren't necessarily Calabi-Yau.

Example: **CP**^{N}

For degree d maps, $\overline{\mathcal{M}}_d = \mathbf{CP}^{(N+1)(d+1)-1}$

$$< x^k >_d \;=\; \int_{\overline{\mathcal{M}}_d} (\text{deg } 2k \text{ form})$$

$$= \begin{cases} q^d & k = \dim_{\mathbf{C}} \overline{\mathcal{M}}_d \\ & = N + d(N+1) \\ 0 & \text{else} \end{cases}$$

where $q = \exp(-\text{Area})$

Quantum cohomology

The results of the previous calculation can be summarized more compactly.

$$< x^{d(N+1)} x^N > = q^d$$

x is identified with generator of $H^2(\mathbf{CP}^N, \mathbf{C})$

so if we identify $\quad x^{N+1} \sim q$

then we can recover the correlation f'ns above from

$$< x^N > = \int_{\mathbf{CP}^N} x^N = 1$$

Quantum cohomology

In effect, we can encode the sum over holomorphic maps in a deformation of the classical cohomology ring,
known as the ``quantum cohomology ring."

Classical cohomology ring for $\mathbf{CP}^N$: $C[x]/(x^{N+1} - 0)$

Quantum cohomology ring for $\mathbf{CP}^N$: $C[x]/(x^{N+1} - q)$

In limit area -> infinity, q -> 0,
=> quantum -> classical

Quantum cohomology

Since I described curve counting in the quintic earlier....

For the quintic hypersurface in $\mathbf{CP}^4$, the quantum cohomology ring is almost the same as the classical cohomology ring, except that a rel'n is modified:

$H^2 = (F_0)\, L$ H hyperplane class

L a line

$F_0 = 5 + 2875q + \ldots$

($F_0 = 5$ is the classical case)

Ordinary mirror symmetry is pretty well understood nowadays.

Purely mathematical description exists (Givental, Yau et al)

However, there are some extensions of mirror symmetry that are still being actively studied.

One example: heterotic mirror symmetry

Heterotic mirror symmetry

is a conjectured generalization that exchanges pairs

$$(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds
and the $\mathcal{E}_i \to X_i$ are holomorphic vector bundles

Constraints: $\mathcal{E}$ stable, $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

Why ``heterotic" ?
b/c appears in heterotic string theories

Heterotic mirror symmetry

The (2d) quantum field theories defining heterotic strings, include those of other (``type II") string theories as special cases.

Hence, heterotic mirror symmetry ought to reduce to ordinary mirror symmetry in a special case, & that turns out to be when

$$\mathcal{E}_i \cong TX_i$$

Heterotic mirror symmetry

Instead of exchanging (p,q) forms, heterotic mirror symmetry exchanges bundle-valued differential forms (= `sheaf cohomology'):

$$H^j(X_1, \Lambda^i \mathcal{E}_1) \leftrightarrow H^j(X_2, (\Lambda^i \mathcal{E}_2)^\vee)$$

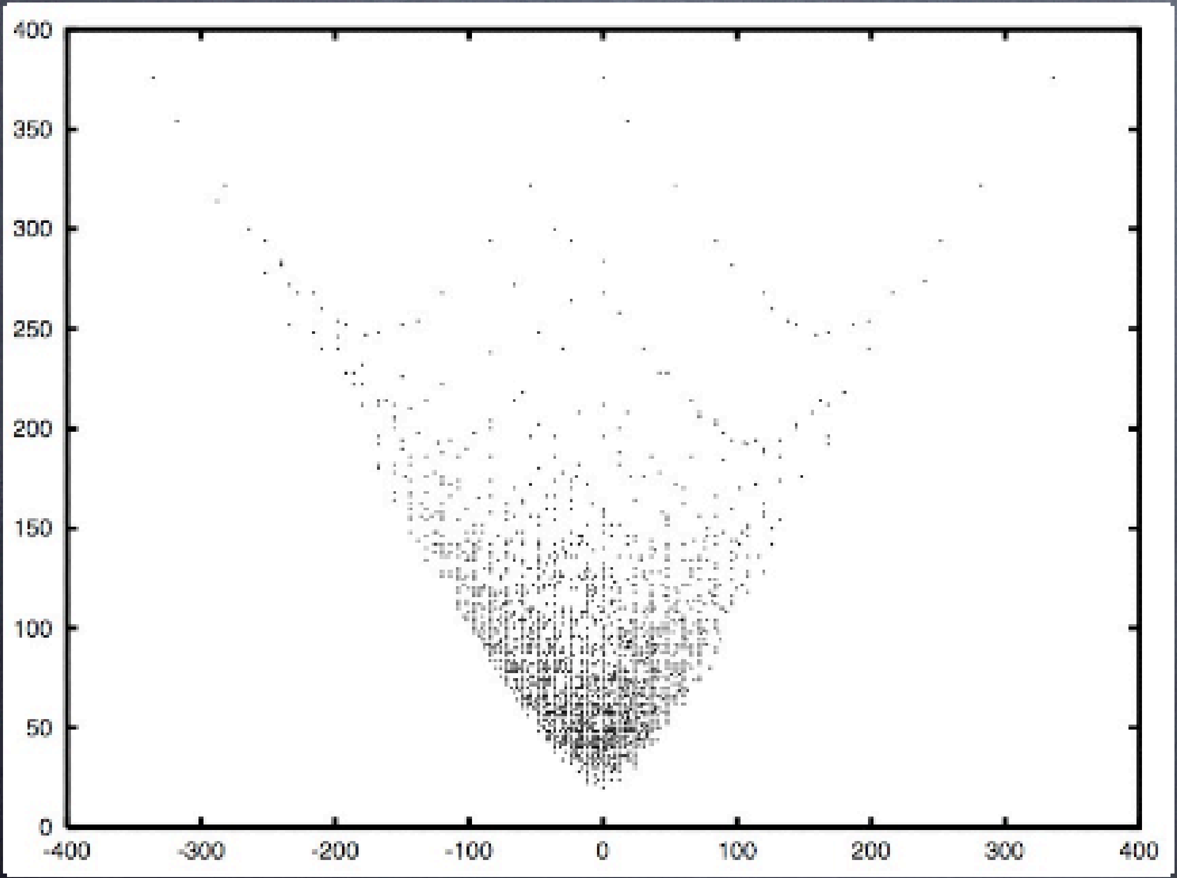
Note when $\mathcal{E}_i \cong TX_i$ this reduces to

$$H^{d-1,1}(X_1) \leftrightarrow H^{1,1}(X_2)$$

(for X_i Calabi-Yau)

Heterotic mirror symmetry

Not much is known about heterotic mirror symmetry, though a few basics have been worked out.



Ex: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^\vee)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^\vee)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

Heterotic mirror symmetry

Mirror constructions:

* an analogue of the Greene-Plesser construction (quotients by finite groups) is known

(Blumenhagen, Sethi, NPB 491 ('97) 263-278)

* but, no known analogue of Batyrev's dual polytopes construction

Heterotic mirror symmetry

Another bit of progress:

Heterotic quantum cohomology rings have been worked out.

(ES, Katz, Sethi, Basu, Guffin, Melnikov, Adams, Distler)

These are a deformation of classical product structures on the groups of bundle-valued differential forms

$$H^{\cdot}(X, \Lambda^{\cdot} \mathcal{E}^{\vee})$$

``quantum sheaf cohomology"

(Combine minimal-area curves & gauge instantons.)

Quantum sheaf cohomology arises from correlation functions in a heterotic generalization of the A model TFT.

Std quantum cohomology:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1,q_1}(\mathcal{M}_d) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M}_d)$$

$$= \sum_d \int_{\mathcal{M}_d} (\mathrm{top} - \mathrm{form})$$

Heterotic quantum cohomology:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1}\left(\mathcal{M}_d, \Lambda^{q_1} \mathcal{F}^\vee\right) \wedge \cdots \wedge H^{p_m}\left(\mathcal{M}_d, \Lambda^{q_m} \mathcal{F}^\vee\right)$$

Use $\left.\begin{array}{c} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array}\right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$

Quantum sheaf cohomology

In computing ordinary quantum cohomology rings, tech issues such as compactifying moduli spaces of holomorphic maps into a cpx manifold arise.

In the heterotic case, there are also sheaves $\mathcal{F}$ over those moduli spaces, which have to be extended over the compactification, in a way consistent with e.g.

$$\Lambda^{\rm top} \mathcal{F}^{\vee} \cong K_{\mathcal{M}}$$

But, this can be done....

Quantum sheaf cohomology

Example:

Take $X = \mathbf{P}^1 \times \mathbf{P}^1$

with $\mathcal{E}$ a deformation of the tangent bundle:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{\left[\begin{array}{cc} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{array} \right]}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

It can be shown the heterotic q. c. is

$$\begin{array}{rcl} \tilde{X}^2 & = & q_2 \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} & = & q_1 \end{array}$$

(a def' of the std q.c. ring of $\mathbf{P}^1 \mathrm{x} \mathbf{P}^1$)

Newer approaches

A more recent approach to these matters is to work with ``Landau-Ginzburg models.”

= strings propagating on spaces with `potential’ (Morse-like) functions

Landau-Ginzburg models

We can replace strings on a space X with strings on a space Y + potential, and if choose Y and potential correctly, get the same correlation functions.

Can give computational advantages.

Example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p π^*s

$\Big\downarrow$

Related by

``renormalization group flow"

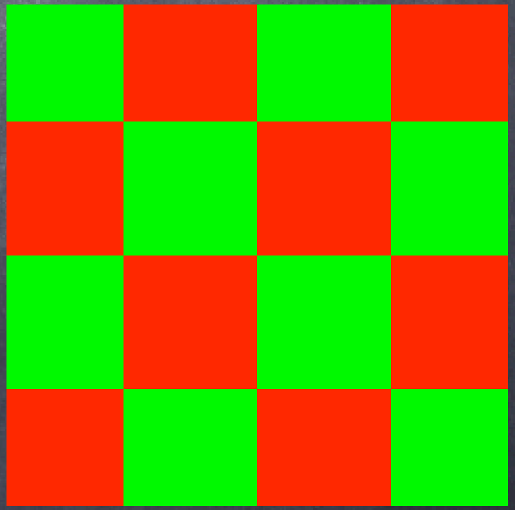
string on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

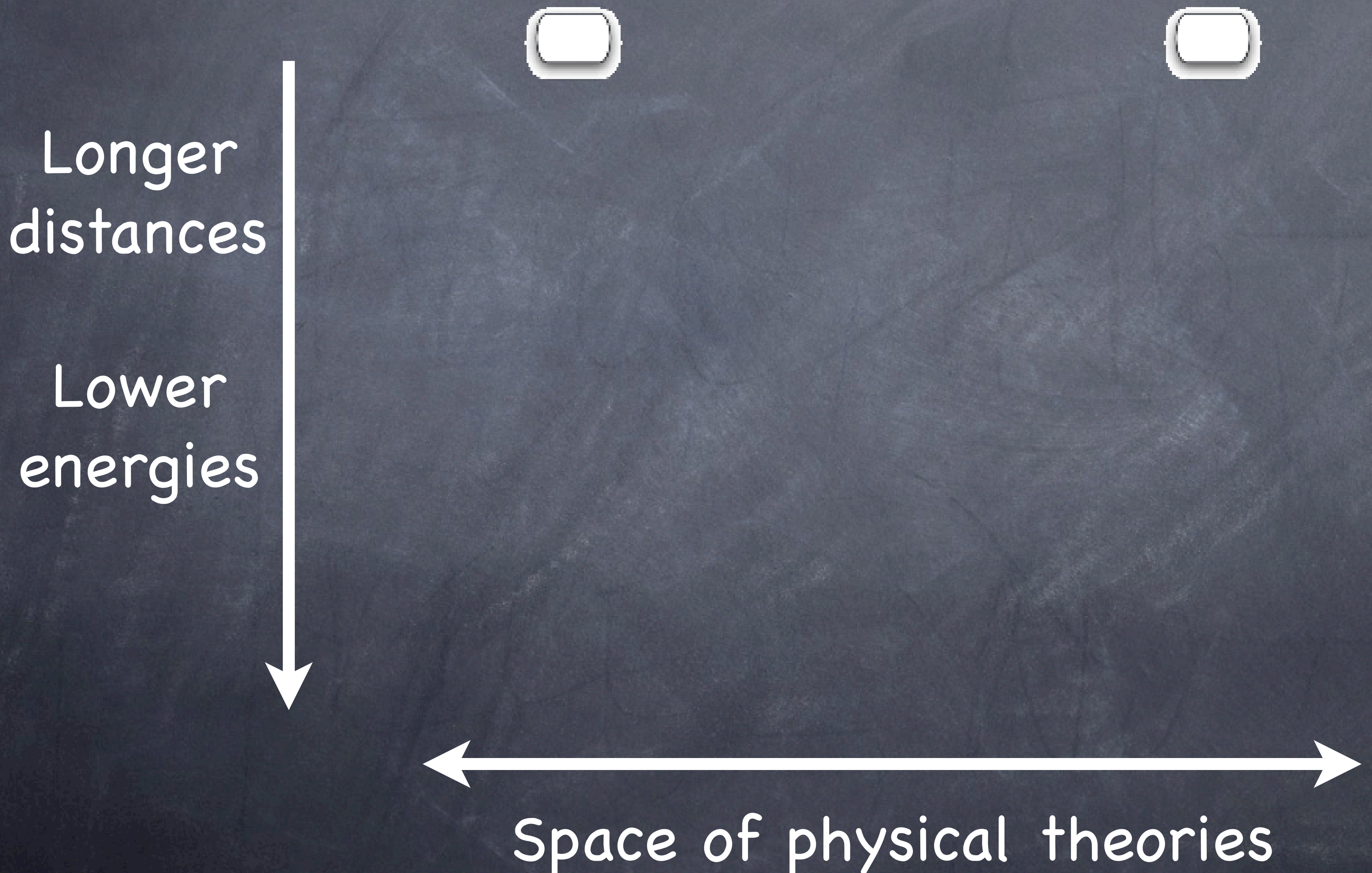
Renormalization group flow

Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.



Renormalization group



Computational advantages:

For example, consider curve-counting in a deg 5 (quintic) hypersurface in $\mathbf{P}^4$
-- need moduli space of curves in quintic, rather complicated

Can replace with LG model on

$$\mathrm{Tot}\left(\mathcal{O}(-5) \to \mathbf{P}^4\right)$$

and here, curve-counting involves moduli spaces of curves on $\mathbf{P}^4$, much easier

(Kontsevich: early '90s; physical LG realization: ES, Guffin, '08)

Application to mirror symmetry:

Instead of directly dualizing spaces,
replace spaces with corresponding LG models,
and dualize the LG models.

(P Clarke, '08)

* Resulting picture is often easier to understand
* Technical advantage: also encapsulates cases in which mirror isn't an ordinary space (but still admits a LG description)

There also exist heterotic LG models:

* a space X
* a holomorphic vector bundle $\mathcal{E} \to X$
  (satisfying same constraints as before)
* some potential-like data:

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^\vee)$$

such that $\displaystyle \sum_a E^a F_a = 0$

(Recover ordinary LG when $\mathcal{E} = TX$,

$E^a \equiv 0$ and $F_i = \partial_i W$)

Heterotic LG models are related to heterotic strings via renormalization group flow.

Example:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$

with $\mathcal{E}' = \pi^* \mathcal{F}_2$ & $F_a \equiv 0, \quad E^a \neq 0$

Renormalization group

A heterotic string on B

with $\mathcal{E} = \mathrm{coker}\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Because heterotic LG models are related to
ordinary heterotic strings via
renormalization group flow,

one can compute many quantities
(quantum sheaf cohomology, elliptic genera, ...)
upstairs in the LG model,
just as in the ordinary case.

One other fun application of LG models:

I've spoken on strings on **spaces**,
but in fact strings can propagate on
more general things.

* stacks

* abstract CFT's without any known
(pseudo-)geometric interpretation at all

& the latter are defined by (RG endpoints of)
certain LG models

The renormalization group (RG) plays an important role in LG models.

Some other occurrences of RG in string theory:

D-branes and derived categories

For any given complex in the derived category,
choose a locally-free resolution,
and map to branes/antibranes.

Problem: different rep's lead to different physics.

Ex: $0 \longrightarrow \mathcal{E} \stackrel{=}{\longrightarrow} \mathcal{E} \longrightarrow 0$ vs. 0

Solution: RG flow....

Another ex: **Stacks in physics**

Nearly every smooth DM stack has a presentation of the form [X/G].

To such a presentation, associate
``G-gauged sigma model on X''

Problem: such presentations not unique

Fix: RG flow:
stacks classify endpoints of RG flow

Mathematics		Physics
Geometry: Gromov-Witten Donaldson-Thomas quantum cohomology etc	⟺	Supersymmetric, topological quantum field theories
Homotopy, categories: derived categories, stacks, etc.	⟺	Renormalization group

Conclusions

Nobody knows whether string theory correctly describes the real world.

However, regardless, it has served as a source of ideas/inspirations for exciting new mathematics.

Where to go for more information?

Thank you for your time!