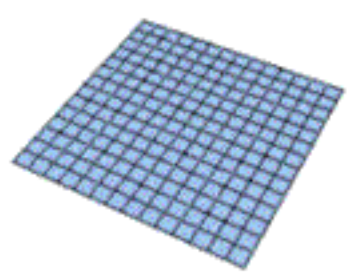
An introduction to mirror symmetry

Eric Sharpe
Virginia Tech

This will be a talk about string theory,
so let me discuss the motivation….

Twentieth-century physics saw two foundational advances:

General relativity
(special relativity)



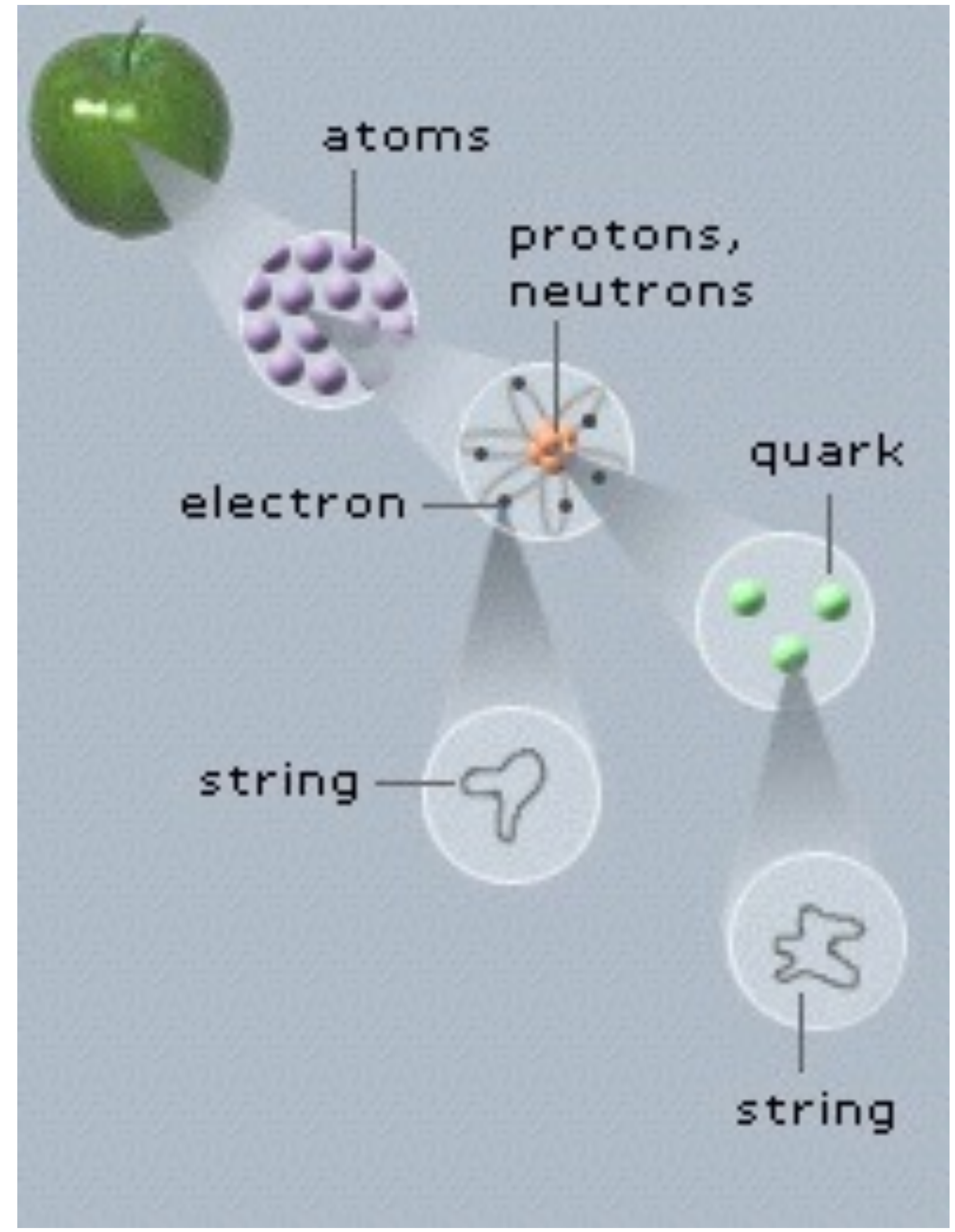
Quantum field theory
(quantum mechanics)

Problem: They contradict each other!
(in the sense that GR is only a low energy effective theory, useless below a certain scale)

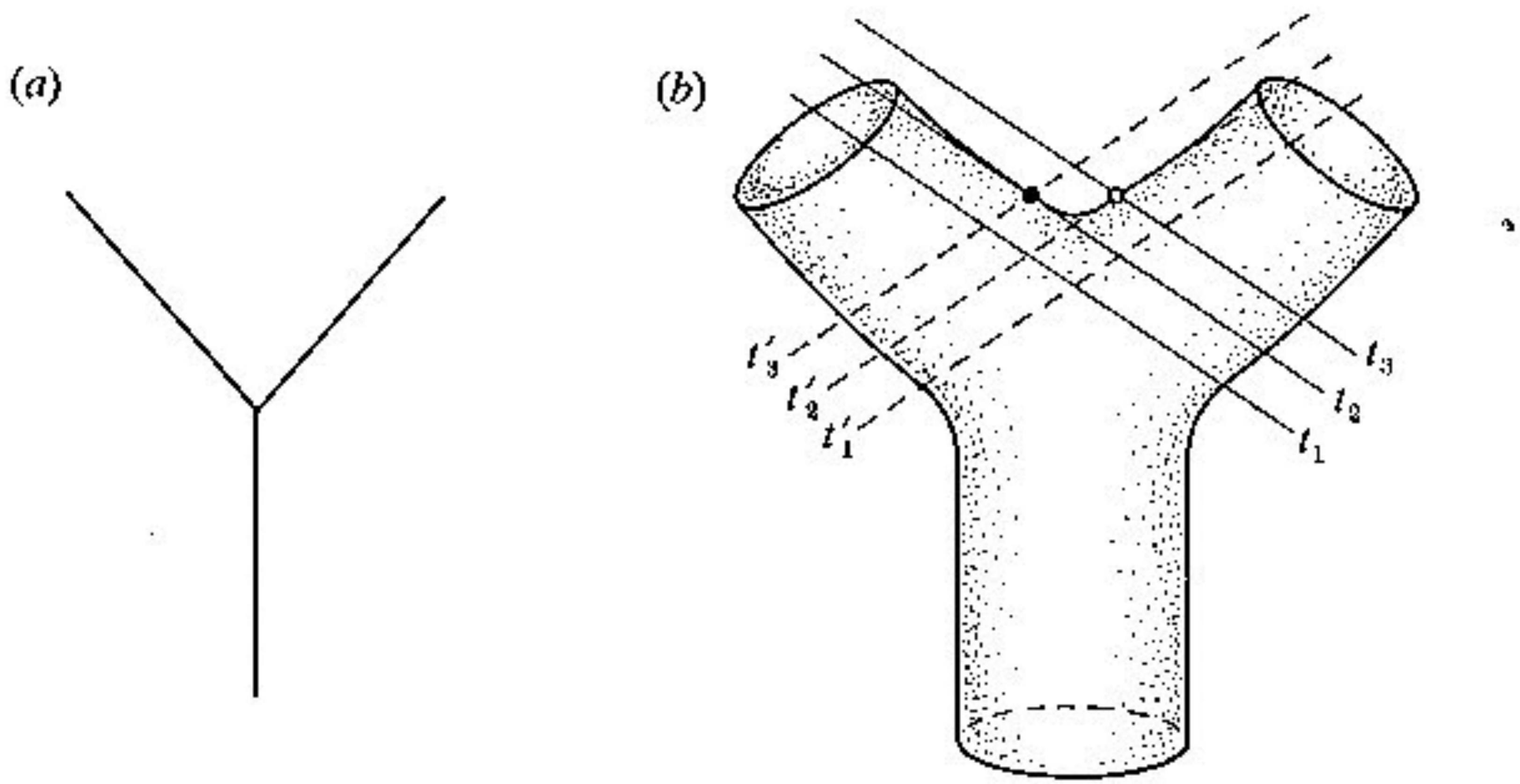
Something else is needed….

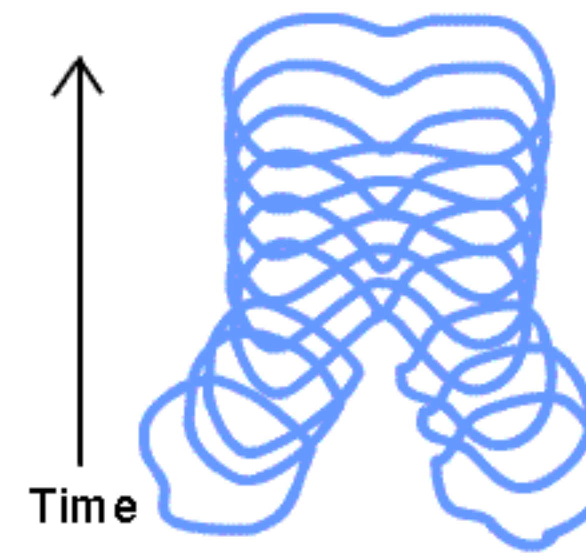
String theory…

… is a physical theory that
reconciles GR & QFT,
by replacing elementary
particles by strings.



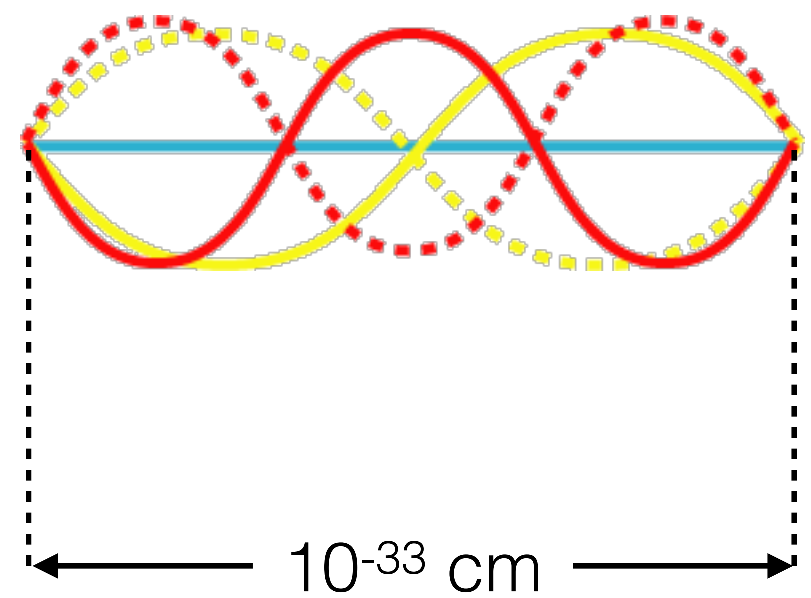
We fatten Feynman diagrams,
which removes QFT-like divergences.





In QFT, those divergences imply
scale-dependence of physics.

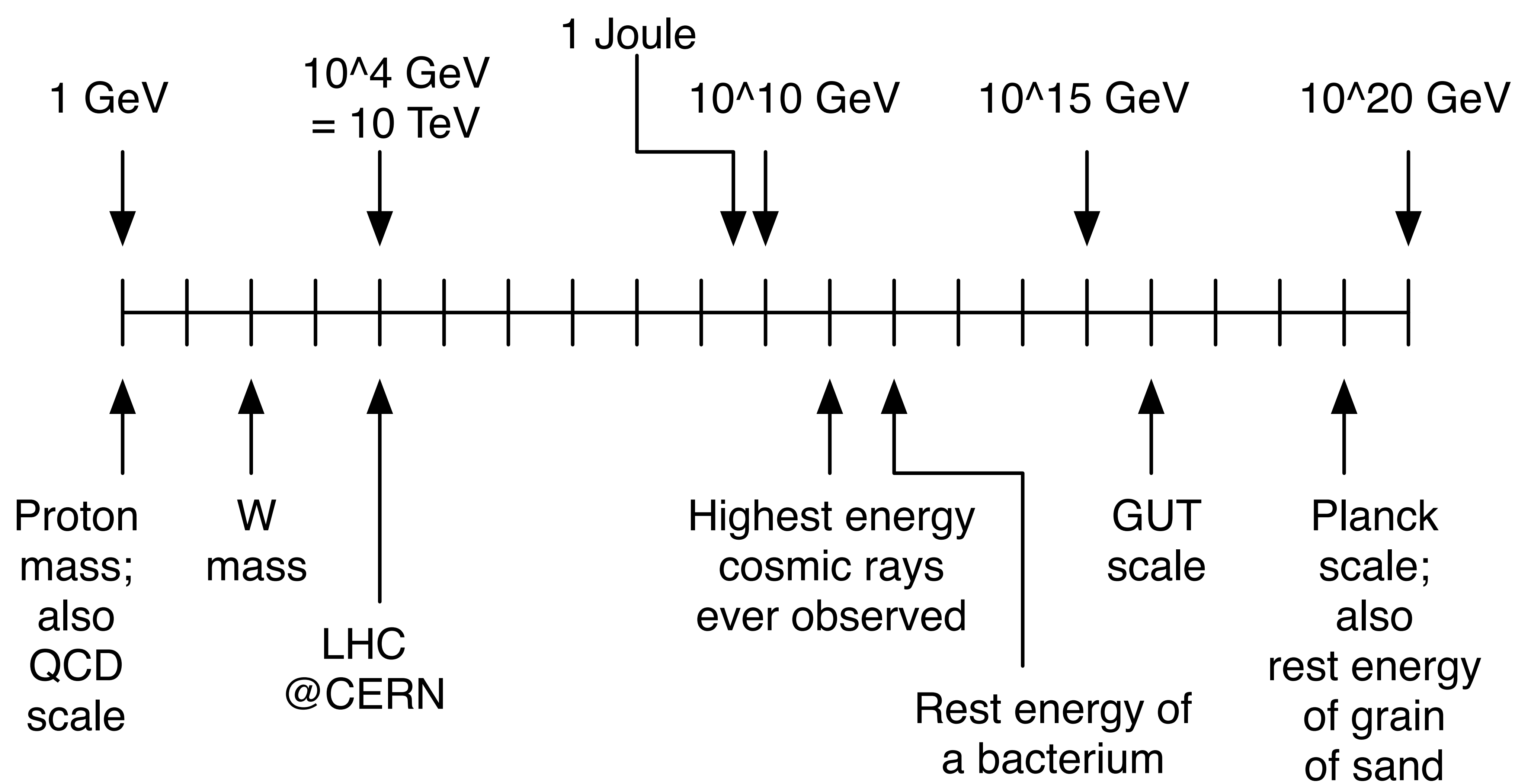
Do not expect scale-dependence in a fundamental theory,
hence no divergences is good.



The typical sizes of the strings are very small — of order the Planck length. To everyday observers, the string appears to be a pointlike object.

From dim'l analysis, typical energy scale for strings is Planck energy = $(h c^5 / G)^{1/2} \sim 10^{19}$ GeV

How big is that?



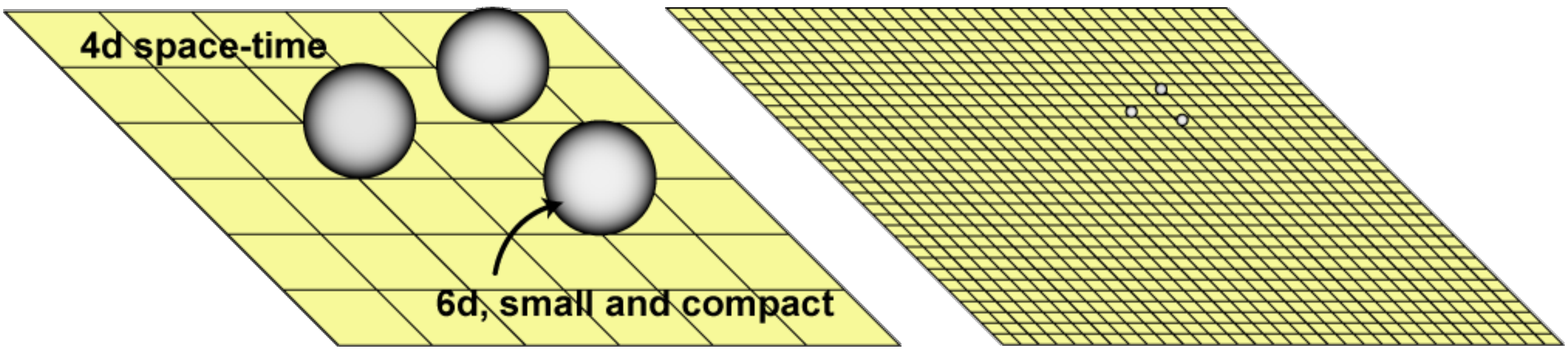
Perturbative (critical) string theory is consistent in 10 dimensions.

Yet, the real world is 4 dimensional
(3 space, 1 time).

So, how can string theory describe the real world?

Compactification scenario

Assume 10d spacetime = $\mathbf{R}^4 \times M$,
where M is some (small) (compact) 6d space.

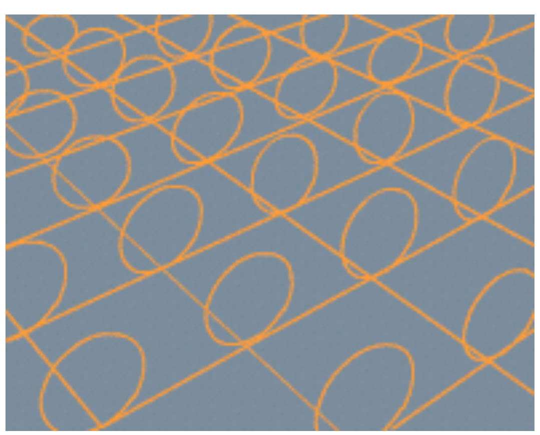


So long as you work at wavelengths much larger than the size of the compact space, you can't see the extra dimensions; spacetime looks like $\mathbf{R}^4$.

Kaluza-Klein theory

This picture of compactifying extra dimensions and getting extra fields in the 4d theory was historically first invented by Kaluza and Klein back in the 1920s.

They proposed a unification of gravity and electromagnetism, viewed as pure gravity in 5d, on **R**^{4} x S^1.



4d metric

U(1) gauge field

$$g^{(5)}_{mn} = \begin{bmatrix} g^{(4)}_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix}$$

scalar field

5d metric

String compactifications are a natural generalization.

What sort of 6-dim'l space can you compactify on?

- Needs to satisfy Einstein's equations for general relativity in vacuum

- To get a \`supersymmetric' low-energy four-dim'l effective theory, need some add'l properties

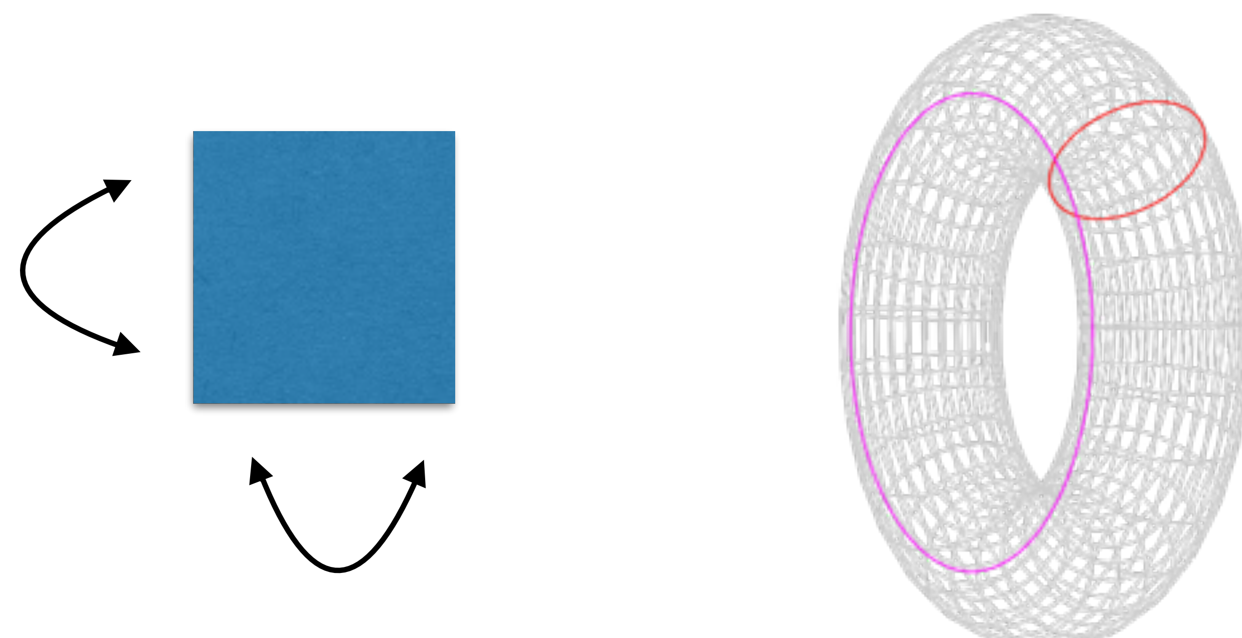
Result, in simplest cases, is that it must be Ricci-flat
plus a few other things;
known as a "**Calabi-Yau**" manifold

What’s a Calabi-Yau ?

Technically: complex Kahler manifold with trivial canonical bundle

= a space that (among other things) satisfies Einstein’s equations for a vacuum (meaning, it’s Ricci-flat).

Example: T^2



But that’s pretty boring;
more complicated examples also exist, in higher dimensions.

What difference can this make?

If that 6d space is too small to be observed,
what impact can it have on observable 4d physics?

Although you can't see the internal 6d space directly, the geometry of that space determines properties of the low-energy 4d theory.

Example: How to count massless spinors in 4d

Recall Dirac equ'n for spinors of mass m:

$$(i\not{D} - m)\psi = 0$$

Start with 10d massless spinors: $\not{D}_{10}\psi = 0$

We can decompose the 10d Dirac operator into

$$\not{D}_{10} = \not{D}_6 + \not{D}_4$$

and so we get 4d massless fermions from sol'ns of

$$\not{D}_6\psi = 0$$

The solutions of $\not{D}_6 \psi = 0$

(which determine 4d massless fermions)

can be characterized in terms of mathematical invariants of the 6-dim'l space, known as ``cohomology groups''

For example, on a Calabi-Yau, there are numbers $h^{p,q}$ = dim's of certain (Dolbeault) cohomology groups.

Math: these are groups of closed complex differential forms

$$\omega_{a_1 \cdots a_p \overline{a}_1 \cdots \overline{a}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\overline{z}^{\overline{a}_1} \wedge \cdots \wedge d\overline{z}^{\overline{a}_q}$$

(mod exact).

In compactifications of type II strings, these count 4d fermions with charges p, q under a pair of U(1) symmetries.

More generally, not just the number of particles but also their couplings, etc, are determined by the geometry of the internal 6d space.

In short, learn about 4d physics by studying mathematical structure of the 6-manifold.

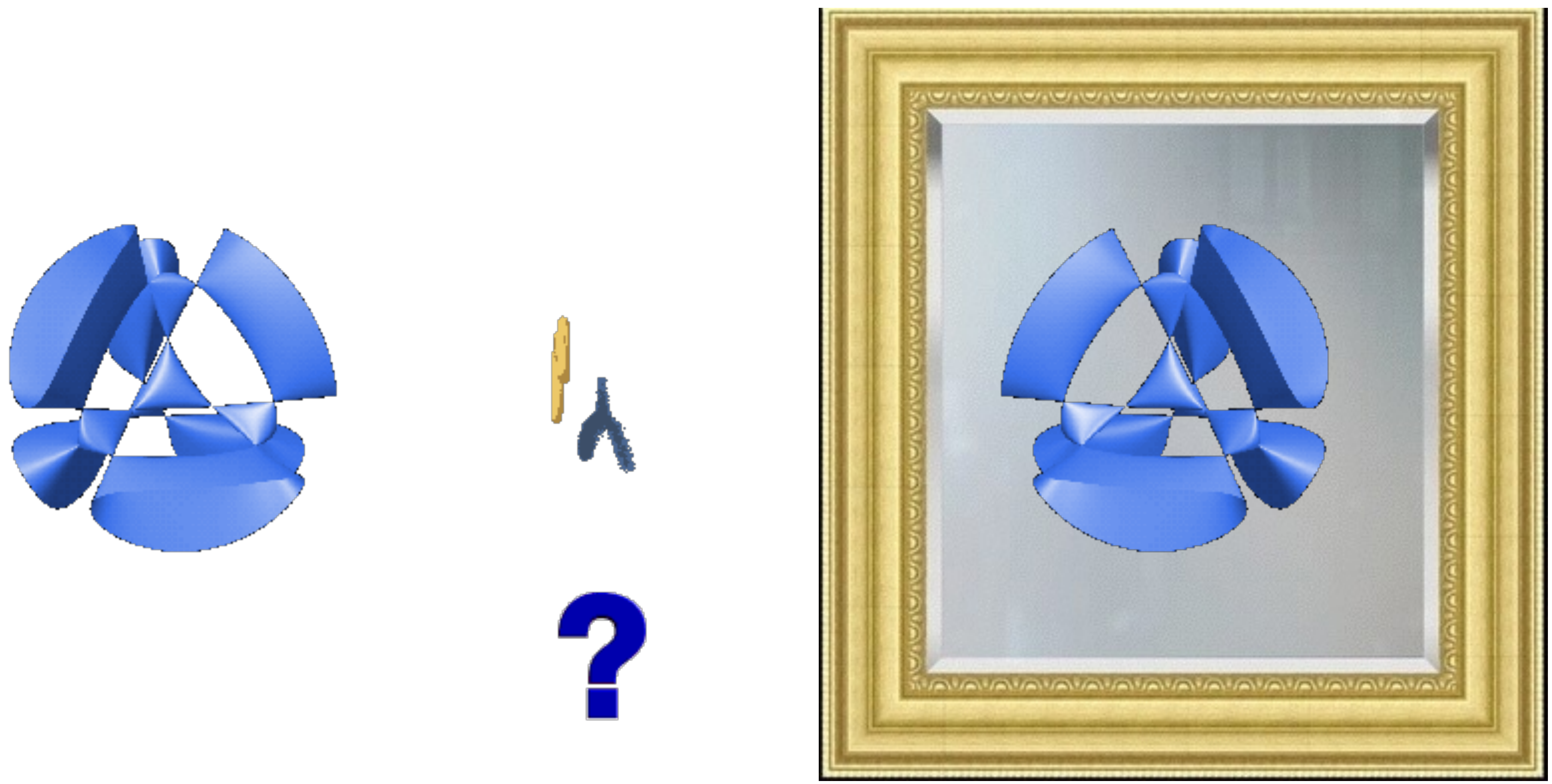
I’ve just told you why math is interesting to physicists, but the reverse has also turned out to be true:

Thinking about the resulting physics has led to new mathematics, which is what I’ll outline today.

Outline

- Overview of mirror symmetry and curve-counting
- Heterotic generalizations:
  - (0,2) mirror symmetry
  - quantum sheaf cohomology

Mirror symmetry



= a duality between 2d QFT's,
first worked out in early 1990s

Pairs of (usually topologically distinct) Calabi-Yau manifolds are described by **same** string theory — strings cannot distinguish.

Mirror symmetry

When two Calabi-Yau manifolds M, W are mirror,
they turn out to be very closely related
(but usually topologically distinct).

Example: $\dim M = \dim W$

After all, if strings are unable to distinguish one from the other,
then the compactified theory should be the same
— in particular, the dimension of the compactified theory had
better not change.

Mirror symmetry

Since the spectrum of light four-dimensional particles is determined by (Dolbeault) cohomology, we can conclude that

$$\sum \dim H^{*,*}(M) = \sum \dim H^{*,*}(W)$$

— total number of 4d particles should be unchanged.

Mirror symmetry

Calabi-Yau spaces are (incompletely) characterized by $h^{p,q}$'s (= dim's of Dolbeault cohomology groups), which compute the number of massless particles.

For example, for a 4-dim'l space, these are

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

"Hodge diamond"

Mirror symmetry acts as a rotation about the diagonal: if X is mirror to Y, then $h^{p,q}(X) = h^{n-p,q}(Y)$.

Example: T^2

T^2 is self-mirror topologically.

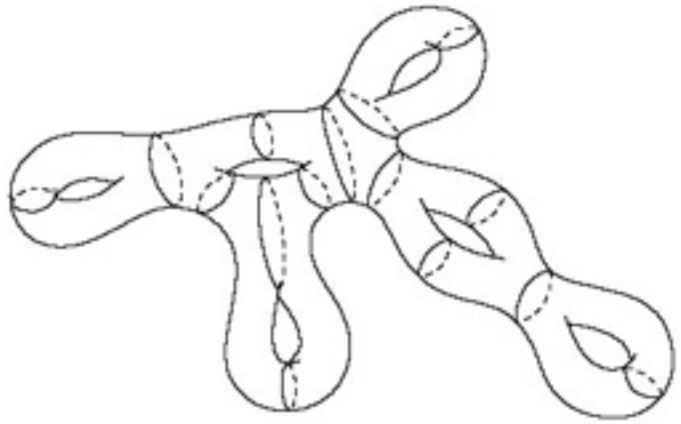
`Diamond' of $h^{p,q}$'s:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

— symmetric under rotation

This symmetry is specific to 2d manifolds with 1 handle; for g handles:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$



Example: Quartics in $\mathbf{P}^3$

(known as K3 manifolds)

K3 is self-mirror topologically;
complex, Kahler structures interchanged

$h^{1,1}$ $h^{1,1}$

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Kummer surface

$$(x^2+y^2+z^2-aw^2)^2-\left(\frac{3a-1}{3-a}\right)pqts=0$$
$$p=w-z-\sqrt{2}x$$
$$q=w-z+\sqrt{2}x$$
$$t=w+z+\sqrt{2}y$$
$$s=w+z-\sqrt{2}y$$
$$a=1.5$$
$$w=1$$

Example: Quintic

The “quintic” (deg 5 hypersurface in $\mathbb{P}^4$) is a nontrivial Calabi-Yau 6-manifold.

Quintic

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Mirror

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Mirror symmetry between spaces M $\longleftrightarrow$ W exchanges:

classical computations on M

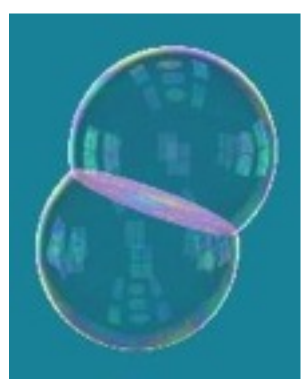
(perturbative in 2d QFT)

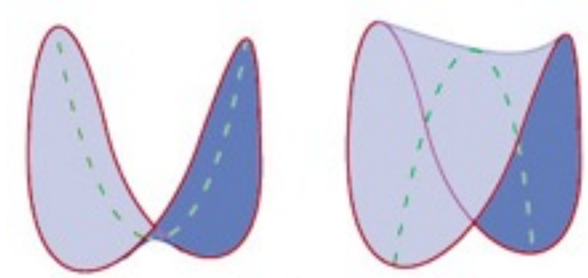
(Feynman diagrams)

sums over minimal area (2d) surfaces on W

(nonperturbative in 2d QFT)

(2d instantons)





A curve and two possible saddle-shaped film surfaces

The fact that this duality exchanges (easy) perturbative effects & (hard) nonperturbative effects makes it very useful for computations!

Degree k	n
1	2875
2	609250
3	317206375

Shown: numbers of minimal area S^2's in one particular Calabi-Yau (the “quintic”), of fixed degree.

These three degrees were the state-of-the-art in mathematics before mirror symmetry
(deg’ 2 in ’86, deg’ 3 in ’91)

Then, b/c of physics, mirror symmetry ~ ’92….

Degree k	n
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
...	...

This launched an army of algebraic geometers….

In math, these surface counts form ‘Gromov-Witten’ theory.

Physically, these numbers, these surface counts, are computing stringy nonperturbative corrections to Yukawa couplings in 4d theories with nonminimal (N=2) supersymmetry.

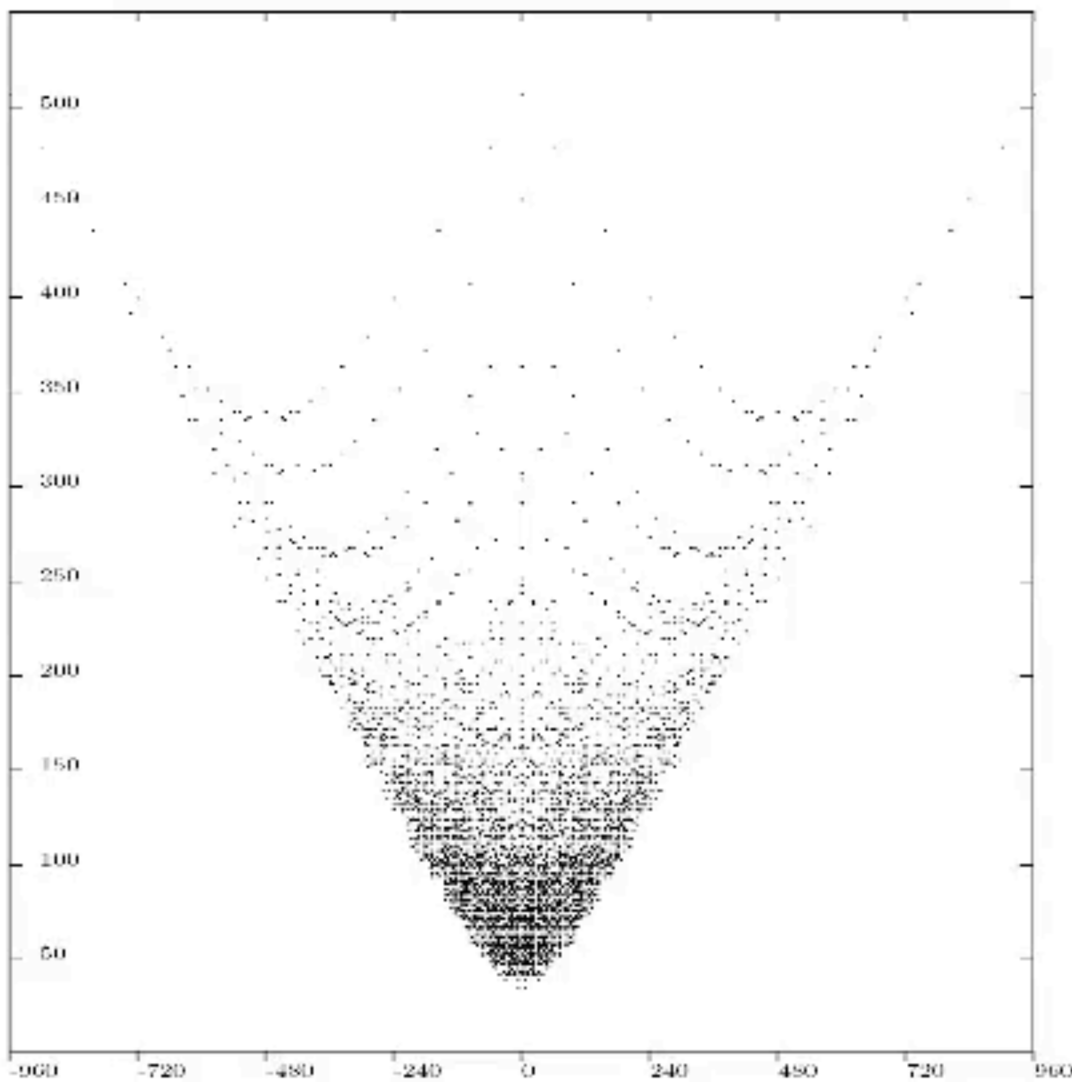
It would be more useful to compute the analogues in 4d theories with minimal (N=1) supersymmetry.
(Or, even better, no supersymmetry at all, but usually we think of getting N=1 at Planck scale, then breaking supersymmetry dynamically.)

We’ll see such an analogue shortly.

First, how many mirrors are there? How are they built?

Numerical checks of mirror symmetry

Plotted below are data for a large number of Calabi-Yau manifolds.



Vertical axis: $h^{1,1} + h^{2,1}$

Horizontal axis: $2(h^{1,1} - h^{2,1})$
= 2 (# Kahler - # cpx def's)

Mirror symmetry exchanges $h^{1,1} \longleftrightarrow h^{2,1}$

==> symm' across vert' axis

(Klemm, Schimmrigk, NPB 411 ('94) 559-583)

Constructions of mirror pairs

One of the original methods:
in special cases, can quotient by a symmetry group.
"Greene-Plesser orbifold construction"

(Greene-Plesser '90)

Example: quintic

$$Q_5 \subset \mathbb{P}^4 \quad \xleftrightarrow{\text{mirror}} \quad \widetilde{Q_5/\mathbb{Z}_5^3}$$

More general methods exist….

Constructions of mirror pairs

Batyrev's construction:

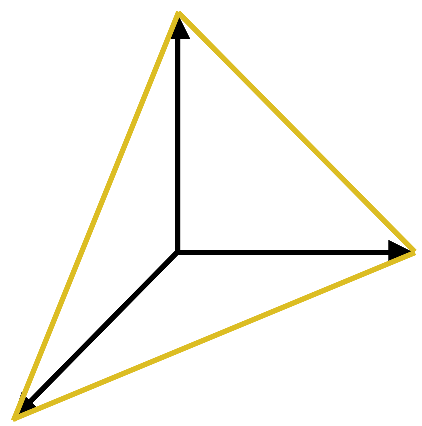
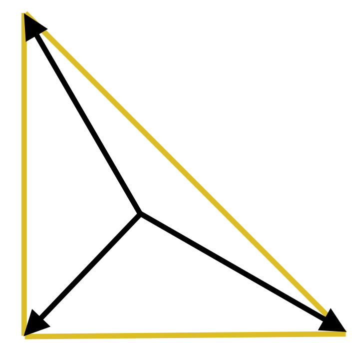
For a hypersurface in a toric variety,
mirror symmetry exchanges

polytope of ambient toric variety	$\longleftrightarrow$	dual polytope for ambient t.v. of mirror

Constructions of mirror pairs

Example of Batyrev's construction:

T^2 as degree 3 hypersurface in $\mathbb{P}^2$

$\mathbb{P}^2 =$  $\longleftrightarrow$  $= \mathbb{P}^2/\mathbb{Z}_3$

$$P^0 = \{y | \langle x, y \rangle \geq -1 \, \forall x \in P\}$$

Result:
degree 3 hypersurface in $\mathbb{P}^2$,
mirror to
$\mathbb{Z}_3$ quotient of degree 3 hypersurface

(matching Greene-Plesser '90)

Ordinary mirror symmetry is pretty well understood nowadays.

- lots of constructions
- both physics and math proofs

  Givental / Yau et al in math

  Morrison-Plesser / Hori-Vafa in physics

However, there are some extensions of mirror symmetry that are still being actively studied.

One example: heterotic mirror symmetry

Pertinent for 4d theories with minimal (N=1) supersymmetry

Heterotic mirror symmetry

is a conjectured generalization involving `heterotic’ strings.

Ordinary mirror symmetry involves `type II’ strings
which are specified by space + metric in 10d.

Heterotic strings are specified by
space + metric + nonabelian gauge field in 10d.

Thus, heterotic mirror symmetry involves not just spaces,
but also nonabelian gauge fields (bundles).

Heterotic mirror symmetry

is a generalization that exchanges pairs

$$(X_1, \mathcal{E}_1) \longleftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds
and the $\mathcal{E}_i$ are bundles / nonabelian gauge fields over X_i .

Constraints: for each $\mathcal{E}$, X,

$$[\mathrm{tr}\, F \wedge F] \;=\; [\mathrm{tr}\, R \wedge R] \;+\; d(\cdots)$$

equivalently: $\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX)$

If nonabelian gauge field = spinor connection,
then $F = R$ $(\mathcal{E} = TX)$ & so satisfied trivially.

Heterotic mirror symmetry

The (2d) quantum field theories defining heterotic strings, include those of other (“type II”) string theories as special cases.

Hence, heterotic mirror symmetry ought to reduce to ordinary mirror symmetry in a special case, & that turns out to be when $\mathcal{E}_i \cong TX_i \;\; (F_i = R_i)$.

Heterotic mirror symmetry

Much as in ordinary mirror symmetry,
dimensions and ranks are closely constrained:

If $(X_1, \mathcal{E}_1)$ is mirror to $(X_2, \mathcal{E}_2)$,
then

$$\dim X_1 \;=\; \dim X_2$$

$$\mathrm{rank}\, \mathcal{E}_1 \;=\; \mathrm{rank}\, \mathcal{E}_2$$

Heterotic mirror symmetry

Here, massless particles are computed by different cohomology groups: $H^q(X, \wedge^p \mathcal{E}^*)$.

Heterotic mirror symmetry exchanges

$$H^q(X_1, \wedge^p \mathcal{E}_1^*) \ \leftrightarrow \ H^q(X_2, \wedge^{r-p} \mathcal{E}_2^*)$$

just as ordinary mirror symmetry exchanges

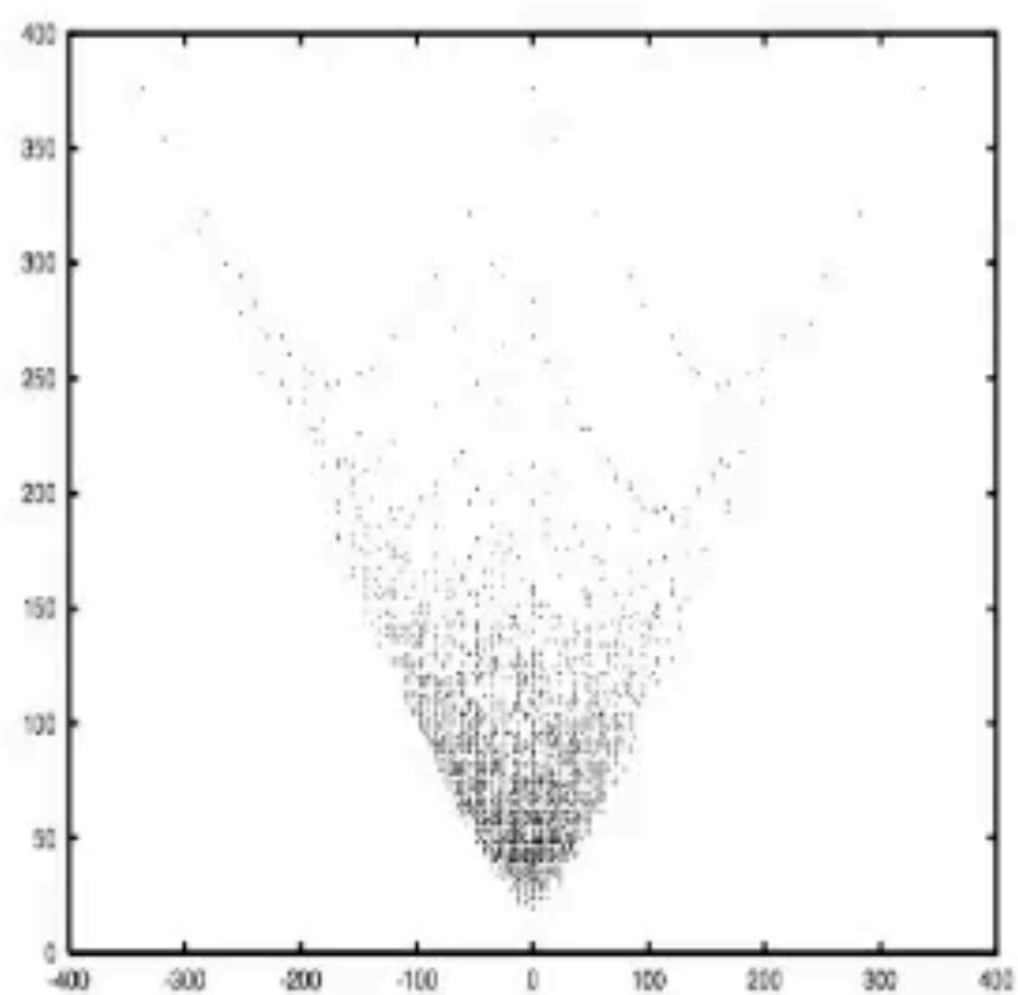
$$H^{p,q}(X_1) \ \leftrightarrow \ H^{n-p,q}(X_2)$$

When $\mathcal{E} = TX$, $H^{p,q}(X) = H^q(X, \wedge^p \mathcal{E}^*)$

& so we see ordinary mirrors as special cases.

Heterotic mirror symmetry

Not as much is known about the heterotic version, though a few basics have been worked out.



Example: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

Heterotic mirror symmetry

Constructions include:

- Blumenhagen-Sethi '96 extended Greene-Plesser orbifold construction to (0,2) models — handy but only gives special cases

- Adams-Basu-Sethi '03 repeated Hori-Vafa-Morrison-Plesser-style GLSM duality in (0,2)

- Melnikov-Plesser '10 extended Batyrev's construction & monomial-divisor mirror map to include def's of tangent bundle, for special ('reflexively plain') polytopes

Progress, but still don't have a general construction.

Heterotic mirror symmetry

Counting minimal area surfaces played
a crucial role in the original mirror symmetry,
and also arises in the heterotic version.

In the heterotic version, it's more complicated
(count minimal area surfaces + take into account the
nonabelian gauge field).

Heterotic version first worked out by
S. Katz, ES in 2004,
& there's been lots of work since then.

(Adams, Anderson, Aspinwall, Distler, Donagi, Ernebjerg, Gray, Lapan, McOrist, Melnikov, Plesser, Quigley, Rahn, Sethi, ….)

Minimal area surfaces:

standard case ("type II strings")

Schematically: For X a space,

$\mathcal{M}$ the space of minimal area S^2's in X

we compute a "correlation function"

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_k$$

where $\mathcal{O}_i \sim \omega_i \in H^{p_i, q_i}(\mathcal{M})$

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

which encodes minimal area surface information.

Such computations are at the heart of `Gromov-Witten' theory in the math community.

Minimal area surfaces:
heterotic case

Schematically: For X a space, $\mathcal{E}$ a bundle on X,
$\mathcal{M}$ the space of minimal area S^2's in X

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k$$

where $\mathcal{O}_i \sim \tilde{\omega}_i \in H^{q_i}\left(\mathcal{M}, \wedge^{p_i} \mathcal{F}^*\right)$

$\mathcal{F}$ = bundle of 2d fermi zero modes over $\mathcal{M}$

anomaly cancellation $\stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} \mathcal{F}^* \cong K_{\mathcal{M}}$

hence, again,

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

(S Katz, ES, 2004)

Correlation functions are often usefully encoded in \`operator products' (OPE's).

Physics: Say $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$ ("operator product")

if all correlation functions preserved:

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_C \cdots \rangle$$

Math: if interpret correlation functions as maps

$$\mathrm{Sym}^\bullet W \longrightarrow \mathbb{C}$$

(where W is the space of $\mathcal{O}$'s)

then OPE's are the kernel, of form $\mathcal{O}_A \mathcal{O}_B - \sum_i \mathcal{O}_i$

Examples:

Ordinary ("type II") case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

where $q, \tilde{q} \sim \exp(-\mathrm{area})$

$\longrightarrow 0$ in classical limit

Looks like a deformation of cohomology ring,
so called "quantum cohomology"

Examples:

Ordinary ("type II") case: $X = \mathbb{P}^1 \times \mathbb{P}^1$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Heterotic case:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \mathcal{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathcal{E}$: $\quad 0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$

where $\quad * = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ $\quad A, B, C, D$ const' 2x2 matrices

$x, \tilde{x}$ vectors of homog' coord's

Here, $W = H^1(X, \mathcal{E}^*) = \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$

OPE's: $\quad \det\left(A\psi + B\tilde{\psi}\right) = q, \quad \det\left(C\psi + D\tilde{\psi}\right) = \tilde{q}$

Check: $\quad \mathcal{E} = TX \quad$ when $\quad A = D = I_{2\times 2}, \quad B = C = 0$

& in this limit, OPE's reduce to those of ordinary case

"quantum sheaf cohomology"

Review of quantum sheaf cohomology

To make this more clear, let's work through the details:

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \to H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \rightarrow W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \rightarrow E \rightarrow 0$$

where $* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \rightarrow H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \rightarrow \wedge^2 E^* \rightarrow \wedge^2 Z \rightarrow Z \otimes W \rightarrow \mathrm{Sym}^2 W \otimes O \rightarrow 0$$

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $\cancelto{0}{H^0(Z \otimes W)} \to H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \to \cancelto{0}{H^1(Z \otimes W)}$

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$

0

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \underset{\delta}{\overset{\sim}{\to}} H^1(S_1) \underset{\delta}{\to} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\mathrm{top}} E^* \cong K_X \\ \mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX}) \end{array}\right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} F^* \cong K_M$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet} \mathrm{W} / (Q^{a+1}, \tilde{Q}^{b+1})$$

where $Q = \det(A\psi + B\tilde{\psi})$, $\tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => OPE's $Q = q, \ \tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Current state of the art:

For spaces called `toric varieties,'
and deformations of the tangent bundle,
heterotic curve corrections encoded in an OPE ring of form

$$\prod_\alpha \left(\det M_\alpha \right)^{Q^a_\alpha} \; = \; q_a$$

(McOrist-Melnikov '07-'08 using GLSM effective actions to give results for `linear' deformations;
Donagi-Guffin-Katz-ES 2011 using mathematical computations valid for all deformations)


The Future
NEXT EXIT

Long-term

More general constructions of (0,2) mirrors,
as current methods are limited

Generalize quantum sheaf cohomology computations to
arbitrary compact Calabi-Yau manifolds

Generalize quantum sheaf cohomology computations to arbitrary compact Calabi-Yau manifolds

To get there, we're currently looking at computations for deformations of tangent bundles of Grassmannians.

- Has some of the technical complexities expected for general case (induced sheaves not locally free, for example)

- But hopefully enough symmetry to guide to a solution.

Summary

- Overview of mirror symmetry and curve-counting
- Heterotic generalizations:
  - (0,2) mirror symmetry
  - quantum sheaf cohomology

Mathematics		**Physics**
Geometry: Gromov-Witten Donaldson-Thomas quantum cohomology etc	⟺	Supersymmetric, topological quantum field theories
Homotopy, categories: derived categories stacks derived spaces categorical equivalence	⟺	D-branes gauge theories sigma models w/ potential renormalization group flow