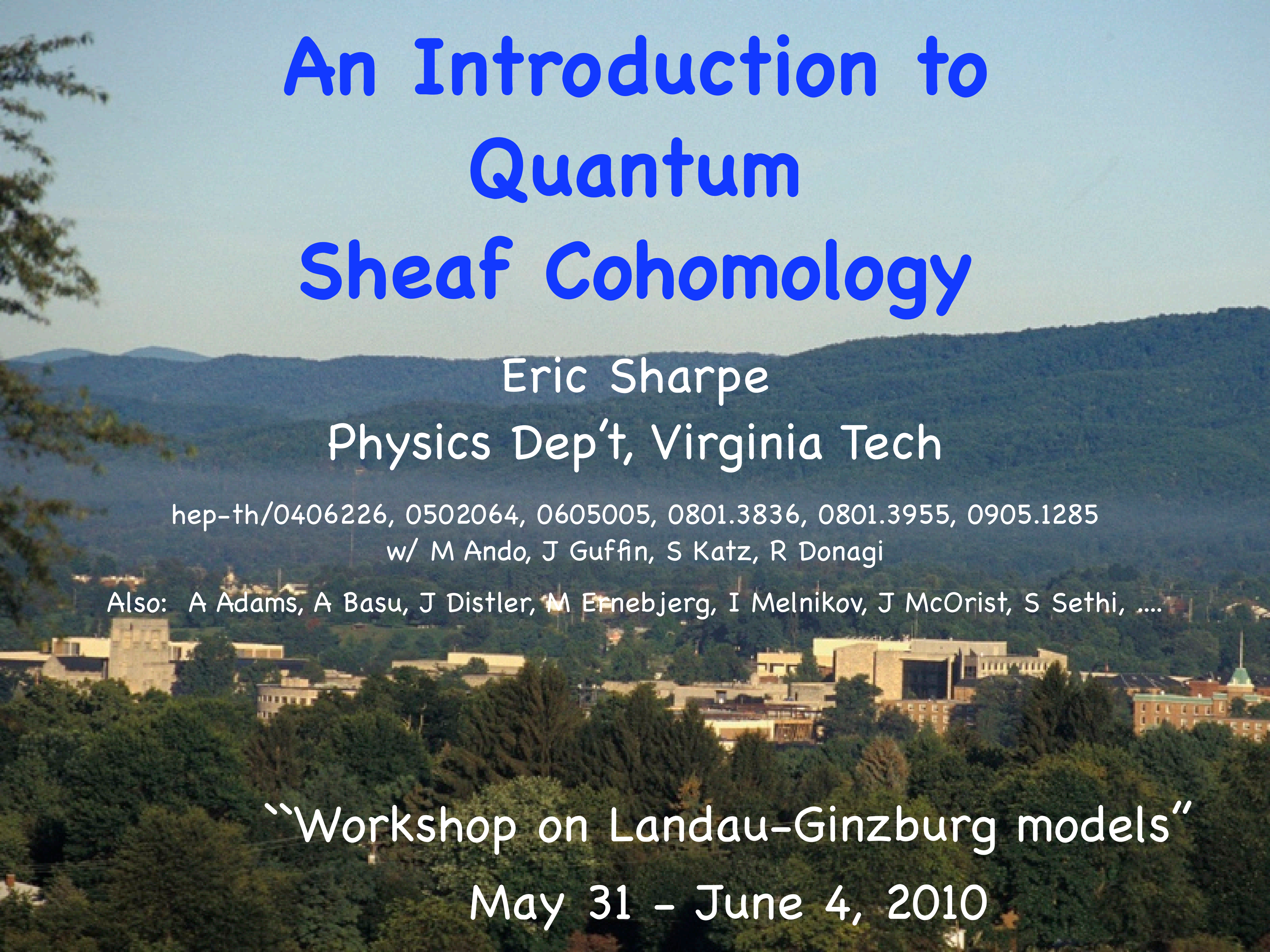

An Introduction to Quantum Sheaf Cohomology

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w/ M Ando, J Guffin, S Katz, R Donagi

Also: A Adams, A Basu, J Distler, M Ernebjerg, I Melnikov, J McOrist, S Sethi, ...

"Workshop on Landau-Ginzburg models"

May 31 - June 4, 2010

Today I'm going to talk about `quantum sheaf cohomology,' an analogue of quantum cohomology that arises in (0,2) mirror symmetry.

As background, what's (0,2) mirror symmetry?

First, recall ordinary mirror symmetry.

Exchanges pairs of Calabi-Yau's $X_1 \leftrightarrow X_2$

so as to flip Hodge diamond.

Ex: The quintic (deg 5) hypersurface in $\mathbf{P}^4$

is mirror to

(res'n of) a deg 5 hypersurface in $\mathbf{P}^4/(\mathbf{Z}_5)^3$

Quintic

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Mirror

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(0,2) mirror symmetry

is a conjectured generalization that exchanges pairs

$$(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds
and the $\mathcal{E}_i \to X_i$ are holomorphic vector bundles

Constraints: $\mathcal{E}$ stable, $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

Reduces to ordinary mirror symmetry when
$\mathcal{E}_i \cong TX_i$

(0,2) mirror symmetry

Instead of exchanging (p,q) forms,
(0,2) mirror symmetry exchanges sheaf cohomology:

$$H^j(X_1, \Lambda^i \mathcal{E}_1) \leftrightarrow H^j(X_2, (\Lambda^i \mathcal{E}_2)^\vee)$$

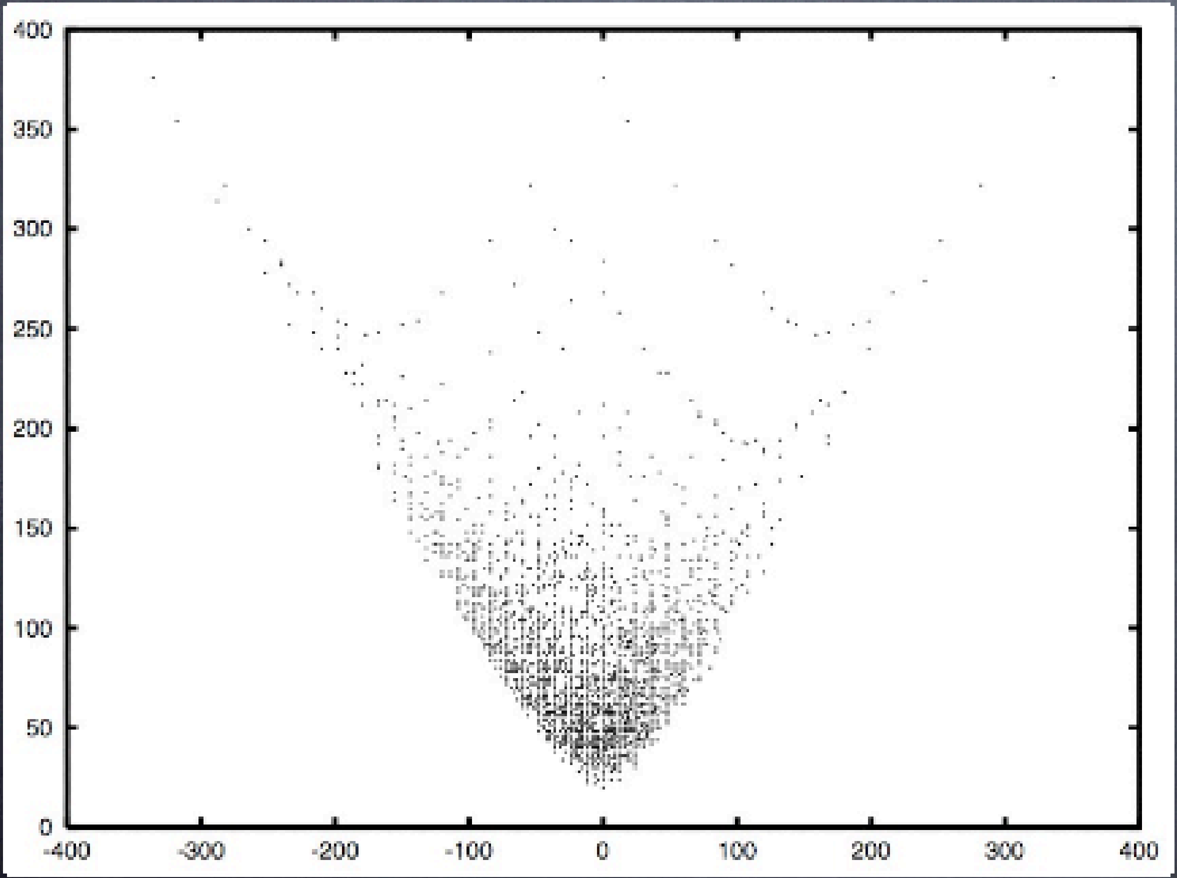
Note when $\mathcal{E}_i \cong TX_i$ this reduces to

$$H^{d-1,1}(X_1) \leftrightarrow H^{1,1}(X_2)$$

(for X_i Calabi-Yau)

(0,2) mirror symmetry

Not much is known about (0,2) mirror symmetry, though basics are known, and more quickly developing.



Ex: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^\vee)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^\vee)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

(0,2) mirror symmetry

A few highlights:

* an analogue of the Greene-Plesser construction (quotients by finite groups) is known

(Blumenhagen, Sethi, NPB 491 ('97) 263-278)

* an analogue of Hori-Vafa (Adams, Basu, Sethi, hepth/0309226)

* analogue of quantum cohomology known since '04

(ES, Katz, Adams, Distler, Ernebjerg, Guffin, Melnikov, McOrist, ...)

* for def's of the tangent bundle, there now exists a (0,2) monomial-divisor mirror map

(Melnikov, Plesser, 1003.1303 & Strings 2010)

(0,2) mirrors are starting to heat up!

Outline of today's talk

Today, I'll going to outline one aspect of (0,2) mirrors,
namely,
quantum sheaf cohomology
(the (0,2) analogue of quantum cohomology),

[Initially developed in '04 by S Katz, ES,
and later work by A Adams, J Distler, R Donagi,
M Ernebjerg, J Guffin, J McOrist, I Melnikov,
S Sethi, ...]

& then discuss (2,2) & (0,2) Landau-Ginzburg models,
and some related issues.

Aside on lingo:

The worldsheet theory for a heterotic string with the
``standard embedding"
(gauge bundle $\mathcal{E}$ = tangent bundle TX)
has (2,2) susy in 2d,
hence ``(2,2) model"

The worldsheet theory for a heterotic string with a
more general gauge connection has (0,2) susy,
hence ``(0,2) model"

Ordinary quantum cohomology is computed physically by the `A model' topological field theory.

The (0,2) analogue of the A model, responsible for `quantum sheaf cohomology,' is called the A/2 model.

We'll review A/2, B/2 models next....

The A/2, B/2 models:

* (0,2) analogues of ((2,2)) A, B models

* No longer strictly TFT's, though become TFT's on the (2,2) locus

* Nevertheless, some correlation functions still have a mathematical understanding

* New symmetries: A/2 on $(X, \mathcal{E})$ same as B/2 on $(X, \mathcal{E}^\vee)$

Next: review/compare A, A/2....

Ordinary A model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + i g_{i\bar{\jmath}}\psi_-^{\bar{\jmath}} D_z \psi_-^i + i g_{i\bar{\jmath}}\psi_+^{\bar{\jmath}} D_{\bar{z}} \psi_+^i + R_{i\bar{\jmath}k\bar{l}}\psi_+^i \psi_+^{\bar{\jmath}} \psi_-^k \psi_-^{\bar{l}}$$

Fermions:

$$\psi_-^i (\equiv \chi^i) \in \overline{\Gamma}((\phi^* T^{0,1}X)^\vee) \qquad \psi_+^i (\equiv \psi_z^i) \in \Gamma(K \otimes \phi^* T^{1,0}X)$$
$$\psi_-^{\bar{\imath}} (\equiv \psi_{\bar{z}}^{\bar{\imath}}) \in \overline{\Gamma}(\overline{K} \otimes \phi^* T^{0,1}X) \qquad \psi_+^{\bar{\imath}} (\equiv \chi^{\bar{\imath}}) \in \Gamma((\phi^* T^{1,0}X)^\vee)$$

Under the scalar supercharge,

$$\begin{array}{ll} \delta\phi^i \propto \chi^i, & \delta\phi^{\bar{\imath}} \propto \chi^{\bar{\imath}} \\ \delta\chi^i = 0, & \delta\chi^{\bar{\imath}} = 0 \\ \delta\psi_z^i \neq 0, & \delta\psi_{\bar{z}}^{\bar{\imath}} \neq 0 \end{array}$$

so the states are

$$\mathcal{O} \sim b_{i_1\cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \chi^{i_1} \cdots \chi^{i_p} \quad \leftrightarrow \quad H^{p,q}(X)$$
$$Q \quad \leftrightarrow \quad d$$

A/2 model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ih_{a\bar{b}}\lambda_-^{\bar{b}}D_z\lambda_-^a + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$$

Fermions:

$$\lambda_-^a \in \overline{\Gamma}(\phi^*\overline{\mathcal{E}}) \qquad \psi_+^i \in \Gamma(K\otimes\phi^*T^{1,0}X)$$

$$\lambda_-^{\bar{b}} \in \overline{\Gamma}(\overline{K}\otimes\phi^*\overline{\mathcal{E}}) \qquad \psi_+^{\bar{\imath}} \in \Gamma((\phi^*T^{1,0}X)^\vee)$$

Constraints:

Green-Schwarz: $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

Another anomaly: $\Lambda^{\mathrm{top}}\mathcal{E}^\vee \cong K_X$

(analogue of the CY condition in the B model)

A/2 model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ih_{a\bar{b}}\lambda_-^{\bar{b}}D_z\lambda_-^a + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$$

Fermions:

$$\lambda_-^a \in \overline{\Gamma}(\phi^*\overline{\mathcal{E}}) \qquad \psi_+^i \in \Gamma(K \otimes \phi^* T^{1,0}X)$$

$$\lambda_-^{\bar{b}} \in \overline{\Gamma}(\overline{K} \otimes \phi^*\overline{\mathcal{E}}) \qquad \psi_+^{\bar{\imath}} \in \Gamma((\phi^* T^{1,0}X)^\vee)$$

Constraints: $\Lambda^{top}\mathcal{E}^\vee \cong K_X, \quad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States:

$$\mathcal{O} \sim b_{\bar{\imath}_1\cdots\bar{\imath}_n a_1\cdots a_p}\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\lambda_-^{a_1}\cdots\lambda_-^{a_p} \leftrightarrow H^n(X, \Lambda^p\mathcal{E}^\vee)$$

When $\mathcal{E} = TX$, reduces to the A model,
since $H^q(X, \Lambda^p(TX)^\vee) = H^{p,q}(X)$

A model classical correlation functions

For X compact, have n $\chi^i, \chi^{\bar{\imath}}$ zero modes,
plus bosonic zero modes $\sim X$, so

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; = \; \int_X H^{p_1,q_1}(X) \wedge \cdots \wedge H^{p_m,q_m}(X)$$

Selection rule from left, right U(1)$_R$'s: $\sum_i p_i = \sum_i q_i = n$

Thus: $\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; \sim \; \int_X (\text{top-form})$

A/2 model classical correlation functions

For X compact, we have n $\psi_+^{\bar{\imath}}$ zero modes and r λ^a zero modes:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int_X H^{q_1}(X, \Lambda^{p_1}\mathcal{E}^\vee) \wedge \cdots \wedge H^{q_m}(X, \Lambda^{p_m}\mathcal{E}^\vee)$$

Selection rule: $\sum_i q_i = n, \;\; \sum_i p_i = r$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X H^{top}(X, \Lambda^{top}\mathcal{E}^\vee)$$

The constraint $\Lambda^{top}\mathcal{E}^\vee \cong K_X$ makes the integrand a top-form.

A model -- worldsheet instantons

Moduli space of bosonic zero modes
= moduli space of worldsheet instantons, $\mathcal{M}$

If no $\psi^i_z, \psi^{\bar{\imath}}_{\bar{z}}$ zero modes, then

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{p_1,q_1}(\mathcal{M}) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M})$$

More generally,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{p_1,q_1}(\mathcal{M}) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M}) \wedge c_{top}(\mathrm{Obs})$$

In all cases: $\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} (\text{top form})$

A/2 model -- worldsheet instantons

The bundle $\mathcal{E}$ on X induces
a sheaf $\mathcal{F}$ (of λ zero modes) on $\mathcal{M}$: $\quad \mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$

where $\pi : \Sigma \times \mathcal{M} \to \mathcal{M}, \quad \alpha : \Sigma \times \mathcal{M} \to X$

On the (2,2) locus, where $\mathcal{E} = TX$, have $\mathcal{F} = T\mathcal{M}$

When no `excess' zero modes,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{F}^\vee)$$

Apply anomaly constraints:

$$\left. \begin{array}{c} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{GRR}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

so again integrand is a top-form.

A/2 model -- worldsheet instantons

General case:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \quad \sim \quad \int_{\mathcal{M}} H^{\sum q_i}\left(\mathcal{M}, \Lambda^{\sum p_i} \mathcal{F}^\vee \right) \wedge$$
$$H^n\left(\mathcal{M}, \Lambda^n \mathcal{F}^\vee \otimes \Lambda^n \mathcal{F}_1 \otimes \Lambda^n (\mathrm{Obs})^\vee \right)$$

where
$$\begin{array}{ll} \psi_+^{\bar{\jmath}} \sim T\mathcal{M} = R^0 \pi_* \alpha^* TX & \lambda_-^a \sim \mathcal{F} = R^0 \pi_* \alpha^* \mathcal{E} \\ \psi_+^i \sim \mathrm{Obs} = R^1 \pi_* \alpha^* TX & \lambda_-^{\bar{b}} \sim \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E} \end{array}$$

(reduces to A model result via Atiyah classes)

Apply anomaly constraints:

$$\left. \begin{array}{c} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{GRR}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \otimes \Lambda^{top} \mathcal{F}_1 \otimes \Lambda^{top} (\mathrm{Obs})^\vee \cong K_{\mathcal{M}}$$

so, again, integrand is a top-form.

To do any computations, we need explicit expressions for the space $\mathcal{M}$ and bundle $\mathcal{F}$.

Will use `linear sigma model' moduli spaces.

Advantage: closely connected to physics

Disadvantage: no universal instanton

$$\alpha : \Sigma \times \mathcal{M} \to X,$$

previous discussion merely formal, need to extend induced sheaves over the compactification divisor.

1st, review linear sigma model (LSM) moduli spaces....

Gauged linear sigma models are 2d gauge theories, generalizations of the $\mathbf{CP}^N$ model, that RG flow in IR to NLSM's.

`Linear sigma model moduli spaces' are therefore moduli spaces of 2d gauge instantons.

The 2d gauge instantons of the gauge theory = worldsheet instantons in IR NLSM

S'pose we want to describe maps into a Grassmannian of k-planes in n-dim'l space, G(k,n).

(for k=1, get $\mathbf{P}^{n-1}$)

Physically, 2d U(k) gauge theory, n fundamentals.

Bundles built physically from (co)kernels of short exact sequences of (special homogeneous) bundles, defined by rep's of U(k).

Lift to nat'l sheaves on $\mathbf{P}^1 \times \mathcal{M}$, pushforward to $\mathcal{M}$.

A few more details.

All the heterotic bundles will be built from (co)kernels of short exact sequences in which all the other elements are bundles defined by reps of U(k).

Ex:

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus^{n} \mathcal{O}(\mathbf{k}) \bigoplus^{k+1} \mathrm{Alt}^2 \mathcal{O}(\mathbf{k}) \longrightarrow \bigoplus^{k-1} \mathrm{Sym}^2 \mathcal{O}(\mathbf{k}) \longrightarrow 0$$

$\mathcal{O}(\mathbf{k})$ is bundle associated to fund' rep' of U(k)

We need to extend pullbacks of such across

$$\mathbf{P}^1 \times \mathcal{M}_{\mathrm{LSM}}$$

Corresponding to $\mathcal{O}(\overline{\mathbf{k}})$ is a
rk k `universal subbundle' S on $\mathbf{P}^1 \times \mathcal{M}$.

Lift all others so as to be a U(k)-rep' homomorphism

Ex:

$$\mathcal{O}(\mathbf{k}) \mapsto S^*$$

$$\mathcal{O}(\mathbf{k}) \otimes \mathcal{O}(\overline{\mathbf{k}}) \mapsto S^* \otimes S$$

$$\mathrm{Alt}^m \mathcal{O}(\mathbf{k}) \mapsto \mathrm{Alt}^m S^*$$

Then pushforward to LSM moduli space, and compute.

Let's do projective spaces in more detail....

Example: $\mathbf{CP}^{N-1}$

Have N chiral superfields $x_1, \cdots, x_N$, each charge 1

For degree d maps, expand:

$$x_i \;=\; x_{i0}u^d \;+\; x_{i1}u^{d-1}v \;+\; \cdots \;+\; x_{id}v^d$$

where u, v are homog' coord's on worldsheet = $\mathbf{P}^1$

Take (x_{ij}) to be homogeneous coord's on $\mathcal{M}$, then

$$\mathcal{M}_{\mathrm{LSM}} \;=\; \mathbf{P}^{N(d+1)-1}$$

Can do something similar to build $\mathcal{F}$.

Example: completely reducible bundle

$$\mathcal{E} \;=\; \oplus_a \mathcal{O}(n_a)$$

Corresponding to $\mathcal{O}(-1) \rightarrow \mathbf{P}^{N-1}$
is the bundle

$$S \equiv \pi_1^* \mathcal{O}_{\mathbf{P}^1}(-d) \otimes \pi_2^* \mathcal{O}_{\mathbf{P}^{N(d+1)-1}}(-1) \;\longrightarrow\; \mathbf{P}^1 \times \mathbf{P}^{N(d+1)-1}$$

Lift of $\mathcal{E}$ is $\oplus_a S^{\otimes -n_a} \longrightarrow \mathbf{P}^1 \times \mathbf{P}^{N(d+1)-1}$
which pushes forward to

$$\mathcal{F} \;=\; \oplus_a H^0\left(\mathbf{P}^1, \mathcal{O}(n_a d)\right) \otimes_{\mathbf{C}} \mathcal{O}(n_a)$$

There is also a trivial extension of this to more general toric varieties.

Example: completely reducible bundle

$$\mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a)$$

Corresponding sheaf of fermi zero modes is

$$\mathcal{F} = \oplus_a H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Check of (2,2) locus

The tangent bundle of a (cpt, smooth) toric variety can be expressed as

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0$$

Applying previous ansatz,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_i H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i)$$

This $\mathcal{F}$ is precisely $T\mathcal{M}_{\rm LSM}$, and $\mathcal{F}_1$ is the obs' sheaf.

Quantum cohomology

… is an OPE ring. For **CP**$^{N-1}$, correl'n f'ns:

$$\langle x^k \rangle = \begin{cases} q^m & \text{if } k = mN + N - 1 \\ 0 & \text{else} \end{cases}$$

Ordinarily need (2,2) susy, but:

* Adams-Basu-Sethi ('03') conjectured (0,2) exs
* Katz-E.S. ('04) computed matching corr'n f'ns
* Adams-Distler-Ernebjerg ('05): gen'l argument
* Guffin, Melnikov, McOrist, Sethi, etc

Quantum cohomology

Example:

ABS studied a (0,2) theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with gauge bundle $\mathcal{E}$ = def' of tangent bundle, expressible as a cokernel:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* \;=\; \left[\begin{array}{cc} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{array} \right]$$

Quantum cohomology

In this example (a (0,2) theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with gauge bundle = def' of tangent bundle),

ABS conjectured:

$$\begin{aligned} \tilde{X}^2 &= \exp(it_2) \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= \exp(it_1) \end{aligned}$$

(a def' of the q.c. ring of $\mathbf{P}^1 \times \mathbf{P}^1$)

Quantum cohomology

Katz-E.S. checked by directly computing, using technology outlined so far:

$$
\begin{aligned}
\langle \tilde{X}^4 \rangle &= \langle 1 \rangle \exp(2it_2) \;=\; 0 \\
\langle X \tilde{X}^3 \rangle &= \langle (X\tilde{X}) \tilde{X}^2 \rangle \\
&= \langle X \tilde{X} \rangle \exp(it_2) \;=\; \exp(it_2) \\
\langle X^2 \tilde{X}^2 \rangle &= \langle X^2 \rangle \exp(it_2) \;=\; (\epsilon_1 - \epsilon_2) \exp(it_2) \\
\langle X^3 \tilde{X} \rangle &= \exp(it_1) \;+\; (\epsilon_1 - \epsilon_2)^2 \exp(it_2) \\
\langle X^4 \rangle &= 2(\epsilon_1 - \epsilon_2) \exp(it_1) \;+\; (\epsilon_1 - \epsilon_2)^3 \exp(it_2)
\end{aligned}
$$

and so forth, verifying the prediction.

Since then:

* Josh Guffin, Sheldon Katz

Checked many more correlation functions, worked out technology for further computations

* Ilarion Melnikov, Jock McOrist, Sav Sethi

Corresponding GLSM computations.

B/2 model

-- also exists

-- classically, can be related to (0,2) A model by exchanging $\mathcal{E}$ and $\mathcal{E}^\vee$

-- but there's a different regularization of the theory. For some special curves, in which

$$\phi^* \mathcal{E} \cong \phi^* \mathcal{E}^\vee$$

the A, B models are classically indistinguishable, but QM'ly are distinguished by their extensions over compactification divisor

(ES, S Katz)

So far:

* outlined A/2, B/2 models

  (first exs of `holomorphic field theories,' rather than `topological field theories')

* outlined quantum sheaf cohomology, old claims of ABS, verification

Next:

(2,2) & (0,2) Landau-Ginzburg models

A **Landau-Ginzburg model** is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

$$S = \int_\Sigma d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\jmath} D_z \psi_-^i + \cdots \right.$$
$$+ g^{i\bar{\jmath}} \partial_i W \partial_{\jmath} \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar{\imath}} \psi_-^{\bar{\jmath}} D_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W}$$

The superpotential $W : X \longrightarrow \mathbf{C}$ is holomorphic, (so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....

(A model: Fan, Jarvis, Ruan, ...; Ito; Guffin, ES)

LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

$$b(\phi)^{j_1 \cdots j_m}_{\overline{\imath}_1 \cdots \overline{\imath}_n} \eta^{\overline{\imath}_1} \cdots \eta^{\overline{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where η, θ are linear comb's of ψ

$$Q \cdot \phi^i = 0, \;\; Q \cdot \phi^{\overline{\imath}} = \eta^{\overline{\imath}}, \;\; Q \cdot \eta^{\overline{\imath}} = 0, \;\; \underline{Q \cdot \theta_j = \partial_j W}, \;\; Q^2 = 0$$

Identify $\quad \eta^{\overline{\imath}} \leftrightarrow d\overline{z}^{\overline{\imath}}, \;\; \theta_j \leftrightarrow \dfrac{\partial}{\partial z^j}, \;\; Q \leftrightarrow \overline{\partial}$

so the states are hypercohomology

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

Quick checks:

1) W=0, standard B-twisted NLSM

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

$$\mapsto H^{\cdot}\left(X, \Lambda^{\cdot} TX\right) \checkmark$$

2) X=$\mathbf{C}^n$, W = quasihomogeneous polynomial

Seq' above resolves fat point {dW=0}, so

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

$$\mapsto \mathbf{C}[x_1, \cdots, x_n]/(dW) \checkmark$$

To A twist, need a U(1) isometry on X w.r.t. which the superpotential is quasi-homogeneous.

Twist by ``R-symmetry + isometry"

Let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) \;=\; \lambda W(\phi_i)$$

then twist: $$\psi \;\mapsto\; \Gamma\left(\text{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K_\Sigma}^{-(1/2)Q_L}\right)$$

where $$Q_{R,L}(\psi) \;=\; Q(\psi) \;+\; \begin{cases} 1 & \psi = \psi_+^i, R \\ 1 & \psi = \psi_-^i, L \\ 0 & \text{else} \end{cases}$$

Example: $X = \mathbf{C}^n$, W quasi-homog' polynomial

Here, to A twist, need to make sense of e.g. $K_\Sigma^{1/r}$

where r = 2(degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)

-- but then usually can't make sense of $K_\Sigma^{1/r}$

I'll work with the latter case.

LG A model:

A twistable example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)
with $W = p\pi^* s,\ s \in \Gamma(B, \mathcal{E})$

Accessible states are Q-closed (mod Q-exact) prod's:

$$b(\phi)_{\overline{\imath}_1 \cdots \overline{\imath}_n j_1 \cdots j_m} \psi_-^{\overline{\imath}_1} \cdots \psi_-^{\overline{\imath}_n} \psi_+^{j_1} \cdots \psi_+^{j_m}$$

where

$$\phi \sim \{s = 0\} \subset B \qquad \psi \sim TB|_{\{s=0\}}$$

$$Q \cdot \phi^i = \psi_+^i,\ \ Q \cdot \phi^{\overline{\imath}} = \psi_-^{\overline{\imath}},\ \ Q \cdot \psi_+^i = Q \cdot \psi_-^{\overline{\imath}} = 0,\ \ Q^2 = 0$$

Identify $\psi_+^i \leftrightarrow dz^i,\ \ \psi_-^{\overline{\imath}} \leftrightarrow d\overline{z}^{\overline{\imath}},\ \ Q \leftrightarrow d$

so the states are elements of $H^{m,n}(B)|_{\{s=0\}}$

Witten equ'n in A-twist:

BRST: $$\delta \psi_-^i = -\alpha \left(\overline{\partial} \phi^i - i g^{i\overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} \right)$$

implies localization on sol'ns of

$$\overline{\partial} \phi^i - i g^{i\overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} = 0$$ (``Witten equ'n")

On complex Kahler mflds, there are 2 independent BRST operators:

$$\delta \psi_-^i = -\alpha_+ \overline{\partial} \phi^i + \alpha_- i g^{i\overline{\jmath}} \partial_{\overline{\jmath}} \overline{W}$$

which implies localization on sol'ns of

$$\begin{aligned} \overline{\partial} \phi^i &= 0 \\ g^{i\overline{\jmath}} \partial_{\overline{\jmath}} \overline{W} &= 0 \end{aligned}$$

which is what we're using.

LG A model, cont'd

In prototypical cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 \, - \, \chi^p dz^i D_i s \, - \, \mathrm{c.c.} \, - \, F_{i\overline{\jmath}} dz^i d\overline{z}^{\overline{\jmath}} \chi^p \chi^{\overline{p}} \right.}_{\mathrm{Mathai-Quillen\ form}}$$

The MQ form rep's a Thom class, so

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \mathrm{Eul}(N_{\{s=0\}/\mathcal{M}}) \\ &= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \end{aligned}$$

-- same as A twisted NLSM on {s=0}

Not a coincidence, as we shall see shortly.

Example:

LG model on Tot(O(-5) --> **P**^{4}),
W = p s

Twisting: $p \in \Gamma(K_\Sigma)$

Degree 0 (genus 0) contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathbf{P}^4} d^2\phi^i \int \coprod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-|s|^2 - \chi^i \chi^p D_i s - \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} - R_{ip\overline{p}\overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

(curvature term ~ curvature of O(-5))

(cont'd)

Example, cont'd

In the A twist (unlike the B twist), the superpotential terms are BRST exact:

$$Q \cdot \left(\psi_-^i \partial_i W \; - \; \psi_+^{\overline{\imath}} \partial_{\overline{\imath}} \overline{W} \right) \; \propto \; -|dW|^2 \; + \; \psi_+^i \psi_-^j D_i \partial_j W \; + \; \mathrm{c.c.}$$

So, under rescalings of W by a constant factor λ, physics is unchanged:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathbf{P}^4} d^2\phi^i \int \prod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-\lambda^2 |s|^2 \; - \; \lambda \chi^i \chi^p D_i s \; - \; \lambda \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} \; - \; R_{ip\overline{p}\overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

Example, cont'd

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathbf{P}^4} d^2\phi^i \int \coprod_i d\chi^i d\chi^{\overline{\imath}} d\chi^p d\chi^{\overline{p}} \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-\lambda^2 |s|^2 \, - \, \lambda \chi^i \chi^p D_i s \, - \, \lambda \chi^{\overline{p}} \chi^{\overline{\imath}} D_{\overline{\imath}} \overline{s} \, - \, R_{i p \overline{p} \overline{k}} \chi^i \chi^p \chi^{\overline{p}} \chi^{\overline{k}} \right)$$

Limits:

1) $\lambda \longrightarrow 0$

Exponential reduces to purely curvature terms;
bring down enough factors to each up χ^p zero modes.

Equiv to, inserting Euler class.

2) $\lambda \longrightarrow \infty$

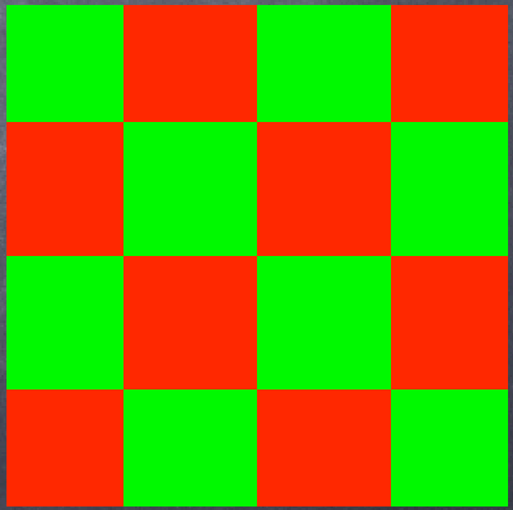
Localizes on $\{ s = 0 \} \subset \mathbf{P}^4$

Equivalent results,
either way.

Renormalization (semi)group flow

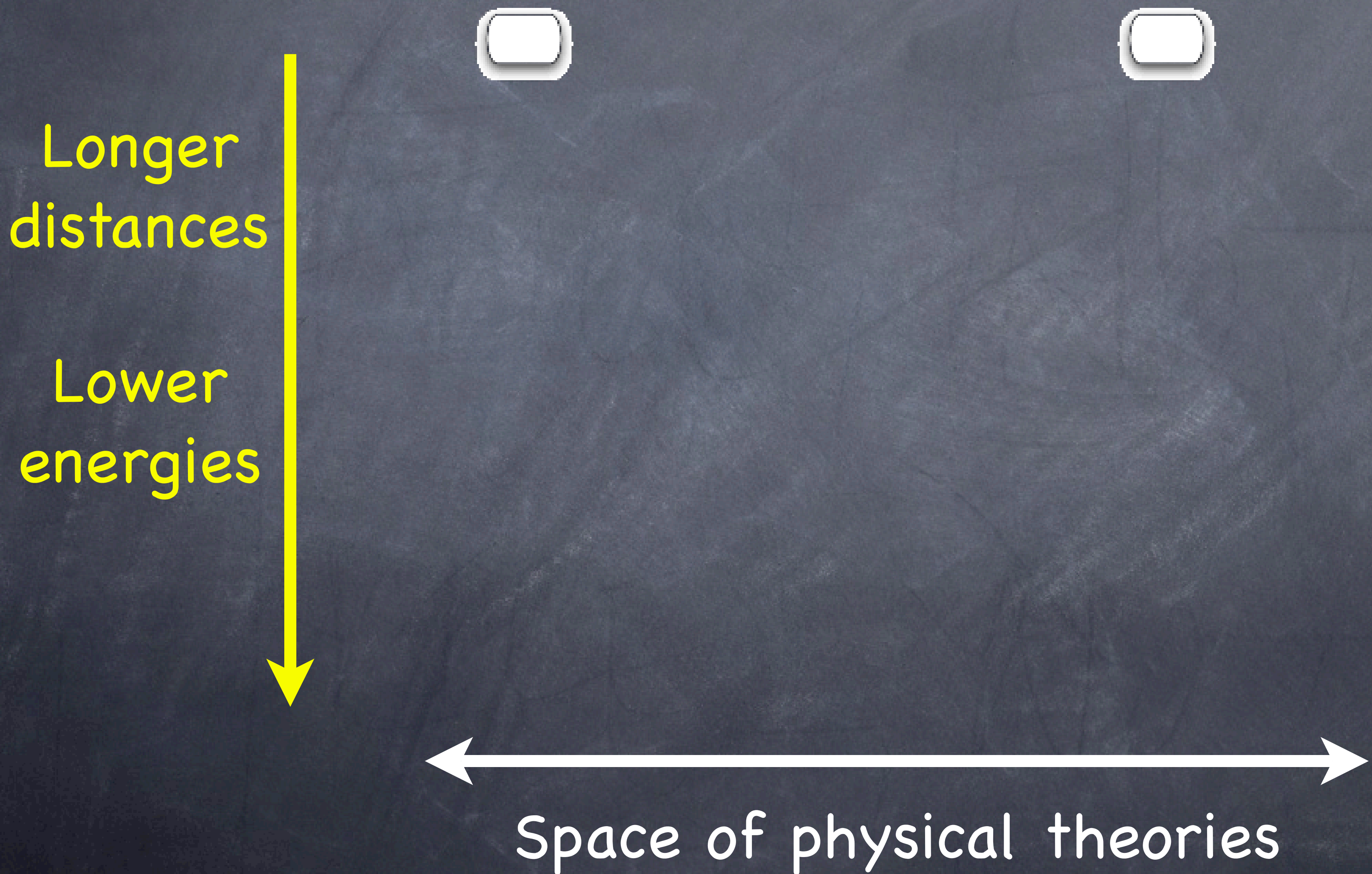
Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.



Problem: cannot follow it explicitly.

Renormalization group



Furthermore, RG preserves TFT's.

If two physical theories are related by RG,
then, correlation functions in a top' twist of one
=
correlation functions in corresponding twist of other.

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p s

$\Big\downarrow$ Renormalization group flow

NLSM on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

This is why correlation functions match.

So far we've outlined (2,2) Landau-Ginzburg models.

Let's now turn to (0,2) Landau-Ginzburg models....

Heterotic Landau-Ginzburg model:

$$S = \int_\Sigma d^2x \Big(g_{i\bar{\jmath}} \partial\phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i h_{a\overline{b}} \lambda_-^{\overline{b}} D_z \lambda_-^a + \cdots$$
$$+ h^{a\overline{b}} F_a \overline{F}_{\overline{b}} + \psi_+^i \lambda_-^a D_i F_a + \text{c.c.}$$
$$+ h_{a\overline{b}} E^a \overline{E}^{\overline{b}} + \psi_+^i \lambda_-^{\overline{a}} D_i E^b h_{\overline{a}b} + \text{c.c.} \Big)$$

Has two superpotential-like pieces of data

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^\vee)$$

such that $\displaystyle\sum_a E^a F_a = 0$

Pseudo-topological twists:

* If $E^a = 0$, then can perform std B/2 twist

$$\psi_+^{\bar{\imath}} \in \Gamma((\phi^* T^{1,0}X)^\vee) \qquad \lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$$

Need $\Lambda^{\text{top}} \mathcal{E} \cong K_X, \ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$

States $\mathbf{H}^{\cdot} \left(\cdots \longrightarrow \Lambda^2 \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{E} \xrightarrow{i_{F_a}} \mathcal{O}_X \right)$

* If $F_a = 0$, then can perform std A/2 twist

$$\psi_+^{i} \in \Gamma(\phi^* T^{1,0}X) \qquad \lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$$

Need $\Lambda^{\text{top}} \mathcal{E}^\vee \cong K_X, \ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$

States $\mathbf{H}^{\cdot} \left(\cdots \longrightarrow \Lambda^2 \mathcal{E}^\vee \xrightarrow{i_{E^a}} \mathcal{E}^\vee \xrightarrow{i_{E^a}} \mathcal{O}_X \right)$

* More gen'ly, must combine with $\mathbf{C}^*$ action.

Heterotic LG models are related to heterotic NLSM's via renormalization group flow.

Example:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$

with $\mathcal{E}' = \pi^* \mathcal{F}_2$ & $F_a \equiv 0, \quad E^a \neq 0$

$$\Big\downarrow \text{Renormalization group}$$

A heterotic NLSM on B

with $\mathcal{E} = \mathrm{coker}\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Adams-Basu-Sethi Example:

Corresponding to NLSM on $\mathbf{P}^1 \times \mathbf{P}^1$ with E′ as cokernel

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$* = \begin{bmatrix} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{bmatrix}$$

have (upstairs in RG) LG model on

$$X = \mathrm{Tot}\left(\mathcal{O} \oplus \mathcal{O} \xrightarrow{\pi} \mathbf{P}^1 \times \mathbf{P}^1 \right)$$

with $$\mathcal{E} = \pi^* \mathcal{O}(1,0)^2 \oplus \pi^* \mathcal{O}(0,1)^2$$

$$F_a \equiv 0 \qquad \begin{array}{ll} E^1 = x_1 p_1 + \epsilon_1 x_1 p_2 & E^3 = \tilde{x}_1 p_1 \\ E^2 = x_2 p_1 + \epsilon_2 x_2 p_2 & E^4 = \tilde{x}_2 p_2 \end{array}$$

Example, cont'd

Since $F_a = 0$, can perform std A twist.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathbf{P}^1 \times \mathbf{P}^1} d^2x \int d\chi^i \int d\lambda^{\overline{a}} \mathcal{O}_1 \cdots \mathcal{O}_n \left(\lambda^{\overline{a}} \tilde{E}_1^{\overline{a}} \right) \left(\lambda^{\overline{b}} \tilde{E}_2^{\overline{b}} \right) f(\tilde{E}_1^{\overline{a}}, \tilde{E}_2^{\overline{a}})$$

which reproduces std results for quantum sheaf cohomology in this example.

One can also compute elliptic genera in these models.

For the given example,
elliptic genus proportional to

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\left(\otimes S_{q^n}((TB)^{\mathbf{C}}) \otimes S_{q^n}((e^{-i\gamma}\mathcal{F}_1)^{\mathbf{C}}) \otimes \Lambda_{-q^n}((e^{-i\gamma}\mathcal{F}_2)^{\mathbf{C}}\right)$$

and there is a Thom class argument that
this matches a corresponding elliptic genus
of the NLSM related by RG flow.

Example in detail: Heterotic string on quintic, bundle = deformation of tangent bundle

LG model on $X = \mathrm{Tot}\left(\mathcal{O}(-5) \rightarrow \mathbf{P}^4\right)$

gauge bundle $\mathcal{E} = TX$

$$E^a \equiv 0 \qquad F_a = (G, p(D_i G + G_i))$$

$G \in \Gamma(\mathcal{O}(5))$ $\qquad$ p fiber coord'

Flows under RG to (0,2) theory on $\{G = 0\} \subset \mathbf{P}^4$

w/ gauge bundle a def of tangent bundle, defined by the G_i

(cont'd)

Perform A/2 twist.

If restrict to zero modes,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$

$$= \int d^2\phi^i \int d\chi^i \int d\lambda^{\overline{\imath}} \int d\chi^{\overline{p}} \int d\lambda^p \, \mathcal{O}_1 \cdots \mathcal{O}_n$$

$$\cdot \exp\left(-|G|^2 - \chi^i \lambda^p D_i G - \chi^{\overline{p}} \lambda^{\overline{\imath}} \left(D_{\overline{\imath}} \overline{G} + \overline{G}_{\overline{\imath}} \right) - R_{i\overline{p}p\overline{k}} \chi^i \chi^{\overline{p}} \lambda^p \lambda^{\overline{k}} \right)$$

Integrate out $\chi^{\overline{p}}$, λ^p:

$$= \int d^2\phi^i \int d\chi^i \int d\lambda^{\overline{\imath}} \mathcal{O}_1 \cdots \mathcal{O}_n \left[\left(\chi^i D_i G \right) \left(\lambda^{\overline{\imath}} \left(D_{\overline{\imath}} \overline{G} + \overline{G}_{\overline{\imath}} \right) \right) + R_{i\overline{p}p\overline{k}} g^{p\overline{p}} \chi^i \lambda^{\overline{k}} \right]$$

$$\cdot \exp\left(-|G|^2 \right)$$

Above is a (0,2) deformation of a Mathai-Quillen form.

More gen'ly, based on GLSM arguments,
Melnikov-McOrist have a formal argument that
A/2 twist should be independent of F's
B/2 twist should be independent of E's

Most general case:

LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \oplus \mathcal{F}_3^\vee \stackrel{\pi}{\longrightarrow} B \right)$

with gauge bundle $\mathcal{E}$ given by

$$0 \longrightarrow \pi^* \mathcal{G}^\vee \longrightarrow \mathcal{E} \longrightarrow \pi^* \mathcal{F}_2 \longrightarrow 0$$

$\Big\downarrow$ Renormalization group

NLSM on $Y \equiv \{ G_\mu = 0 \} \subset B$ $\quad G_\mu \in \Gamma(\mathcal{G})$

with bundle $\mathcal{E}'$ given by cohom' of the monad

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

(2,2) locus: $\mathcal{F}_1 = 0, \ \mathcal{F}_2 = TB, \ \mathcal{F}_3 = \mathcal{G}$

Heterotic GLSM phase diagrams:

Heterotic GLSM phase diagrams are famously different from (2,2) GLSM phase diagrams;
however,
the analysis of earlier still applies.

A LG model on X, with bundle E,
can be on the same Kahler phase diagram as
a LG model on X', with bundle E',
if X birat'l to X', and E, E' match on the overlap.

(necessary, not sufficient)

Example:

NLSM on $\{G=0\} \subset W\mathbf{P}^4_{w_1,\cdots,w_5}$ $\qquad G \in \Gamma(\mathcal{O}(d))$

with bundle $\mathcal{E}'$ given by

$$0 \longrightarrow \mathcal{E}' \longrightarrow \oplus\mathcal{O}(n_a) \longrightarrow \mathcal{O}(m) \longrightarrow 0$$

is described (upstairs in RG) by a LG model on

$$X \;=\; \mathrm{Tot}\left(\mathcal{O}(-m) \stackrel{\pi}{\longrightarrow} W\mathbf{P}^4\right)$$

with bundle $\;0 \longrightarrow \pi^*\mathcal{O}(d) \longrightarrow \mathcal{E} \longrightarrow \oplus\pi^*\mathcal{O}(n_a) \longrightarrow 0$

and is related to LG on

$$\mathrm{Tot}\left(\oplus\mathcal{O}(-w_i) \longrightarrow B\mathbf{Z}_m\right) \;=\; [\mathbf{C}^5/\mathbf{Z}_m]$$

with ~ same bundle.

Summary:

-- overview of progress towards (0,2) mirrors;
starting to heat up!

-- outline of quantum sheaf cohomology
(part of (0,2) mirrors story)

-- (2,2) and (0,2) Landau-Ginzburg models over
nontrivial spaces

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