Duality in two-dimensional nonabelian gauge theories

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B Jia, ES, R Wu, arXiv: 1401.1511

My talk today concerns dualities in two-dimensional gauge theories, two-dimensional analogues of Seiberg duality.

Let's recall some basics of Seiberg duality.

Idea: systematic procedure/examples of susy nonabelian gauge theories which look very different,
but which are believed to flow to the same point in the IR.

Since RG is lossy, the fact that there exist theories which flow to the same point is not surprising;
but, Seiberg duality gives nontrivial examples,
and so was of great interest when it was first discovered.

Basic example of Seiberg duality in 4d:

Electric theory:

$SU(N_c)$ gauge theory

N_f chiral mult's Q in fund'

N_f chiral mult's $\tilde{Q}$ in antifund'

Magnetic theory:

$SU(N_f\text{-}N_c)$ gauge theory

N_f chiral mult's q in fund'

N_f chiral mult's $\tilde{q}$ in antifund'

N_f^2 neutral chirals M

$$W = Mq\tilde{q}$$

Both have global symmetry $SU(N_f)$ x $SU(N_f)$ x $U(1)_B$ x $U(1)_R$

Flow to same IR fixed point, at least for $3N_c/2 < N_f < 3\ N_c$

Check:

- 't Hooft anomalies
- compare baryons & mesons
- moduli spaces

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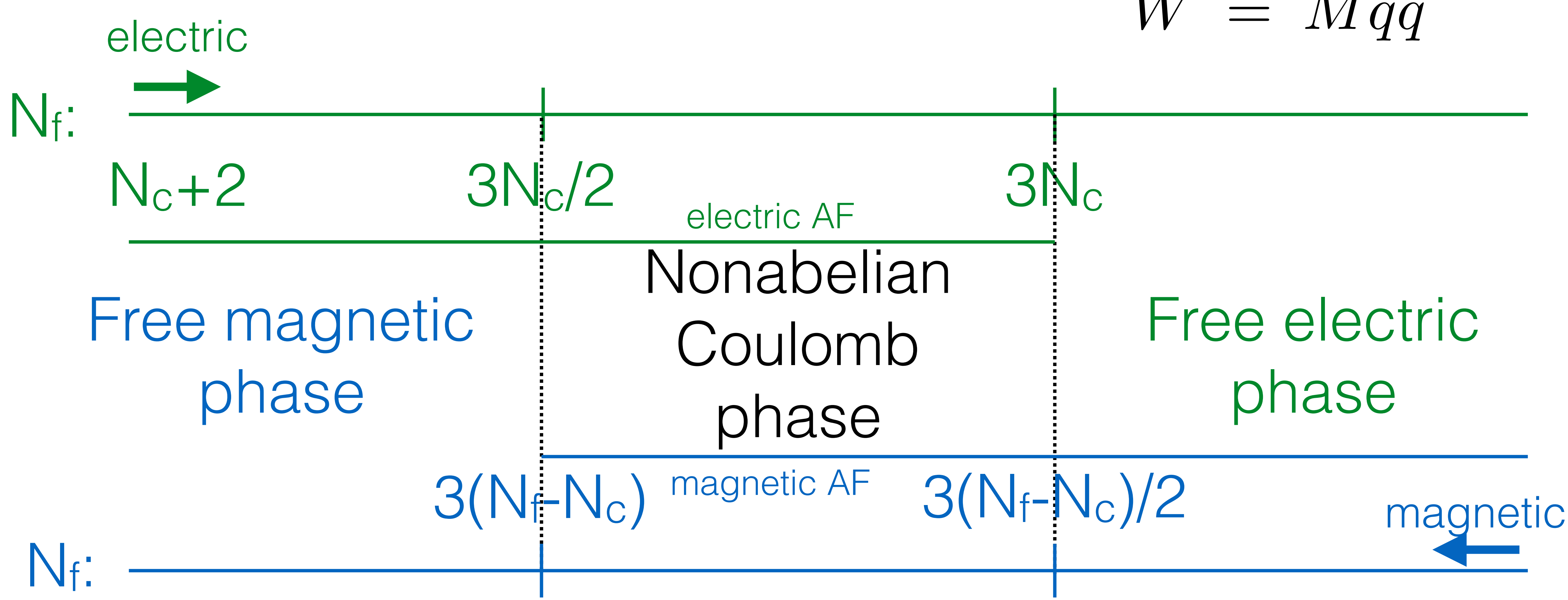

Basic example of Seiberg duality in 4d:

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Magnetic theory:

SU(N_f-N_c) gauge theory

N_f chiral mult's q in fund'

N_f chiral mult's $\tilde{q}$ in antifund'

N_f^2 neutral chirals M

$$W = Mq\tilde{q}$$

Gauge inv't chiral ops include:

Baryons $\quad B_{i_1\cdots i_{N_c}} = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1}\cdots Q^{i_{N_c}}_{a_{N_c}} \qquad b_{i_1\cdots i_{N_f-N_c}} = \epsilon^{a_1\cdots a_{N_f-N_c}} q^{i_1}_{a_1}\cdots q^{i_{N_f-N_c}}_{a_{N_f-N_c}}$

Mesons $\quad Q^i_a \tilde{Q}^a_j$

Within the \`conformal window' $3N_c/2 < N_f < 3N_c$,

$$\epsilon^{i_1\cdots i_{N_c} i_{N_c+1}\cdots i_{N_f}} B_{i_1\cdots i_{N_c}} = \epsilon^{a_1\cdots a_{N_f-N_c}} q^{i_{N_c+1}}_{a_1}\cdots q^{i_{N_f}}_{a_{N_f-N_c}} \quad \text{or} \quad *B = b$$

$$M^i_j = Q^i_a \tilde{Q}^a_j$$

My talk today concerns analogues of Seiberg duality in two-dimensional nonabelian gauge theories with (2,2) and (0,2) supersymmetry.

These theories arise as part of recent ongoing developments in gauged linear sigma models (GLSM's).

I'll outline constructions of a number of nonabelian theories and their duals, when known.

Theme: dualities derived from geometry

Outline

- (2,2) theories:
  - review $\mathbb{CP}^N$ model, hypersurfaces, Grassmannian
  - Theories w/ both fundamentals and antifundamentals - Benini-Cremonesi duality, and first application of geometry to derive gauge theory dualities
  - Abelian/nonabelian duality: G(2,4) vs $\mathbb{P}^5[2]$
  - Pfaffian constructions and more dualities
- (0,2) theories:
  - `gauge bundle dualization duality'
  - Gadde-Gukov-Putrov triality via geometry,
  - abelian/nonabelian examples, Pfaffian examples
- Obstructions to some dualities
- Decomposition

Gauge duality from geometry

Two-dimensional gauge theories are very different from four-dimensional gauge theories.

Crucial difference: no gauge dynamics.

In effect, in 2d,
gauge fields = Lagrange multipliers.

As a result, all gauge effects can be understood as low-energy NLSM effects.

Ex: gauge instantons =>
worldsheet instantons in low-energy NLSM

In principle, makes 2d Seiberg duality a lot easier.

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Analyze semiclassical low-energy behavior:

Potential $V = D^2$

where $$D = \sum_i |\phi_i|^2 - r$$

r = Fayet-Iliopoulos parameter

When $r \gg 0$, $\{V=0\} = S^{2n+1}$

so semiclassical Higgs moduli space is $\{V=0\}/U(1) = \mathbb{CP}^n$

Prototypical example: $\mathbb{CP}^n$ **model** **((2,2) susy)**

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Semiclassical Higgs moduli space is $\{V=0\}/U(1) = \mathbb{CP}^n$

Of course, that doesn't tell the whole story.

r is renormalized at one-loop:

$$\Delta r \propto \sum_i q_i \qquad \text{here, } = n+1$$

so the $\mathbb{CP}^n$ shrinks to strong coupling under RG.

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Summary:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

$\Big\Downarrow$ RG

Nonlinear sigma model on $\mathbb{CP}^n$

$\Big\Downarrow$ RG

Hypersurfaces **((2,2) susy)**

For later use, it will be handy to describe hypersurfaces.

S'pose want NLSM on $\{G=0\} \subset \mathbb{CP}^n$

where G is a homogeneous polynomial of degree d.

Try: U(1) gauge theory, n+1 chiral multiplets charge +1,
superpotential W = G

But that superpotential is not gauge invariant,
so this isn't the answer.

Correct method….

Hypersurfaces **((2,2) susy)**

Want gauge theory with low energy limit

= NLSM on $\{G=0\} \subset \mathbb{CP}^n$

Answer:

U(1) gauge theory, n+1 chiral multiplets charge +1,
1 chiral multiplet P charge -d,
superpotential W = P *G*

Gauge-invariant superpotential

$r \gg 0: \quad D = \sum_i |\phi_i|^2 - d\,|p|^2 - \mathrm{r}$ implies ϕ_i not all zero

$G=0, \quad pdG=0$ imply, for smooth hypersurface,

$p=0, \quad G=0$

Result is desired NLSM at low energies
(modulo r renormalization)

Next example: nonabelian version **((2,2) susy)**

Gauge theory:

U(k) gauge group,

matter: n chiral multiplets in fund' **k**, n > k

Similar analysis:

Nonlinear sigma model on G(k,n)

Consistency check: when k=1, G(k,n) = $\mathbb{CP}^{n-1}$

Next example: nonabelian version ((2,2) susy)

Dualities:

Mathematically, $G(k,n) = G(n-k,n)$

Since IR limits are same,

U(k) gauge group,
matter: n chiral multiplets in fund' **k**, n > k

is Seiberg dual to

U(n-k) gauge group,
matter: n chiral multiplets in fund' **k**
(n > n-k trivially)

Automatic: same chiral rings, same anomalies,
same Higgs moduli space

Next example: nonabelian version ((2,2) susy)

What if we add antifundamentals ?

Answer (Benini-Cremonesi, '12):

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' $\mathbf{k}^*$, A<n

is Seiberg dual to

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**, A chirals P in antifund' $\mathbf{k}^*$,
nA neutral chirals M,
superpotential: $W = M \Phi P$

B-C justified by checking elliptic genera;
we will justify with geometry momentarily….

Next example: nonabelian version ((2,2) susy)

We can understand that case geometrically.

U(k) gauge group,
matter: n chirals in fund' **k**, A chirals in antifund' **k***

Nonlinear sigma model on $\mathrm{Tot}\left(S^{A} \rightarrow G(k,n)\right)$

Duality: $= \mathrm{Tot}\left((Q^{*})^{A} \rightarrow G(n\text{-}k,n)\right)$

generalizing G(k,n) = G(n-k,n)

But how to realize $\mathrm{Tot}\left((Q^{*})^{A} \rightarrow G(n\text{-}k,n)\right)$?

Next example: nonabelian version ((2,2) susy)

How to realize $\mathrm{Tot}\left((Q^*)^A \rightarrow G(n\text{-}k,n)\right)$ in physics?

Trick: S, Q are related:

$$0 \rightarrow S \xrightarrow{\Phi} O^n \rightarrow Q \rightarrow 0$$

so we build Q using S, O^n,
and a superpotential realizing the map.

Here: A antifundamentals P, to realize S^A

nA neutrals M, to realize A copies of O^n

superpotential $W = M\Phi P$

— matching B-C dual

Next example: nonabelian version ((2,2) susy)

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' **k***, A<n

Seiberg
$\longleftrightarrow$
dual

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' **k***,
nA neutral chirals M,
superpotential: W = MΦ P

$\downarrow$ $\downarrow$

$$\mathrm{Tot}\left(S^A \to G(k,n)\right) = \mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k,n)\right)$$

In this fashion, we can understand this 2d version of Seiberg duality purely geometrically.

Next example: nonabelian version ((2,2) susy)

What about more general matter representations?
Adjoints, higher tensors, etc?

In 4d, demanding asymptotic freedom would exclude most arbitrarily complicated matter representations.

In 2d, no such constraint in principle.

However, we will argue later that there may be different constraints in 2d that make dualities for more complicated matter representations, rare.

Abelian/nonabelian dualities ((2,2) susy)

In 2d there are also Seiberg-like dualities between abelian and nonabelian theories.

Simple example: $G(1,n) = G(n-1,n)$

LHS = $U(1)$ gauge theory, n chiral multiplets

RHS = $U(n-1)$ gauge theory, n chiral multiplets

More fun example next….

Abelian/nonabelian dualities ((2,2) susy)

A more interesting example is motivated by the geometry

$$G(2,4) \quad = \quad \text{degree 2 hypersurface in } \mathbb{P}^5$$

Abelian/nonabelian dualities ((2,2) susy)

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

$\longleftrightarrow$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

RG $\downarrow$ RG $\downarrow$

G(2,4) = degree 2 hypersurface in $\mathbb{P}^5$

$= \mathbb{C}^{2\cdot 4} // GL(2)$

$= \{ z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23} \} \subset \mathbb{C}^6 // \mathbb{C}^\times$

RG $\downarrow$

…

The physical duality implied at top relates abelian & nonabelian gauge theories, which in 4d for ex would be surprising.

Abelian/nonabelian dualities ((2,2) susy)

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

$\longleftrightarrow$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

Relation: $z_{ij} = \epsilon_{\alpha\beta} \phi_i^\alpha \phi_j^\beta$

Consistency checks:

Compare symmetries: GL(4) action

$$\phi_i^\alpha \mapsto V_i^{\ j} \phi_j^\alpha \qquad\qquad z_{ij} \mapsto V_i^{\ k} V_j^{\ \ell} z_{k\ell}$$

Chiral rings, anomalies, Higgs moduli space match automatically.

Can also show elliptic genera match, applying computational methods of Benini-Eager-Hori-Tachikawa '13, Gadde-Gukov '13.

Abelian/nonabelian dualities ((2,2) susy)

Brief outline of elliptic genus of $\mathbb{P}^5[2]$:

By applying susy localization, can derive exact expressions in terms of iterated residues.

(Benini, Eager, Hori, Tachikawa '13; Gadde, Gukov '13)

Here,

$$Z = \frac{2\pi\eta(q)^3}{\theta_1(q,y^{-1})} \oint du \left(\prod_{i,j} \frac{\theta_1(q, y^{-1} x e^{2\pi i(\zeta_i+\zeta_j)})}{\theta_1(q, x e^{2\pi i(\zeta_i+\zeta_j)})} \right) \frac{\theta_1(q, x^{-2} e^{2\pi i(-\zeta_1-\zeta_2-\zeta_3-\zeta_4)})}{\theta_1(q, y x^{-2} e^{2\pi i(-\zeta_1-\zeta_2-\zeta_3-\zeta_4)})}$$

where the ζ_i are fugacities for $(\mathbb{C}^\times)^4 \subset GL(4)$ symmetry

Can show with eg Mathematica that the residues match those of corresponding flavored elliptic genus of G(2,4).

Abelian/nonabelian dualities ((2,2) susy)

This little game is entertaining,
but why's it useful ?

Standard physics methods rely on matching global symmetries and corresponding 't Hooft anomalies between prospective gauge duals.

However, generic superpotentials break all symmetries.

Identifying gauge duals as different presentations of the same geometry allows us to construct duals when standard physics methods do not apply.

Abelian/nonabelian dualities ((2,2) susy)

A simple set of examples in which global symmetry broken:

$$G(2,4)[d_1, d_2, \cdots] \qquad = \qquad \mathbb{P}^5[2, d_1, d_2, \cdots]$$

Abelian/nonabelian dualities ((2,2) susy)

A simple set of examples in which global symmetry broken:

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**
chirals p_a of charge $-d_a$
under det U(2)
superpotential

$$W = \sum_a p_a f_a(\epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta)$$

$\Big\downarrow$ RG

$$G(2,4)[d_1, d_2, \cdots]$$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
chirals P_a of charge $-d_a$,
superpotential

$$W = P(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}) + \sum_a P_a f_a(z_{ij})$$

$\Big\downarrow$ RG

$$\mathbb{P}^5[2, d_1, d_2, \cdots]$$

$$G(2,4)[d_1, d_2, \cdots] = \mathbb{P}^5[2, d_1, d_2, \cdots]$$

$$\epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta = z_{ij}$$

Straightforward extrapolation of previous duality,
as one might hope.

Pfaffians ((2,2) susy)

There exist more exotic dualities implied by geometry.
To justify them, need to outline construction of Pfaffians.

Let A be an nxn matrix,
each entry a homogeneous poly' over a proj' space
(or other toric variety), call it V.

A Pfaffian variety is defined by the locus on V where
$\mathrm{rank}\, A \leq k$ for some k.

- Not a hypersurface or a complete intersection in general.
- Only recently has anyone figured out how to describe such spaces with GLSM's. (Hori-Tong '06, Hori '11, Jockers et al '12)

Pfaffians ((2,2) susy)

Two constructions of Pfaff $= \{\mathrm{rank}\, A \leq k\}$

- PAX model

U(n-k) gauge theory,
chirals X_a in n copies of fundamental,
chirals P_a in n copies of antifundamental,
$W = \mathrm{tr}\, PA(\Phi)X$ (plus data for V)

- PAXY model

U(k) gauge theory,
chirals $\tilde{X}_a$ in n copies of fundamental,
chirals $\tilde{Y}^a$ in n copies of antifundamental,
nxn matrix of neutral chirals $\tilde{P}^a_b$,
$W = \mathrm{tr}\, \tilde{P}\left(A(\Phi) - \tilde{Y}\tilde{X}\right)$ (plus data for V)

These two constructions are dual to one another….

Pfaffians **((2,2) susy)**

Duality between PAX, PAXY constructions:

Start with PAX model:

U(n-k) gauge theory,

chirals X_a in n copies of fundamental,

chirals P_a in n copies of antifundamental,

$$W = \mathrm{tr}\, PA(\Phi)X$$ (plus data for V)

Apply B-C duality:

U(k) gauge theory,

chirals $\tilde{X}_a$ in n copies of fundamental,

chirals $\tilde{Y}^a$ in n copies of antifundamentals

n^2 neutral chirals $\tilde{P}^a_b = (XP)^a_b$,

plus a new superpotential term for total

$$W = \mathrm{tr}\left(A\tilde{P} + \tilde{P}\tilde{Y}\tilde{X} \right)$$ (plus data for V)

original term new term

Result = PAXY model

Pfaffians ((2,2) susy)

Start with standard math result:

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It’s then natural to propose….

Pfaffians ((2,2) susy)

U(2) gauge theory,
n chirals in fundamental

$\longleftrightarrow$

U(n-2)xU(1) gauge theory,
n chirals X in fundamental of U(n-2),
n chirals P in antifundamental of U(n-2)
(n choose 2) chirals z_{ij} = - z_{ji}
each of charge +1 under U(1),
W = tr PAX

RG RG

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(using description of Pfaffians of
Hori '11, Jockers et al '12)

Even more complicated possibilities exist.

So far, I've outlined various dualities in 2d (2,2) susy theories.

Next: 2d (0,2)

I'll begin by describing

- a frequently occurring duality
- dynamical susy breaking in (0,2)
- (0,2) superspace

and then discuss various gauge-theoretic dualities.

Gauge bundle dualization duality ((0,2) susy)

(Nope, not a typo….)

Nonlinear sigma models with (0,2) susy defined by space X, with gauge bundle $E \to X$

Duality: $\mathrm{CFT}(X, E) = \mathrm{CFT}(X, E^*)$

ie, replacing the gauge bundle with its dual seems to be an invariance of the theory.

(Parts in ES '06, complete in Gadde-Gukov-Putrov '13, Jia-ES-Wu '14)

We'll use this duality, but first, some checks….

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Action invariant:

$$L = \frac{1}{2} g_{\mu\nu} \partial\phi^\mu \overline{\partial}\phi^\nu + \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\overline{z}} \psi_+^\nu + \frac{i}{2} h_{\alpha\beta} \lambda_-^\alpha D_z \lambda_-^\beta + F_{i\overline{j}a\overline{b}} \psi_+^i \psi_+^{\overline{j}} \lambda_-^a \lambda_-^{\overline{b}}$$

Under $E \leftrightarrow E^*$, $\lambda_-^a \leftrightarrow \lambda_-^{\overline{b}}$ & $F \leftrightarrow -F$

so we see the Lagrangian is invariant.

Consistency conditions:

$\mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX})$ invariant under $E \leftrightarrow E^*$

Massless spectra:

$h^\bullet(X, \wedge^\bullet E)$, $h^\bullet(X, \mathrm{End}\, E)$ invariant under $E \leftrightarrow E^*$

using $h^p(X, \wedge^q E^*) \cong h^{n-p}(X, \wedge^{r-q} E)$ (Serre duality on CY)

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Bundle must be `stable': $g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0$

Math result: a bundle is stable iff its dual is stable.

Can also show:

- elliptic genera match
- compatible with worldsheet instantons

In fact, at some level, this is ~ trivial on worldsheet; just flipping complex structure on left movers.

Let's move on….

Review of (0,2) multiplets

Next I'll describe some (0,2) gauge theories,
so let me here briefly review (0,2) susy multiplets:

(2,2) chiral: $(\phi, \psi_+, \psi_-, F)$

(0,2) chiral: (ϕ, ψ_+)

(0,2) Fermi: (ψ_-, F)

(2,2) vector: $(A_\mu, \sigma, \lambda_+, \lambda_-, D)$ (WZ gauge)

(0,2) vector: (A_μ, λ_-, D)

(0,2) twisted chiral: (σ, λ_+)

Gadde-Gukov-Putrov triality **((0,2) susy)**

This is a Seiberg-like duality,
that closes after 3 steps instead of 2.

Let's walk through it.

Start: U(k) gauge theory,

matter: n chirals Φ in fund' **k**, n>k,
A Fermi's in antifund' $\mathbf{k}^*$,
B chirals P in antifund' $\mathbf{k}^*$,
nB neutral Fermi's Γ,

$$W = \Gamma \Phi P$$

There is a potential gauge anomaly,
which can be cancelled if $B = 2k - n + A$.

Let's analyze the geometry….

Gadde-Gukov-Putrov triality ((0,2) susy)

U(k) gauge theory,

matter: n chirals Φ in fund' **k**, n>k,

A Fermi's in antifund' $\mathbf{k}^*$,

B chirals P in antifund' $\mathbf{k}^*$,

nB neutral Fermi's Γ,

$$W = \Gamma \Phi P$$

$r \gg 0:$		$r \ll 0:$
Space: $G(k,n)$	$\xleftrightarrow{\quad r \quad}$	Space: $G(k,B)$
Bundle: $S^A \oplus (Q^*)^B$		Bundle: $(S^*)^A \oplus (Q^*)^n$

Can apply duality to either side….

Gadde-Gukov-Putrov triality ((0,2) susy)

$r \gg 0:$

Space: $G(k,n)$

Bundle: $S^A \oplus (Q^*)^B$

$$\xleftrightarrow{\quad r \quad}$$

$r \ll 0:$

Space: $G(k,B)$

Bundle: $(S^*)^A \oplus (Q^*)^n$

Let's look at geometric equivalences, on LHS:

$$G(k,n) = G(n-k,n)$$
$$S_k = Q^*_{n-k}$$
$$Q^*_k = S_{n-k}$$

So:

Space: $G(k,n)$ = Space: $G(n-k,n)$

Bundle: $S^A \oplus (Q^*)^B$ = Bundle: $(Q^*)^A \oplus S^B$

which implies a statement about (0,2) gauge theories.

Gadde-Gukov-Putrov triality ((0,2) susy)

In this fashion, we get a chain of dualities:

$$
\begin{tikzcd}
S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \arrow[r, dash, dotted, "\mathrm{r}"] \arrow[d, leftrightarrow, "="'] & (S^*)^A \oplus (Q^*)^n \to G(k,2k+A-n) \\
(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \arrow[r, dash, dotted, "\mathrm{r}"] & (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k,A) \arrow[d, leftrightarrow, "="] \\
(S^*)^n \oplus Q^A \to G(A-n+k,2k+A-n) \arrow[r, dash, dotted, "\mathrm{r}"] \arrow[d, leftrightarrow, "="'] & S^n \oplus Q^{2k+A-n} \to G(A-n+k,A) \\
Q^n \oplus (S^*)^A \to G(k,2k+A-n) \arrow[r, dash, dotted, "\mathrm{r}"] & Q^{2k+A-n} \oplus (S^*)^A \to G(k,n)
\end{tikzcd}
$$

But applying gauge bundle dualization duality,
last line = first line,
so there is a 3-step sequence.

Triality

Prelude to other examples: ((0,2) susy)

U(2) representation conventions

Our next examples will involve gauge bundles defined by more general representations of U(2), so let me take just a moment to outline conventions.

Will describe an irrep of U(2) by $(a,b), \; a \geq b$

(0,0) = trivial

(1,0) = **2**

(0,-1) = **2***

(1,-1) + (0,0) = ad

(a,a) = rep of det U(2)

dim (a,b) = a - b + 1, $\mathrm{Cas}_1(a,b) = a + b$

Abelian/nonabelian dualities ((0,2) susy)

Let's build on our previous duality

$$G(2,4)[d_1,d_2,\cdots]=\mathbb{P}^5[2,d_1,d_2,\cdots]$$

by extending to heterotic cases.

Example:

Bundle $0\to E\to\oplus^8 O(1,1)\to O(2,2)\oplus^2 O(3,3)\to 0$

on the CY G(2,4)[4].

Described by

U(2) gauge theory

4 chirals in fundamental

1 Fermi in (-4,-4) (hypersurface)

8 Fermi's in (1,1) (gauge bundle E)

1 chiral in (-2,-2) (gauge bundle E)

2 chirals in (-3,-3) (gauge bundle E)

plus superpotential

Abelian/nonabelian dualities ((0,2) susy)

Example:

Bundle $0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$

on the CY G(2,4)[4].

Geometry predicts this is dual to GLSM for

Bundle $0 \to E \to \oplus^8 O(1) \to O(2) \oplus^2 O(3) \to 0$

on the CY $\mathbb{P}^5[2,4]$

Checks:
- both satisfy anomaly cancellation
- elliptic genera match

To make the duality work, we used the fact that reps defining bundle all lives in det U(2)….

Abelian/nonabelian dualities ((0,2) susy)

Another example:

Bundle $0 \to E \to O(1,0) \oplus^5 O(2,1) \to O(3,1) \oplus^2 O(3,2) \to 0$

on the CY G(2,4)[4].

- Satisfies anomaly cancellation.
- No idea if there's an abelian dual on $\mathbb{P}^5[2,4]$.

Pfaffians **((0,2) susy)**

It's also possible to build (0,2) models on Pfaffians.

Deformations off (2,2) locus:

PAX:

$$W = \mathrm{tr}\left(\Lambda_P A(\Phi) X + P A(\Phi) \Lambda_X + P\left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} + G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha X \right)$$

PAXY:

$$W = \mathrm{tr}\Big(\Lambda_{\tilde{P}} A(\Phi) + \tilde{P}\left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} + G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha + \Lambda_{\tilde{P}} \tilde{X} \tilde{Y} + \tilde{P} \Lambda_{\tilde{X}} \tilde{Y} + \tilde{P} \tilde{X} \Lambda_{\tilde{Y}} \Big)$$

In both cases, $G_\alpha(\Phi)$ (satisfying certain conditions) define deformations off (2,2) locus.

These (0,2) PAX, PAXY models are related by Seiberg / B-C-like gauge duality.

Pfaffians ((0,2) susy)

More (0,2) models on Pfaffians.

Example: PAX model, Pfaffian $\{\mathrm{rank}\, A \leq 2\} \subset \mathbb{P}^7$

Bundle

$$0 \to E \to \oplus^5 O((0,0)_{-1}) \oplus^2 O((2,2)_0) \to \oplus^2 O((2,2)_{-1}) \oplus O((1,-1)_{-1}) \to 0$$

Described by

U(2)xU(1) gauge theory

4 chirals in $(0,-1)_0$ 4 Fermi's in $(1,0)_{-1}$

8 chirals in $(0,0)_{+1}$ (defining $\mathbb{P}^7$)

5 Fermi's in $(0,0)_{-1}$ 2 chirals in dual of $(2,2)_{-1}$

2 Fermi's in $(2,2)_0$ 1 chiral in dual of $(1,-1)_{-1}$

defines E

+ superpotential

- anomaly free
- dual not known

Possible obstructions to duality **((0,2) susy)**

So far we have discussed dualities in two-dimensional gauge theories with (anti)fundamentals.

What about more general matter representations?

From a geometric perspective,
our dualities have all boiled down to exchanging

$$G(k,n) \leftrightarrow G(n-k,n)$$

$$(S \to G(k,n)) \leftrightarrow (Q^* \to G(n-k,n))$$

What would be the analogue for more general matter reps ?

Possible obstructions to duality ((0,2) susy)

Geometrically, to dualize more general rep's, must construct resolutions of corresponding bundles.

Example: U(k) gauge theory, Fermi's in $\wedge^2 \overline{\mathbf{k}}$

(plus fundamental chirals….)

Pertinent bundle: $\wedge^2 S \rightarrow G(k,n)$

Dual: $\wedge^2 Q^* \rightarrow G(n-k,n)$

Q* cannot be realized directly, only indirectly w/ sequence.

To realize $\wedge^2 Q^*$ use

$$0 \rightarrow \wedge^2 Q^* \rightarrow \wedge^2 O^n \rightarrow S^* \otimes O^n \rightarrow \mathrm{Sym}^2 S^* \rightarrow 0$$

Potential Problem: how to realize that sequence physically.

Possible obstructions to duality ((0,2) susy)

Example, cont'd

To realize dual $\wedge^2 Q^*$ use

$$0 \to \wedge^2 Q^* \to \wedge^2 O^n \to S^* \otimes O^n \to \mathrm{Sym}^2 S^* \to 0$$

In open strings, this is easy, and implicitly I'm describing a prescription for dualizing arbitrary matter reps on boundaries.

But in (0,2) , we only know how to realize 3-term sequences.
To realize the dual above,
I'd need to realize a 4-term sequence,
and no one knows how to do that in (0,2).

Possible obstructions to duality ((0,2) susy)

This analysis suggests that it may be difficult to find a Seiberg-like gauge theoretic dual to a (0,2) theory with random matter representations.

Basic obstruction: we only know how to realize 3-term sequences in (0,2); we'd need to realize longer sequences.

Other hand: existence of (0,2) mirrors implies there should be gauge dualities **not** understandable as different presentations of same geometry.

Open string boundaries: no such obstruction, this gives instead a prescription for construction of duals.

Let’s conclude with one other worldsheet duality:
“decomposition”

Decomposition

In a 2d orbifold or gauge theory,
if a finite subgroup of the gauge group acts trivially on all matter, the theory decomposes as a disjoint union.

(Hellerman et al '06)

Ex: $\mathrm{CFT}([X/\mathbb{Z}_2]) = \mathrm{CFT}\left(X \coprod X\right)$

On LHS, the $\mathbb{Z}_2$ acts triv'ly on X,
hence there are dim' zero twist fields.
Projection ops are lin' comb's of dim 0 twist fields.

Ex: $\mathrm{CFT}([X/D_4])$ where $\mathbb{Z}_2 \subset D_4$ acts trivially on X

$$= \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

where $D_4 / \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$

This is what's meant by `decomposition'….

Decomposition

Decomposition is also a statement about mathematics.
Dictionary:

2d Physics	Math
D-brane	Derived category
Gauge theory	Stack
Gauge theory w/ trivially acting subgroup	Gerbe
Universality class of renormalization group flow	Categorical equivalence

Decomposition

Decomposition is also a statement about mathematics.
Dictionary:

2d Physics	Math
D-brane	Derived category
Gauge theory	Stack
Gauge theory w/ trivially acting subgroup	**Gerbe**
Universality class of renormalization group flow	Categorical equivalence

Decomposition is a statement about physics of strings on gerbes, summarized in the decomposition conjecture….

Decomposition

Decomposition conjecture: (version for banded gerbes)

(Hellerman et al ’06)

string on gerbe

$$\mathrm{CFT}(\text{G-gerbe on X}) = \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

string on disjoint union of spaces

where the B field is determined by the image of

characteristic class

$$H^2(X, Z(G)) \xrightarrow{Z(G) \mapsto U(1)} H^2(X, U(1))$$

flat B field

Consistent with:

- multiloop orbifold partition f’ns
- q.c. ring rel’ns as derived from GLSM’s
- D-branes, K theory, sheaves on gerbes

Applications:

- predictions for GW inv’ts, checked by H H Tseng et al ’08-‘10
- understand GLSM phases, via giving a physical realization of Kuznetsov’s homological projective duality for quadrics (Caldararu et al ’07, Hori ’11, Halverson et al ’13…)

Decomposition

$$\mathrm{CFT}(\text{G-gerbe on X}) = \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

Checking this statement in orbifolds involved comparing e.g. multiloop partition functions, state spaces, D-branes, …

In gauge theories, there are further subtleties.

Example:

Ordinary $\mathbb{CP}^n$ model = U(1) gauge theory with n+1 chiral superfields, each of charge +1

Gerby $\mathbb{CP}^n$ model = U(1) gauge theory with n+1 chiral superfields, each of charge +k, k>1

Require physics of charge k > 1 different from charge 1 — but how can multiplying the charges by a factor change anything?

Decomposition

Require physics of charge k > 1 different from charge 1
— but how can multiplying the charges by a factor change anything?

For physics to see gerbes, there must be a difference,
but why isn't this just a convention?
How can physics see this?

Answer: nonperturbative effects

Noncompact worldsheet: distinguish via θ periodicity

Compact worldsheet: define charged field via specific bundle

(Adams-Distler-Plesser, Aspen '04)

Decomposition has been extensively checked for *abelian*
gauge theories and orbifolds;
nonabelian gauge theories much more recent….

Decomposition

Extension of decomposition to nonabelian gauge theories:

Since 2d gauge fields don't propagate,
analogous phenomena should happen in nonabelian gauge theories with center-invariant matter.

Proposal: **(ES, '14)**

For G semisimple, with center-inv't matter,
G gauge theories decompose into a sum of theories with variable discrete theta angles:

Ex: $SU(2) = SO(3)_{+} + SO(3)_{-}$

— SO(3)'s have different discrete theta angles

Decomposition

Extension of decomposition to nonabelian gauge theories:

Aside: discrete theta angles

(Gaiotto-Moore-Neitzke '10, Aharony-Seiberg-Tachikawa '13, Hori '94)

Consider 2d gauge theory, group $G = \tilde{G} / K$

$\tilde{G}$ compact, semisimple, simply-connected

K finite subgroup of center of $\tilde{G}$

The theory has a degree-two K-valued char' class w

For λ any character of K, can add a term to the action

$$\lambda(w)$$

— discrete theta angles, classified by characters

Ex: $SO(3) = SU(2) / \mathbb{Z}_2$ has 2 discrete theta angles

Decomposition

Ex: SU(2) = SO(3)$_+$ + SO(3)$_-$

Let's see this in pure nonsusy 2d QCD.

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps
that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$

Decomposition

More general statement of decomposition for 2d nonabelian gauge theories with center-invariant matter:

For G semisimple, K a finite subgp of center of G,

$$G = \sum_{\lambda \in \hat{K}} (G/K)_\lambda$$

indexes discrete theta angles

Other checks include 2d susy partition functions, utilizing Benini-Cremonesi '12, Doroud et al '12; arguments there revolve around cocharacter lattices.

Summary: duality from geometry

- (2,2) theories:
  - review $\mathbb{CP}^N$ model, hypersurfaces, Grassmannian
  - Theories w/ both fundamentals and antifundamentals - Benini-Cremonesi duality, and first application of geometry to derive gauge theory dualities
  - Abelian/nonabelian duality: G(2,4) vs $\mathbb{P}^5[2]$
  - Pfaffian constructions and more dualities
- (0,2) theories:
  - `gauge bundle dualization duality'
  - Gadde-Gukov-Putrov triality via geometry,
  - abelian/nonabelian examples, Pfaffian examples
- Obstructions to some dualities
- Decomposition