Abelian GLSM's, gerbes, and homological projective duality

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T Pantev, ES, hepth/0502027, 0502044, 0502053

S Hellerman, A Henriques, T Pantev, ES, M Ando, hepth/0606034

R Donagi, ES, arxiv: 0704.1761

A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855

N Addington, E Segal, ES, to appear

In this talk, I'm going to describe how some examples of Kuznetov's homological projective duality (hpd) (for complete intersections of quadrics) are realized physically, as phases of abelian GLSM's.

GLSM = `gauged linear sigma model'
These are the bread-and-butter tools used by physicists to describe families of spaces and related aspects of string compactifications.

Hpd taught us a great deal about GLSM's and other physics, and that's what I'll discuss today.

WARNING: physics talk

What did hpd teach us?

Prior to ~ 2006, it was (falsely) believed that:

* GLSM's could only describe global complete intersections,

* which could only arise physically as critical locus of a superpotential, and

* GLSM Kahler `phases' are all birational to one another

The papers

Hori-Tong hep-th/0609032, Donagi-ES 0704.1761, Caldararu et al 0709.3855

provided counterexamples to each statement above, all special cases of hpd.

I won't describe homological projective duality itself, instead I'm going to focus on physics examples.

Prototype:

A complete intersection of k quadrics in $\mathbf{P}^n$,

$$\{Q_1 = \cdots = Q_k = 0\}$$

is hpd to

a (nc resolution of a) branched double cover of $\mathbf{P}^{k-1}$, branched over the locus

$$\{\det A = 0\}$$

where $$\sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

We'll see how examples of this form,
(CI quadrics vs branched double covers),
are realized physically as phases of abelian GLSM's.

To understand those GLSM's, we'll detour through
the physics of stacks & $\mathbf{Z}_2$ gerbes.

We'll begin with easy examples,
and get into more interesting cases,
for example in which nc resolutions arise physically.

We'll begin with the GLSM for $\mathbf{P}^3[2,2]$ (=T^2):

GLSM's are families of 2d gauge theories
that RG flow to families of CFT's.
In this case:

$r \gg 0$

one-parameter
Kahler moduli space

$r \ll 0$

NLSM on
$\mathbf{P}^3[2,2]$

r

LG
point
= branched
double
cover

GLSM for $\mathbf{P}^3[2,2]$ (=T^2):

Briefly, the GLSM consists of:

* 4 chiral superfields $\Phi_i = (\phi_i, \psi_i, F_i)$, one for each homogeneous coordinate on $\mathbf{P}^3$, each of charge 1 w.r.t. a gauged U(1)

* 2 chiral superfields $P_a = (p_a, \psi_{pa}, F_{pa})$, (one for each of the $\{Q_a = 0\}$), each of charge -2

-- Kentaro's language: matter $\mathbf{C}(1)^4 \oplus \mathbf{C}(-2)^2$

* a superpotential

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

The GLSM describes a symplectic quotient:

Moment map (D term):

$$\sum_i |\phi_i|^2 - 2\sum_a |p_a|^2 = r$$

$r \gg 0 :$ ϕ_i not all zero

Critical locus of superpotential $W = \sum_a p_a Q_a(\phi)$ is

$$p_a = Q_a = 0$$

NLSM on CY CI = $\mathbf{P}^3[2,2]$ = T^2

The other limit is more interesting....

Moment map (D term):

$$\sum_i |\phi_i|^2 - 2\sum_a |p_a|^2 = r$$

$r \ll 0 :$ p_a not all zero

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

implies that ϕ_i massive (since deg 2)

NLSM on $\mathbf{P}^1$????

That can't be right, since other phase is CY.

The correct analysis of the $r \ll 0$ limit is more subtle.

One subtlety is that the ϕ_i are not massive everywhere.

Write $$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

then they are only massive away from the locus

$$\{\det A = 0\} \subset \mathbf{P}^1$$

But that just makes things more confusing....

A more important subtlety is the fact that the p's have nonminimal charge,

so over most of the $\mathbf{P}^1$ of p vevs,

we have a nonminimally-charged abelian gauge theory,

meaning massless fields have charge -2,

instead of 1 or -1.

Mathematically, this is a string on a $\mathbf{Z}_2$ gerbe.

Let's briefly review gerbes, to understand implications.

How to define the QFT for a string on a stack?

Every* (smooth, Deligne-Mumford) stack can be presented as a global quotient $[X/G]$, for X a space and G a group.

To such a presentation, associate a G-gauged sigma model on X.

Use RG flow in 2d to wash out presentation-dependence. (Now thoroughly checked in 2d.)

A gerbe is defined by a quotient $[X/G]$, in which a subgroup of G acts trivially on X .

(* with minor caveats)

For the special case of stacks that are gerbes, there are further issues.

A gerbe is defined by a G-gauge theory in which a subgroup of G acts trivially.

First issue:

Physically, why is such a gauge theory
any different at all
from a gauge theory in which one quotients by the effectively-acting coset?

Answer: nonperturbative effects

To illustrate, imagine an analogue of the $\mathbf{CP}^{N-1}$ model but in which all chiral superfields have charge k instead of charge 1.

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$
$$\text{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} > = q^d$$
$$\text{Here}: \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$
$$\text{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies
$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

For noncpt worldsheets, analogous argument exists.

(Distler, Plesser, Aspen 2004 & hepth/05......; Seiberg, Banks-Seiberg 2010)

Strings on gerbes, cont'd

So far, we've outlined how physics sees ineffective group actions (via nonperturbative effects) -- so physics distinguishes gerbes from spaces.

Second issue:

The resulting theories violate `cluster decomposition', one of the foundational axioms of QFT. How is that consistent?

Answer:

strings on gerbes = strings on disjoint unions of spaces

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

stack

disjoint union of spaces

We now believe, for (2,2) CFT's,

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K \right] \right)$$

(together with some B field), where

$\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \text{gerbe on } Y) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (Y, B) \right)$$

$$(Y \;=\; [X/K])$$

where the B field is determined by the image of

$$H^2(Y, Z(G)) \stackrel{Z(G) \to U(1)}{\longrightarrow} H^2(Y, U(1))$$

Basic point:

Maps into $\mathbf{Z}_k$ gerbe over X
= maps into X of degree divisible by k

Path integral into disjoint union of k copies of X,
with variable B fields:

* if degree not divisible by k,
then proportional to sum over kth roots of unity
= 0 -- cancel out

* if degree is divisible by k,
then add instead of cancelling out

Result is same as path integral on gerbe.

Quick consistency check:

A sheaf on a banded G-gerbe
is the same thing as
a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

This implies a decomposition of D-branes (~ sheaves), which is precisely consistent with the decomposition conjecture.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K \right]$

Checked by H-H Tseng, Y Jiang, et al in
0812.4477, 0905.2258, 0907.2087, 0912.3580, 1001.0435, 1004.1376, ...

GLSM's

Let's now return to our analysis of GLSM's.

Example: $\mathbf{CP}^3[2,2]$

Superpotential: $$\sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

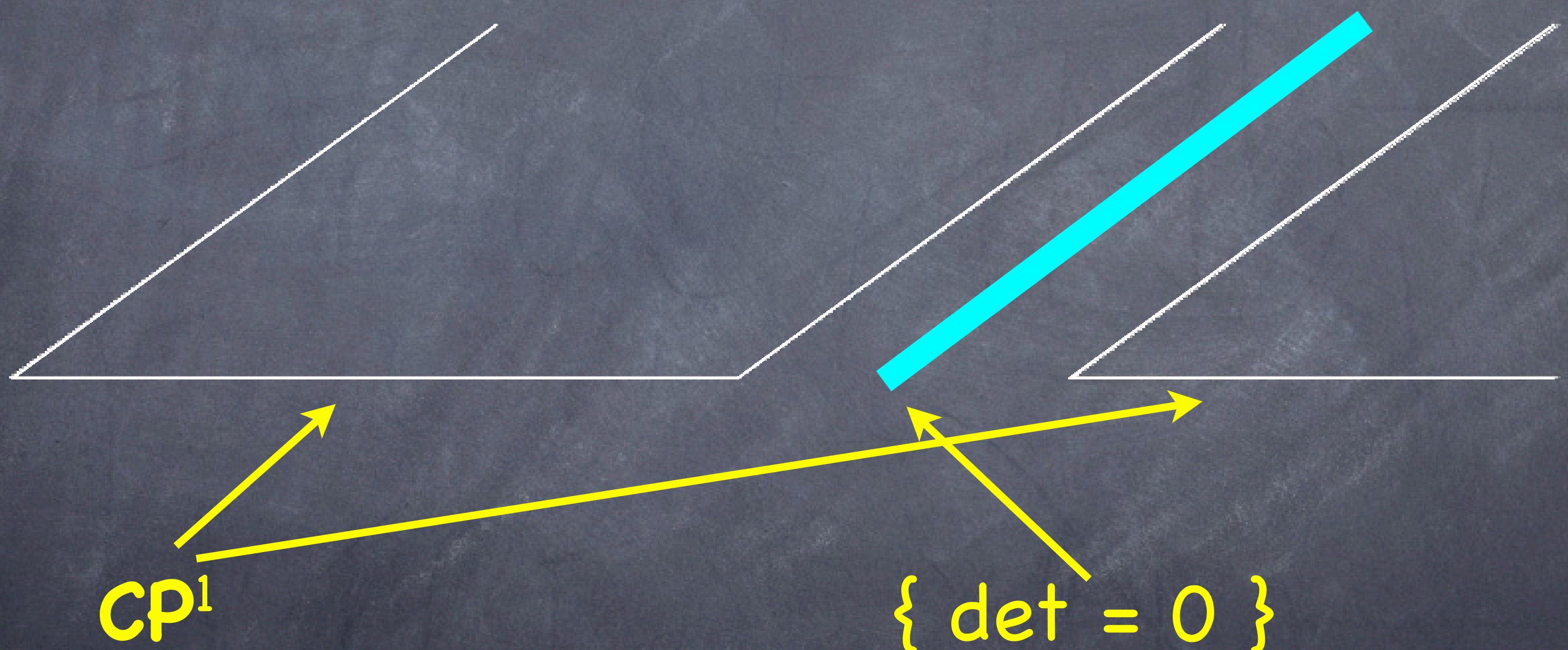
$r \ll 0:$

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

* leaves just the p fields, of charge -2

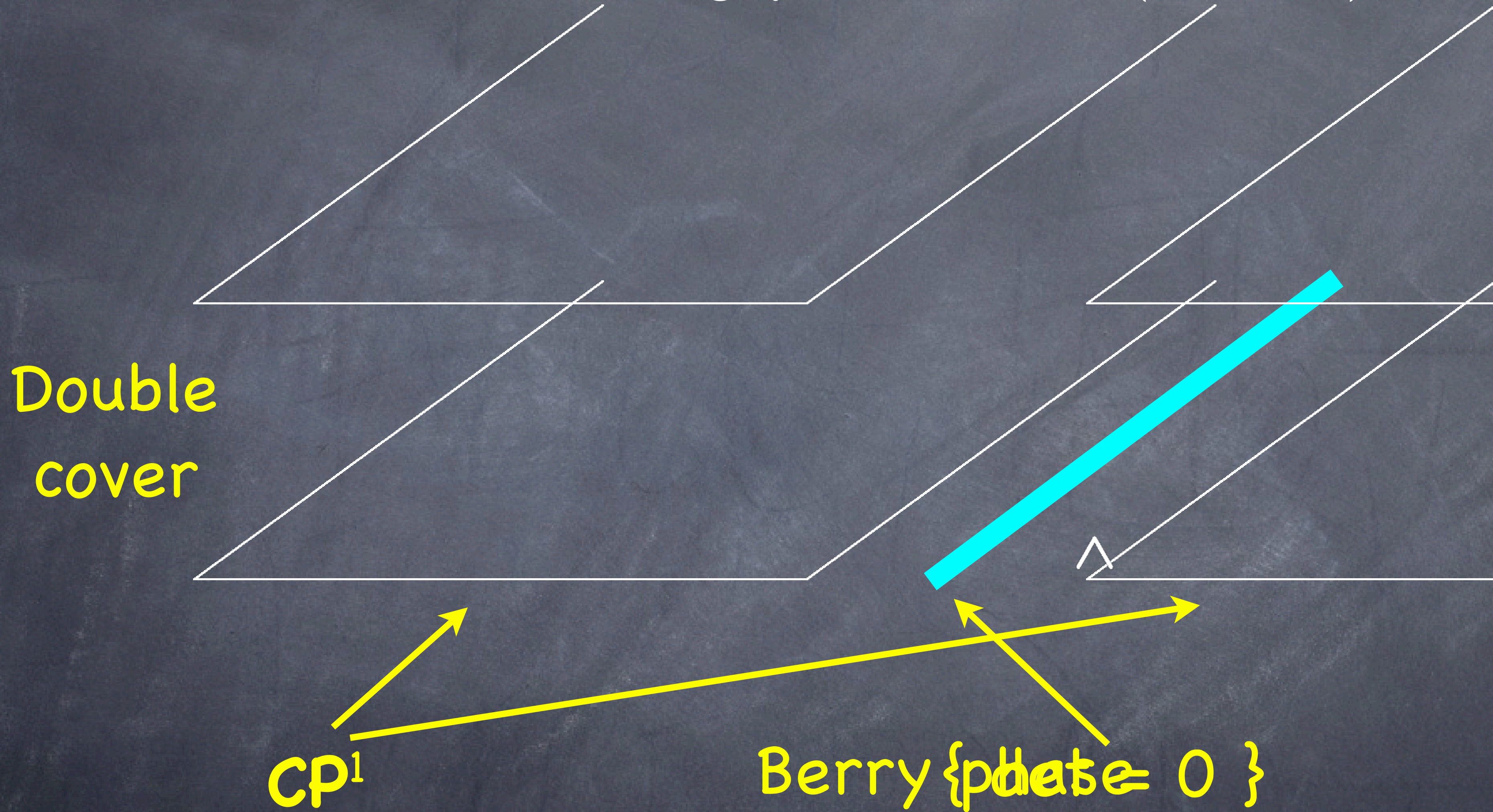
* $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg point: $(r \ll 0)$



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{CP}^1$....

The Landau-Ginzburg point: $(r \ll 0)$



Result: branched double cover of **CP**^{1}

So far:

The GLSM realizes:

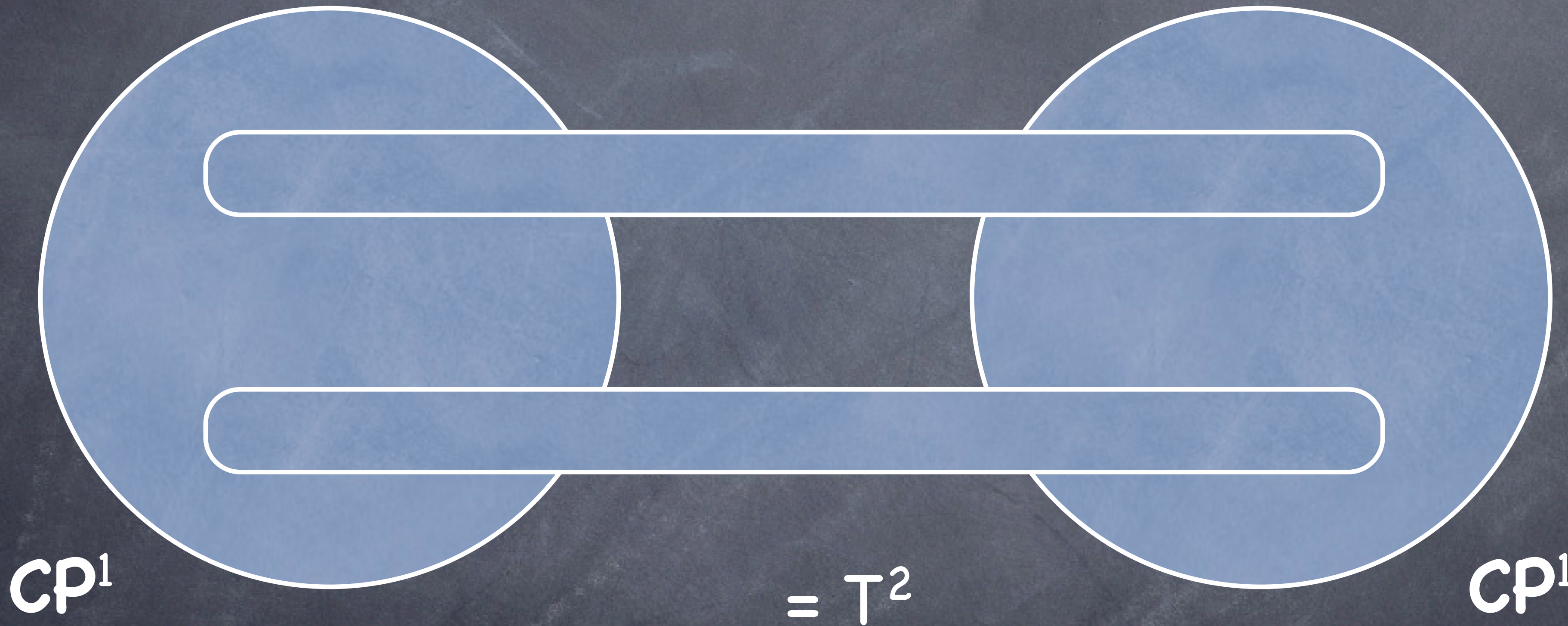
$\mathbf{CP}^3[2,2]$ $\overset{\text{Kahler}}{\longleftrightarrow}$ branched double cover of $\mathbf{CP}^1$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

* novel realization of geometry (as something other than CI)

Branched double cover of $\mathbf{CP}^1$ over deg 4 locus



So our GLSM for $\mathbf{CP}^3[2,2]$ relates

$T^2 \xleftrightarrow{\text{Kahler}} T^2$ (no surprise)

Next simplest example:

GLSM for $\mathbf{CP}^5[2,2,2]$ = K3

At LG point, have a branched double cover of $\mathbf{CP}^2$,
branched over a degree 6 locus
--- another K3

$$K3 \xleftrightarrow{\text{Kahler}} K3$$

(no surprise)

So far:

* easy low-dimensional examples of hpd

* geometry realized at LG,

but **not** as the critical locus of a superpotential.

For physics, this is already neat, but there are much more interesting examples yet....

The next example in the pattern is more interesting.

GLSM for $\mathbf{CP}^7[2,2,2,2]$ = CY 3-fold

At LG point,
naively, same analysis says
get branched double cover of $\mathbf{CP}^3$,
branched over degree 8 locus.

-- another CY
(Clemens' octic double solid)

Here, **different** CY's;
not even birational

However, the analysis that worked well in lower dimensions, hits a snag here:

The branched double cover is singular,
but the GLSM is smooth at those singularities.

Hence, we're not precisely getting a branched double cover; instead, we're getting something slightly different.

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover, as hpd implies in this case.

Check that we are seeing K's noncomm' resolution:

Here, K's noncomm' res'n is defined by ($\mathbf{P}^3$,B) where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the GLSM superpotential.

B is analogous to the structure sheaf; other sheaves are B-modules.

Physics?......

Physics picture of K's noncomm' space:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Summary so far:

This GLSM realizes:

$\mathbf{CP}^7[2,2,2,2] \xleftrightarrow{\text{Kahler}}$ nc res'n of branched double cover of $\mathbf{CP}^3$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Physical realization of a nc resolution

Geometry realized differently than critical locus

More examples:

CI of
n quadrics in $\mathbf{P}^{2n-1}$

$\overset{\text{Kahler}}{\longleftrightarrow}$

(possible nc res'n of)
branched double
cover of $\mathbf{P}^{n-1}$,
branched over deg 2n
locus

Both sides CY

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$\overset{\text{Kahler}}{\longleftrightarrow}$

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved | Negatively curved

r flows: ········→ ········→ ········→

Based on both these examples of abelian GLSM's,
realizing examples of hpd,
and also nonabelian GLSM's realizing other examples of hpd,

it's natural to conjecture that phases of GLSM's are related by hpd (replacing `birational').

This seems to be borne out by recent work, eg:
Ballard, Favero, Katzarkov, 1203.6643

D-brane probes of nc resolutions

Let's now return to the branched double covers and nc resolutions thereof.

I'll outline next some work-in-progress on D-brane probes of those nc resolutions.

(w/ N Addington, E Segal)

Idea: `D-brane probe' = roving skyscraper sheaf; by studying spaces of such, can sometimes gain insight into certain abstract CFT's.

Setup:

To study D-brane probes at the LG points, we'll RG flow the GLSM a little bit, to build an `intermediate' Landau-Ginzburg model. (D-brane probes = certain matrix fact'ns in LG)

$\mathbf{P}^n[2,2,..,2]$ (k intersections) is hpd to

LG on $\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]} \right)$

with superpotential

$$W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

Our D-brane probes of this Landau-Ginzburg theory will consist of (sheafy) matrix factorizations:

$$\mathcal{E}_0 \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{E}_1$$

where

$$P \circ Q, Q \circ P = W \, \mathrm{End}$$

up to a constant shift

(equivariant w.r.t. $\mathbf{C}^*_R$)

In a NLSM, a D-brane probe is a skyscraper sheaf. Here in LG, idea is that we want MF's that RG flow to skyscraper sheaves.

That said, we want to probe nc res'ns (abstract CFT's), for which this description is a bit too simple.

First pass at a possible D-brane probe:
(wrong, but usefully wrong)

$$\mathcal{O}_x \rightleftarrows 0$$

where x is any point.

Since $W|_x$ is constant, $0 = W|_x$ up to a const shift, hence skyscraper sheaves define MF's.

This has the right `flavor' to be pointlike, but we're going to need a more systematic def'n....

When is a matrix factorization `pointlike'?

One necessary condition:
contractible off a pointlike locus.

Example: X = **C**^{2} $\quad W = xy$

$$\mathcal{O} \overset{x}{\underset{y}{\rightleftarrows}} \mathcal{O}$$

is contractible on $\{y \neq 0\}$:

There exist maps s, t s.t. $1 = ys + tx$

namely $t = 0$, $s = y^{-1}$

Sim'ly, contractible on $\{x \neq 0\}$

hence support lies on $\{x = y = 0\}$

When is a matrix factorization `pointlike'?

Demanding contractible off a point,
gives set-theoretic pointlike support,
but to distinguish fat points, need more.

To do this, compute Ext groups.

Say a matrix factorization is `homologically pointlike'
if has same Ext groups as a skyscraper sheaf:

$$\dim \mathrm{Ext}^k_{\mathrm{MF}}(\mathcal{E}, \mathcal{E}) \; = \; \left(\begin{array}{c} n \\ k \end{array} \right)$$

We're interested in Landau-Ginzburg models on

$$\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]}\right)$$

with superpotential $W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$

For these theories, it can be shown that the `pointlike' matrix factorizations are of the form

$$\mathcal{O}_U \rightleftarrows 0$$

where U is an isotropic subspace of a single fiber.

Let's look at some examples, fiberwise, to understand what sorts of results these D-brane probes will give.

Example: Fiber $[\mathbf{C}^2/\mathbf{Z}_2]$, $W|_F = xy$

Two distinct matrix factorizations:

$$\mathcal{O}_{\{y=0\}} \;\; \sim \;\; \begin{array}{c} \mathcal{O} \\ x \downarrow \uparrow y \\ \mathcal{O}(1/2) \end{array} \qquad \text{where } \mathcal{O}_{\{y=0\}} = \begin{array}{c} \mathcal{O}_{\{y=0\}} \\ \downarrow \uparrow \\ 0 \end{array}$$

and

$$\mathcal{O}_{\{x=0\}} \;\; \sim \;\; \begin{array}{c} \mathcal{O} \\ y \downarrow \uparrow x \\ \mathcal{O}(1/2) \end{array} \qquad \text{where } \mathcal{O}_{\{x=0\}} = \begin{array}{c} \mathcal{O}_{\{x=0\}} \\ \downarrow \uparrow \\ 0 \end{array}$$

D-brane probes see 2 pts over base => double cover

Example: Family $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}_\alpha$

$$W = x^2 - \alpha^2 y^2$$

Find branch locus:

$$A = \left[\begin{array}{cc} 1 & 0 \\ 0 & -\alpha^2 \end{array} \right] \qquad \det A = -\alpha^2$$

When $\alpha \neq 0$,

there are 2 distinct matrix factorizations:

$$(\mathcal{O}_{\{x=\alpha y\}} \rightleftharpoons 0), \quad (\mathcal{O}_{\{x=-\alpha y\}} \rightleftharpoons 0)$$

Over the branch locus $\{\alpha = 0\}$, there is only one.

=> **branched** double cover

Global issues:

Over each point of the base, we've picked an isotropic subspace U of the fibers, to define our ptlike MF's.

These choices can only be glued together up to an overall $\mathbf{C}^*$ automorphism,
so globally there is a $\mathbf{C}^*$ gerbe.

Physically this ambiguity corresponds to gauge transformation of the B field;
hence, characteristic class of the B field should match that of the $\mathbf{C}^*$ gerbe.

So far:

When the LG model flows in the IR to a smooth branched double cover,
D-brane probes see that branched double cover
(and even the cohomology class of the B field).

Case of an nc resolution:

Toy model: $$[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$$

$$W \;=\; ax^2 \;+\; bxy \;+\; cy^2$$

Branch locus:

$$A = \left[\begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right] \qquad \det A \;\propto b^2 - 4ac \;\equiv\; \Delta$$

Generically on $\mathbf{C}^3$, have 2 MF's, quasi-iso to

$$\mathcal{O}_F \underset{2ax+by-\sqrt{\Delta}y}{\overset{2ax+by+\sqrt{\Delta}y}{\rightleftarrows}} \mathcal{O}_F(1/2) \quad , \quad \mathcal{O}_F \underset{2ax+by+\sqrt{\Delta}y}{\overset{2ax+by-\sqrt{\Delta}y}{\rightleftarrows}} \mathcal{O}_F(1/2)$$

Gen'ly on branch locus, become a single MF,
but something special happens at $\{a = b = c = 0\}$....

Case of an nc resolution, cont'd:

Toy model: $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$

$$W = ax^2 + bxy + cy^2$$

At the point $\{a = b = c = 0\}$

there are 2 families of ptlike MF's:

$$\mathcal{O}_F \underset{\phi}{\overset{0}{\rightleftarrows}} \mathcal{O}_F(1/2) \qquad\qquad \mathcal{O}_F \underset{0}{\overset{\phi}{\rightleftarrows}} \mathcal{O}_F(1/2)$$

where ϕ is any linear comb' of x, y (up to scale)

* 2 small resolutions (stability picks one)

I’m glossing over details,
but the take-away point is that for
nc resolutions
(naively, singular branched double covers),
D-brane probes see small resolutions.

Often these small resolutions will be non-Kahler,
and hence not Calabi-Yau.

(closed string geometry $\neq$ probe geometry;
also true in eg orbifolds)

Summary:

* physical realization of hpd

CI quadrics $\longleftrightarrow$ (nc res'n of) branched double cover

as phases of abelian GLSM

* detour through physics of gerbes

* D-brane probes