Predictions for GW inv'ts of a noncommutative resolution

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Recently, an efficient method for extracting Gromov-Witten invariants from gauged linear sigma models (GLSM's) was developed,

(Jockers, Morrison, Romo et al, 1208.6244)

utilizing exact results for partition functions of GLSM's on S^2's developed just a few months prior.

(Benini, Cremonesi, 1206.2356; Doroud et al, 1206.2606)

Briefly, in this talk I'll apply those methods to some recent GLSM's....

Apply to GLSM's....

Historically, it was thought that GLSM's could only realize geometries in one particular way: as the critical locus of a superpotential.

However, counterexamples were found about six years ago:

Hori, Tong, hep-th/0609032 Donagi, ES, arXiv: 0704.1761	nonabelian
Caldararu et al, arXiv: 0709.3855	abelian

The abelian GLSM ex's also included physical realizations of `noncommutative resolutions'.

My goal today,
is to apply the new GW computational methods of Jockers et al,
to the abelian GLSM's just described,
in which geometry is realized differently than as the critical locus of a superpotential,
and in which nc res'ns arise.

Outline:

* Describe GLSM for $\mathbf{P}^7[2,2,2,2]$ <--> nc res'n

  -- Explain GLSM analysis

  for simpler analogue $\mathbf{P}^3[2,2]$

  - Decomposition conjecture

  -- Back to $\mathbf{P}^7[2,2,2,2]$, explain nc res'n

  -- other examples; `homological projective duality'

  -- brane probes of nc res'ns

* Outline new method for computing GW inv'ts

* Results for nc res'n

I'm going to apply Jockers et al's computations to a class of GLSM's relating geometries of the following form:

r >> 0:

A complete intersection of k quadrics in **P**^{n},

$$\{Q_1 = \cdots = Q_k = 0\}$$

r << 0:

a (nc resolution of a) branched double cover of **P**$^{k-1}$, branched over the locus

$$\{\det A = 0\}$$

where $$\sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

The branched double cover structure is realized via some ~ novel physics.

Kentaro may describe other examples of GLSM's whose interpretation requires understanding some novel physics, in his talk.

I'll begin with the simplest toy example,

the GLSM for $\mathbf{P}^3[2,2]$ (=T^2):

GLSM's are families of 2d gauge theories that RG flow to families of CFT's.

In this case:

$r \gg 0$

one-parameter Kahler moduli space

$r \ll 0$

NLSM on $\mathbf{P}^3[2,2]$

r

LG point = branched double cover

GLSM for $\mathbf{P}^3[2,2]$ (=T^2):

Briefly, the GLSM consists of:

* 4 chiral superfields $\Phi_i = (\phi_i, \psi_i, F_i)$, one for each homogeneous coordinate on $\mathbf{P}^3$, each of charge 1 w.r.t. a gauged U(1)

* 2 chiral superfields $P_a = (p_a, \psi_{pa}, F_{pa})$, (one for each of the $\{Q_a = 0\}$), each of charge -2

* a superpotential

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

The GLSM describes a symplectic quotient:

Moment map (D term):

$$\sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 = r$$

$r \gg 0 :$ ϕ_i not all zero

Critical locus of superpotential $W = \sum_a p_a Q_a(\phi)$ is

$$p_a \frac{\partial Q_a}{\partial \phi_i} = Q_a = 0$$

but smooth => Q_a , $\frac{\partial Q_a}{\partial \phi_i}$ not both zero, hence

$p_a = Q_a = 0 :$ NLSM on CY CI = $\mathbf{P}^3[2,2] = T^2$

The other limit is more interesting....

Moment map (D term):

$$\sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 = r$$

$r \ll 0 :$ p_a not all zero

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

implies that ϕ_i massive (since deg 2)

NLSM on $\mathbf{P}^1$????

That can't be right, since other phase is CY, and GLSM's must relate CY's <----> CY's.

The correct analysis of the $r \ll 0$ limit is more subtle.

One subtlety is that the ϕ_i are not massive everywhere.

Write $$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

then they are only massive away from the locus

$$\{\det A = 0\} \subset \mathbf{P}^1$$

But that just makes things more confusing....

A more important subtlety is the fact that the p's have nonminimal charge,

so over most of the $\mathbf{P}^1$ of p vevs,
we have a nonminimally-charged abelian gauge theory,
meaning massless fields have charge/weight -2,
instead of 1 or -1.

-- gauging a trivially-acting $\mathbf{Z}_2$

Mathematically, this is a string on a $\mathbf{Z}_2$ gerbe,
which physics sees as a double cover.

Let's quickly review how this works....

Strings on gerbes:

Present a (smooth DM) stack as [X/H].

String on stack = H-gauged sigma model on X.

(presentation-dependence washed out w/ renormalization group)

If a subgroup G acts trivially, then this is a G-gerbe.

Physics questions:

* Does physics know about G?

(Yes, via nonperturbative effects -- Adams, Plesser, Distler.)

* The result violates cluster decomposition; why consistent?

(B/c equiv to a string on a disjoint union....)

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

gerbe

We now believe, for (2,2) CFT's,

disjoint union of spaces

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K \right] \right)$$

(together with some B field), where

$\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on\ } Y) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (Y, B) \right)$$

$$(Y \;=\; [X/K])$$

where the B field is determined by the image of

$$H^2(Y, Z(G)) \stackrel{Z(G) \rightarrow U(1)}{\longrightarrow} H^2(Y, U(1))$$

Quick consistency check:

A sheaf on a banded G-gerbe
is the same thing as
a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

This implies a decomposition of D-branes (~ sheaves), which is precisely consistent with the decomposition conjecture.

Another quick consistency check:

Prediction:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K\right]$

and this has been checked in

H-H Tseng, Y Jiang, et al,
0812.4477, 0905.2258, 0907.2087, 0912.3580, 1001.0435, 1004.1376, ...

GLSM's

Let's now return to our analysis of GLSM's.

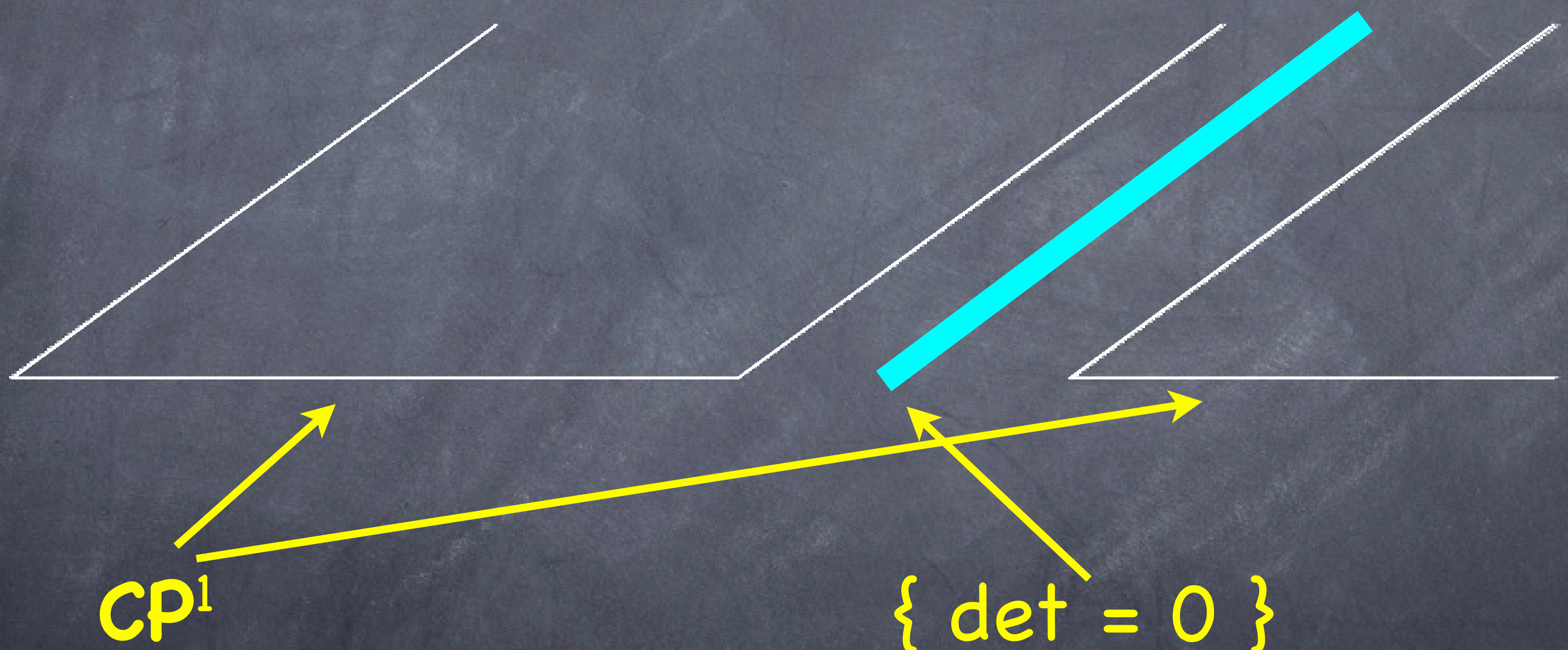
Example: **CP**^{3}[2,2]

Superpotential: $$\sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

$r \ll 0:$

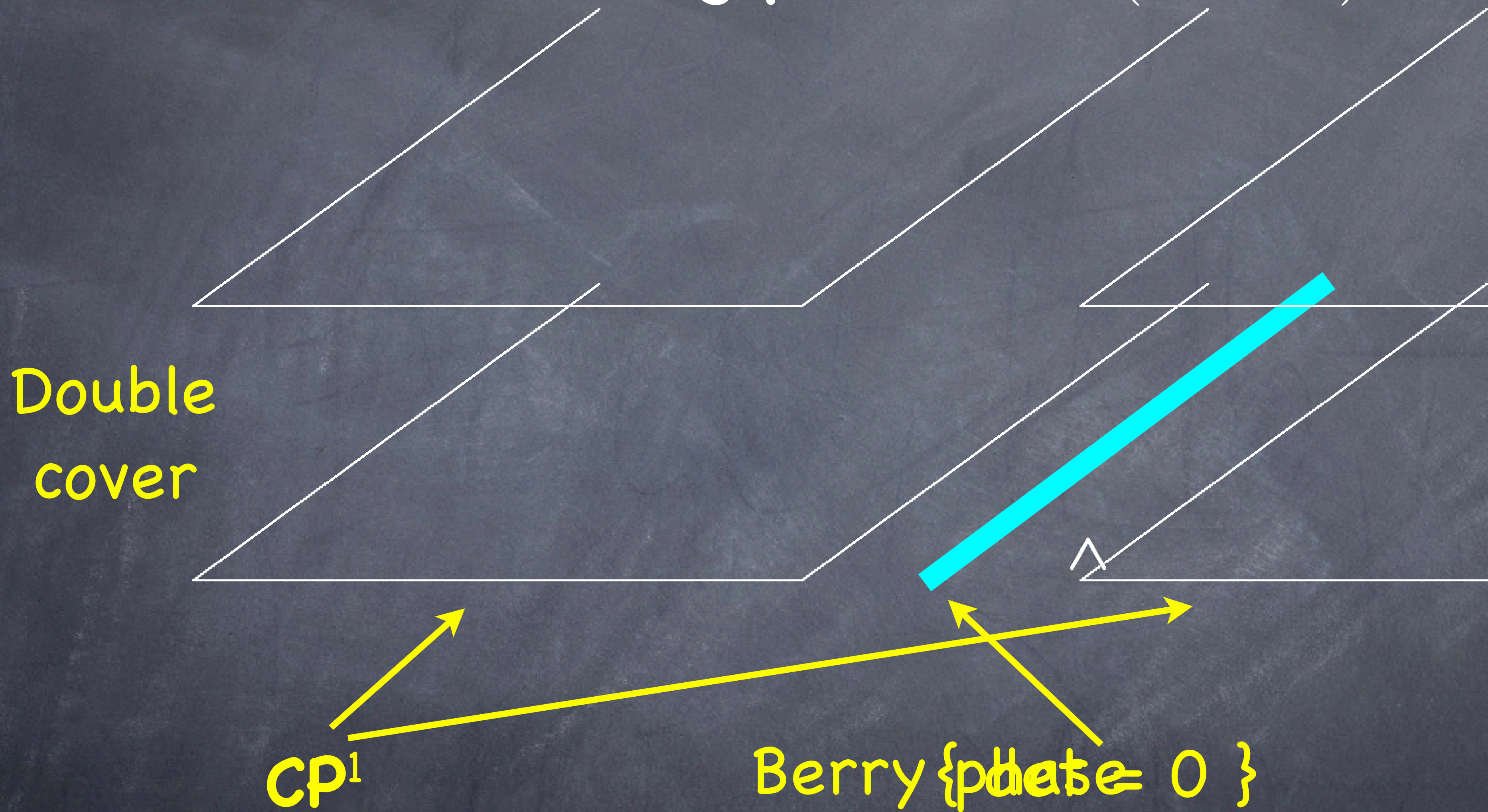
* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.
* leaves just the p fields, of charge/weight -2
* **Z**$_2$ gerbe, hence double cover

The Landau-Ginzburg point: $(r \ll 0)$



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{CP}^1$....

The Landau-Ginzburg point: $(r \ll 0)$



Result: branched double cover of $\mathbf{CP}^1$

So far:

The GLSM realizes:

$CP^3[2,2]$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{CP}^1$, deg 4 locus $= T^2$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

* novel physical realization of geometry (as something other than critical locus of W)

Next simplest example:

GLSM for $\mathbf{CP}^5[2,2,2]$ = K3

At LG point, have a branched double cover of $\mathbf{CP}^2$, branched over a degree 6 locus
--- another K3

$$K3 \overset{\text{Kahler}}{\longleftrightarrow} K3$$

(no surprise)

So far:

* easy low-dimensional examples of hpd

* geometry realized at LG,

but **not** as the critical locus of a superpotential.

For physics, this is already neat, but there are much more interesting examples yet....

The next example in the pattern is more interesting.

GLSM for $\mathbf{CP}^7[2,2,2,2]$ = CY 3-fold

At LG point,
naively, same analysis says
get branched double cover of $\mathbf{CP}^3$,
branched over degree 8 locus.

-- another CY
(Clemens' octic double solid)

Here, **different** CY's

However, the analysis that worked well in lower dimensions, hits a snag here:

The branched double cover is singular,
but the GLSM is smooth at those singularities.

Hence, we're not precisely getting a branched double cover; instead, we're getting something slightly different.

We believe the GLSM is actually describing
a `noncommutative resolution'
of the branched double cover,
one described by Kuznetsov.

Check that we are seeing K's noncomm' resolution:

Here, K's noncomm' res'n is defined by $(\mathbf{P}^3, B)$
where B is the sheaf of even parts of Clifford
algebras associated with the universal quadric over $\mathbf{P}^3$
defined by the GLSM superpotential.

B is analogous to the structure sheaf;
other sheaves are B-modules.

Physics?......

Physics picture of K's noncomm' space:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Summary so far:

This GLSM realizes:

$\mathbf{CP}^7[2,2,2,2]$ $\xleftrightarrow{\text{Kahler}}$ nc res'n of branched double cover of $\mathbf{CP}^3$

where RHS realized at LG point via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Physical realization of a nc resolution

Geometry realized differently than critical locus

More examples:

CI of n quadrics in $\mathbf{P}^{2n-1}$ $\overset{\text{Kahler}}{\longleftrightarrow}$ (possible nc res'n of) branched double cover of $\mathbf{P}^{n-1}$, branched over deg 2n locus

Both sides CY

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$\longleftrightarrow$ Kahler

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases, 4 of each of the above.

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved		Negatively curved

r flows: ⇢ ⇢ ⇢

All of these pairs of geometries are related by Kuznetsov's ``homological projective duality'' (hpd).

It's natural to conjecture that all phases of GLSM's are related by hpd.

This seems to be borne out by recent work, eg:
Ballard, Favero, Katzarkov, 1203.6643

More Kuznetsov duals:

Another class of examples, also realizing Kuznetsov's h.p.d., were realized in GLSM's by Hori-Tong.

$G(2,7)[1^7]$ $\xleftrightarrow{\text{Kahler}}$ Pfaffian CY

(Rodland, Kuznetsov, Borisov-Caldararu, Hori-Tong)

* unusual geometric realization
(via strong coupling effects in nonabelian GLSM)

* non-birational

More Kuznetsov duals:

$G(2,N)[1^m]$ (N odd) $\xleftrightarrow{\text{Kahler}}$ vanishing locus in P^{m-1} of Pfaffians

Check r flow:

$K = O(m-N)$ $\qquad\qquad$ $K = O(N-m)$

Opp sign, as desired,
so all flows in same direction.

D-brane probes of nc resolutions

Let's now return to the branched double covers and nc resolutions thereof.

I'll outline next some work on D-brane probes of those nc resolutions.

(w/ N Addington, E Segal)

Idea: `D-brane probe' = roving skyscraper sheaf; by studying spaces of such, can sometimes gain insight into certain abstract CFT's.

Setup:

To study D-brane probes at the LG points,
we'll RG flow the GLSM a little bit,
to build an `intermediate' Landau-Ginzburg model.
(D-brane probes = certain matrix fact'ns in LG)

$\mathbf{P}^n[2,2,..,2]$ (k intersections) is hpd to

LG on $\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]} \right)$

with superpotential

$$W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

Our D-brane probes of this Landau-Ginzburg theory will consist of (sheafy) matrix factorizations:

$$\mathcal{E}_0 \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{E}_1$$

where

$$P \circ Q, Q \circ P = W \, \mathrm{End}$$

up to a constant shift

(equivariant w.r.t. $\mathbf{C}^*_{R}$)

In a NLSM, a D-brane probe is a skyscraper sheaf. Here in LG, idea is that we want MF's that RG flow to skyscraper sheaves.

That said, we want to probe nc res'ns (abstract CFT's), for which this description is a bit too simple.

First pass at a possible D-brane probe:
(wrong, but usefully wrong)

$$\mathcal{O}_x \rightleftarrows 0$$

where x is any point.

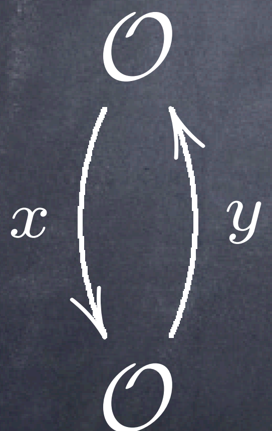
Since $W|_x$ is constant, $0 = W|_x$ up to a const shift, hence skyscraper sheaves define MF's.

This has the right `flavor' to be pointlike, but we're going to need a more systematic def'n....

When is a matrix factorization `pointlike'?

One necessary condition:
contractible off a pointlike locus.

Example: X = **C**^{2} $W = xy$



is contractible on $\{y \neq 0\}$:

There exist maps s, t s.t. $1 = ys + tx$

namely $t = 0$, $s = y^{-1}$

Sim'ly, contractible on $\{x \neq 0\}$

hence support lies on $\{x = y = 0\}$

When is a matrix factorization `pointlike'?

Demanding contractible off a point,
gives set-theoretic pointlike support,
but to distinguish fat points, need more.

To do this, compute Ext groups.

Say a matrix factorization is `homologically pointlike'
if has same Ext groups as a skyscraper sheaf:

$$\dim \mathrm{Ext}^k_{\mathrm{MF}}(\mathcal{E}, \mathcal{E}) \;=\; \left(\begin{array}{c} n \\ k \end{array} \right)$$

We're interested in Landau-Ginzburg models on

$$\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]}\right)$$

with superpotential $W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$

For these theories, it can be shown that the `pointlike' matrix factorizations are of the form

$$\begin{array}{c} \mathcal{O}_U \\ \downarrow \uparrow \\ 0 \end{array}$$

where U is an isotropic subspace of a single fiber.

Let's look at some examples, fiberwise, to understand what sorts of results these D-brane probes will give.

Example: Fiber $[\mathbf{C}^2/\mathbf{Z}_2]$, $W|_F = xy$

Two distinct matrix factorizations:

$$\begin{array}{c} \mathcal{O}_{\{y=0\}} \\ \downarrow \uparrow \\ 0 \end{array} \sim \begin{array}{c} \mathcal{O} \\ x \downarrow \uparrow y \\ \mathcal{O}(1/2) \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{O}_{\{x=0\}} \\ \downarrow \uparrow \\ 0 \end{array} \sim \begin{array}{c} \mathcal{O} \\ y \downarrow \uparrow x \\ \mathcal{O}(1/2) \end{array}$$

D-brane probes see 2 pts over base => double cover

Example: Family $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}_\alpha$

$$W = x^2 - \alpha^2 y^2$$

Find branch locus:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha^2 \end{bmatrix} \qquad \det A = -\alpha^2$$

When $\alpha \neq 0$,

there are 2 distinct matrix factorizations:

$$(\mathcal{O}_{\{x=\alpha y\}} \rightleftharpoons 0), \quad (\mathcal{O}_{\{x=-\alpha y\}} \rightleftharpoons 0)$$

Over the branch locus $\{\alpha = 0\}$, there is only one.

=> **branched** double cover

Global issues:

Over each point of the base, we've picked an isotropic subspace U of the fibers, to define our ptlike MF's.

These choices can only be glued together up to an overall **C*** automorphism,
so globally there is a **C*** gerbe.

Physically this ambiguity corresponds to gauge transformation of the B field;
hence, characteristic class of the B field should match that of the **C*** gerbe.

So far:

When the LG model flows in the IR to a smooth branched double cover,
D-brane probes see that branched double cover
(and even the cohomology class of the B field).

Case of an nc resolution:

Toy model: $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$

$$W \;=\; ax^2 \;+\; bxy \;+\; cy^2$$

Branch locus:

$$A = \left[\begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right] \qquad \det A \;\propto b^2 - 4ac \;\equiv\; \Delta$$

Generically on $\mathbf{C}^3$, have 2 MF's, quasi-iso to

$$\mathcal{O}_F \underset{2ax+by-\sqrt{\Delta}y}{\overset{2ax+by+\sqrt{\Delta}y}{\rightleftarrows}} \mathcal{O}_F(1/2) \quad , \quad \mathcal{O}_F \underset{2ax+by+\sqrt{\Delta}y}{\overset{2ax+by-\sqrt{\Delta}y}{\rightleftarrows}} \mathcal{O}_F(1/2)$$

Gen'ly on branch locus, become a single MF,

but something special happens at $\{a=b=c=0\}$....

Case of an nc resolution, cont'd:

Toy model: $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$

$$W = ax^2 + bxy + cy^2$$

At the point $\{a = b = c = 0\}$

there are 2 families of ptlike MF's:

$$\mathcal{O}_F \underset{\phi}{\overset{0}{\rightleftarrows}} \mathcal{O}_F(1/2) \qquad\qquad \mathcal{O}_F \underset{0}{\overset{\phi}{\rightleftarrows}} \mathcal{O}_F(1/2)$$

where ϕ is any linear comb' of x, y (up to scale)

* 2 small resolutions (stability picks one)

I’m glossing over details,
but the take-away point is that for
nc resolutions
(naively, singular branched double covers),
D-brane probes see small resolutions.

Often these small resolutions will be non-Kahler,
and hence not Calabi-Yau.

(closed string geometry $\neq$ probe geometry;
also true in eg orbifolds)

Now, let's finally turn to GW inv'ts.

Basic idea: (Jockers, Morrison, Romo et al, 1208.6244)

Partition function of GLSM on S^2 can be computed exactly, for example:

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \left(\frac{\Gamma(\mathsf{q} - i\sigma - m/2)}{\Gamma(1 - \mathsf{q} + i\sigma - m/2)} \right)^8 \left(\frac{\Gamma(1 - 2\mathsf{q} + 2i\sigma + 2m/2)}{\Gamma(2\mathsf{q} - 2i\sigma + 2m/2)} \right)^4$$

(Benini, Cremonesi, 1206.2356; Doroud et al, 1206.2606)

After normalization, this becomes exp(-K):

$$\begin{aligned} \frac{Z}{\text{stuff}} &= \exp(-K) \\ &= -\frac{i}{6}\kappa(t-\bar{t})^3 + \frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n \left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right) \\ &\quad - \frac{i}{(2\pi i)^2}\sum_n N_n \left(\mathrm{Li}_2(q^n) + \mathrm{Li}_2(\overline{q}^n)\right) n(t-\bar{t}) \end{aligned}$$

... and then read off the N_n's

Let's work through this in more detail.

For a U(1) gauge theory,

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \prod_i Z_{\Phi,i}$$

where

$$Z_\Phi = \frac{\Gamma(\mathrm{Q}/2 - Q(i\sigma + m/2))}{\Gamma(1 - \mathrm{Q}/2 + Q(i\sigma - m/2))}$$

Q = gauge U(1) charge

Q defines hol' Killing vector that combines with $U(1)_R$

Q vs Q :

To explain the difference, it's helpful to look at a NLSM lagrangian on S^2:

$$g_{i\bar{\jmath}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{\jmath}} - ig_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\gamma^m\mathcal{D}_m\psi^i + g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} - F^i(\frac{1}{2}g_{i\bar{\jmath},\bar{k}}\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{k}} - W_i)$$

$$- \bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\psi^j\psi^k - \bar{W}_{\bar{\imath}}) - \frac{1}{2}W_{ij}\psi^i\psi^j - \frac{1}{2}\bar{W}_{\bar{\imath}\bar{\jmath}}\bar{\psi}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}} + \frac{1}{4}g_{i\bar{\jmath},k\bar{l}}\psi^i\psi^k\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{l}}$$

$$- \frac{1}{4r^2}g_{i\bar{\jmath}}X^iX^{\bar{\jmath}} + \frac{i}{4r^2}K_iX^i - \frac{i}{4r^2}K_{\bar{\imath}}X^{\bar{\imath}} - \frac{i}{2r}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\nabla_jX^i\psi^j$$

(B. Jia, 2013, to appear)

Specific to S^2

r = radius of S^2

X = holomorphic Killing vector (defines Q of previous slide)

Constraints: $2W = -iX^i\partial_iW$

so if $W \neq 0$ then $X \neq 0$ -- important for GLSM

As a warm-up,
let's outline the GW computation at $r >> 0$,
on $\mathbf{P}^7[2,2,2,2]$,
where the answer is known,
and then afterwards we'll turn to the $r << 0$ limit.

For the GLSM for $\mathbf{P}^7[2,2,2,2]$:

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \underbrace{\left(\frac{\Gamma(\mathsf{q} - i\sigma - m/2)}{\Gamma(1 - \mathsf{q} + i\sigma - m/2)} \right)^8}_{\Phi,\, Q = 1,\ \mathsf{Q} = 2\mathsf{q}} \underbrace{\left(\frac{\Gamma(1 - 2\mathsf{q} + 2i\sigma + 2m/2)}{\Gamma(2\mathsf{q} - 2i\sigma + 2m/2)} \right)^4}_{P,\, Q = -2,\ \mathsf{Q} = 2 - 4\mathsf{q}}$$

For r >> 0, close contour on left.

Define $f(\epsilon) = \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1+2k-2\epsilon)^4}{\Gamma(1+k-\epsilon)^8} \right|^2$

then

$$Z = \oint \frac{d\epsilon}{2\pi i} (z\overline{z})^{\mathsf{q}-\epsilon} \pi^4 \frac{(\sin 2\pi\epsilon)^4}{(\sin \pi\epsilon)^8} f(\epsilon)$$

$$= \frac{8}{3}(z\overline{z})^{\mathsf{q}} \left[-\ln(z\overline{z})^3 f(0) - 8\pi^2 f'(0) + 3\ln(z\overline{z})^2 f'(0) + \ln(z\overline{z}) \left(8\pi^2 f(0) - 3f''(0) \right) + f^{(3)}(0) \right]$$

$\mathbf{P}^7[2,2,2,2]$, cont'd

In principle,

$$Z \propto \exp(-K)$$
$$= -\frac{i}{6}\kappa(t-\bar{t})^3 + \frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n \left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right)$$
$$- \frac{i}{(2\pi i)^2}\sum_n N_n \left(\mathrm{Li}_2(q^n) + \mathrm{Li}_2(\overline{q}^n)\right) n(t-\bar{t})$$

We know $\kappa = 2^4 = 16$ and

$$t = \frac{\ln z}{2\pi i} + (\text{terms invariant under } z \mapsto z e^{2\pi i})$$

so we can solve for the normalization of Z, then plug in and compute the N_n's.

P⁷[2,2,2,2], cont'd

Details:

$$Z = \frac{8}{3}(z\overline{z})^q \left[-\ln(z\overline{z})^3 f(0) \; - \; 8\pi^2 f'(0) \; + \; 3\ln(z\overline{z})^2 f'(0) \right.$$
$$\left. + \ln(z\overline{z}) \left(8\pi^2 f(0) - 3f''(0) \right) \; + \; f^{(3)}(0) \right]$$

also $\propto -\frac{i}{6}\kappa(t-\overline{t})^3 + \frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n \left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n) \right)$

$$- \frac{i}{(2\pi i)^2}\sum_n N_n \left(\mathrm{Li}_2(q^n) \; + \; \mathrm{Li}_2(\overline{q}^n) \right) n(t-\overline{t})$$

Expect $t - \overline{t} = \frac{\ln(z\overline{z})}{2\pi i} + \frac{\Delta(z) + \overline{\Delta}(\overline{z})}{2\pi i}$ for some $\Delta(z)$

so we use the ln(z z*)³ term to normalize.

$\mathbf{P}^7[2,2,2,2]$, cont'd

After normalization,

$$e^{-K} \;=\; -i\frac{16}{6}\left[\frac{\ln(z\overline{z})^3}{(2\pi i)^3} \;+\; \frac{8\pi^2}{(2\pi i)^3}\frac{f'(0)}{f(0)} \;-\; \frac{3}{2\pi i}\frac{\ln(z\overline{z})^2}{(2\pi i)^2}\frac{f'(0)}{f(0)}\right.$$

$$\left. -\frac{\ln(z\overline{z})}{2\pi i}\left(\frac{8\pi^2}{(2\pi i)^2} - \frac{3}{(2\pi i)^2}\frac{f''(0)}{f(0)}\right) - \frac{1}{(2\pi i)^3}\frac{f^{(3)}(0)}{f(0)}\right]$$

also $$= -\frac{i}{6}\kappa(t-\bar{t})^3 \;+\; \frac{\zeta(3)}{4\pi^3}\chi(X) \;+\; \frac{2i}{(2\pi i)^3}\sum_n N_n\left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right)$$

$$-\frac{i}{(2\pi i)^2}\sum_n N_n\left(\mathrm{Li}_2(q^n) \;+\; \mathrm{Li}_2(\overline{q}^n)\right) n(t-\bar{t})$$

Expect $$t-\bar{t} = \frac{\ln(z\overline{z})}{2\pi i} \;+\; \frac{\Delta(z)+\overline{\Delta}(\overline{z})}{2\pi i}$$ for some $\Delta(z)$

so from ln(z z*)² term, $$\Delta + \overline{\Delta} \;=\; -\;\frac{\partial}{\partial\epsilon}\ln f(\epsilon)\Big|_{\epsilon=0}$$

$\mathbf{P}^7[2,2,2,2]$, cont'd

So far,

$$q = \exp(2\pi i t) = z e^{2\pi i C}\left(1 + 64z + 7072z^2 + 991232z^3 + 158784976z^4 + \cdots\right)$$

Invert:

$$z \; = \; q e^{-2\pi i C} \; - \; 64 q^2 e^{-4\pi i C} \; + \; 1120 q^3 e^{-6\pi i C} \; - \; 38912 q^4 e^{-8\pi i C} \; + \; \cdots$$

Plug into remaining equations:

$$-\frac{i}{(2\pi i)^2}\sum_n n N_n \left(\mathrm{Li}_2(q^n) \; + \; \mathrm{Li}_2(\overline{q}^n)\right) = -i\frac{16}{6}\frac{1}{(2\pi i)^2}\left[3\left(\frac{\partial}{\partial\epsilon}\right)^2 \ln f(\epsilon)\Big|_{\epsilon=0} \; - \; 8\pi^2\right.$$

$$\frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n \left(\mathrm{Li}_3(q^n) \; + \; \mathrm{Li}_3(\overline{q}^n)\right) = i\frac{16}{6}\frac{1}{(2\pi i)^3}\left(\frac{\partial}{\partial\epsilon}\right)^3 \ln f(\epsilon)\Big|_{\epsilon=0}$$

-- 2 equ'ns for the genus 0 GW inv'ts

Result for $\mathbf{P}^7[2,2,2,2]$:

n	N
1	512
2	9728
3	416256
4	25703936
5	1957983744
6	170535923200

matches Hosono et al, hep-th/9406055

Now, let's consider the opposite limit, r << 0.

In order for the previous analysis to work, we needed

$$t \;=\; \frac{\ln z}{2\pi i} \;+\; (\text{terms invariant under } z \mapsto z e^{2\pi i})$$

-- characteristic of large-radius

-- don't typically expect to be true of LG models (so, computing Fan-Jarvis-Ruan using these methods will be more obscure),

but the present case is close enough to geometry that this should work, and indeed, one can extract integers.

Applying the same method, one finds

Compare GW inv'ts of smooth br' double cover

n	N	N
1	64	29504
2	1216	128834192
3	52032	1423720545880
4	3212992	23193056024793312

(Morrison, in "Mirror Symmetry I")

Interpretation?

We've found a set of integers, that play the same role as GW inv'ts, but for a nc res'n.

I don't know of a notion of GW theory for nc res'ns, but there's work on DT invt's

(see e.g. Szendroi, Nagao, Nakajima, Toda)

Perhaps some version of GW/DT can be used to define a set of integers that ought to be GW inv'ts?

Summary:

* Reviewed GLSM's for complete intersections of quadrics.

At LG, get (pseudo-)geometries:
(nc res'ns of) branched double covers.

* Applied recent methods of Jockers et al to compute GW inv'ts for $\mathbf{P}^7[2,2,2,2]$, and also to compute corresponding integers at LG.

Result is prediction for GW of nc res'n.

Thank you for your time!