An Introduction to Quantum Sheaf Cohomology

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w/ J Guffin, S Katz, R Donagi

Also: A Adams, A Basu, J Distler, M Ernebjerg, I Melnikov, J McOrist, S Sethi, ...

arXiv: 1110.3751, 1110.3752,
also hep-th/0605005, 0502064, 0406226

Today I'm going to talk about nonperturbative corrections to chiral rings
in 2d theories with (0,2) susy.
These are described by `quantum sheaf cohomology,' an analogue of quantum cohomology that arises in (0,2) mirror symmetry.

As background, what's (0,2) mirror symmetry?

Quantum cohomology?

Ordinary mirror symmetry?

Background: ordinary mirror symmetry

This is a symmetry in which 2d NLSM's on two (usually topologically-distinct) Calabi-Yau's (Ricci-flat spaces with cov const spinors) are described by the **same** 2d CFT.

* analogue of T-duality

* exchanges perturbative info in one NLSM, with nonperturbative info in the other NLSM

Another property of ordinary mirror symmetry is that it exchanges cohom' of (p,q) differential forms

$$\omega_{i_1\cdots i_p\bar{\imath}_1\cdots\bar{\imath}_q}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge d\bar{z}^{\bar{\jmath}_q}$$

with that of (n-p,q) differential forms, where n = cpx dim of CY.

We organize the dimensions of the spaces of (p,q) forms, denoted $h^{p,q}$, into diamond-shaped arrays.

Ex: space of cpx dim 2:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

Mirror symmetry acts as a rotation about diagonal

Example: T^2

T^2 is self-mirror topologically;
cpx, Kahler structures interchanged

$h^{0,1}$ ↗ (cpx) $h^{1,1}$ ↖ (Kahler)

$h^{p,q}$'s:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

Note this symmetry is
specific to genus 1;
for genus g:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Example: quartics in P^3

(known as K3 mflds)

K3 is self-mirror topologically;

cpx, Kahler structures interchanged

$h^{1,1}$ ↗ ↖ $h^{1,1}$

$h^{p,q}$'s:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Kummer surface

$$\left(x^2 + y^2 + z^2 - aw^2\right)^2 - \left(\frac{3a-1}{3-a}\right) pqts = 0$$
$$p = w - z - \sqrt{2}x$$
$$q = w - z + \sqrt{2}x$$
$$t = w + z + \sqrt{2}y$$
$$s = w + z - \sqrt{2}y$$
$$a = 1.5$$

Example: the quintic

The quintic (deg 5) hypersurface in $\mathbf{P}^4$
is mirror to
(res'n of) a deg 5 hypersurface in $\mathbf{P}^4/(\mathbf{Z}_5)^3$

Quintic

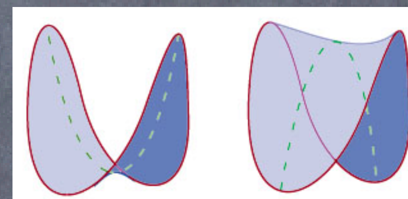
			1			
		0		0		
	0		1		0	
1		101		101		1
	0		1		0	
		0		0		
			1			

Mirror

			1			
		0		0		
	0		101		0	
1		1		1		1
	0		101		0	
		0		0		
			1			

The most important aspect of mirror symmetry is the fact that it exchanges perturbative & nonperturbative contributions.

Nonperturbative effects: ``worldsheet instantons" which are minimal-area (=holomorphic) curves.



A curve and two possible saddle-shaped film surfaces

Physically, these generate corrections to 2d OPE's, and also spacetime superpotential charged-matter couplings.

The impact on mathematics was impressive....

Deg k	n_k
1	2875
2	609250
3	317206375

Shown: numbers of minimal S^2's in one particular Calabi-Yau (the quintic in $\mathbf{P}^4$), of fixed degree.

These three degrees were the state-of-the-art before mirror symmetry (deg 2 in '86, deg 3 in '91)

Then, after mirror symmetry ~ '92, the list expanded...

Deg k	n_k
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
...	...

These nonperturbative effects generate e.g. corrections to OPE rings of chiral operators.

Resulting OPE rings called **quantum cohomology**.

Ex: NLSM on $\mathbf{CP}^{N-1}$

Chiral ring generated by x;
b/c of nonperturbative effects, OPE: $x^N \sim q$
(We'll see this explained in more detail later.)

Compare classical cohomology ring of $\mathbf{CP}^{N-1}$, which says $x^N = 0$ for x a 2-form
(since x^N a 2N-form, but $\mathbf{CP}^{N-1}$ only 2N-2-dim'l)

Thus, ``quantum" cohomology.

There are close analogues in 4d:

4d N=1 pure SU(N) SYM	**2d susy CP^{N-1} model**
$S = \lambda\lambda + \cdots$	$x = \psi_+^i \psi_-^{\bar{\jmath}} + \cdots$
$S^N = \Lambda^{3N}$	$x^N = q$
$W = S(1 + \log(\Lambda^{3N}/S^N))$ (VY)	$W = \Sigma(1 + \log(\Lambda^N/\Sigma^N))$
Konishi	Konishi

At this point in time,
mirror symmetry itself & ordinary quantum cohomology are considered to be fairly old and well-developed ideas.

Modern interest revolves around generalizations.
I'll talk about such next.

(0,2) mirror symmetry

So far, I've discussed symmetry properties of 2d (2,2) susy CFT's -- specified by a (Calabi-Yau) space.

``(0,2) mirror symmetry'' is a symmetry property of 2d (0,2) susy CFT's.

To specify one of these, need space plus also bundle/gauge field over that space.

Not any space/bundle pair will do; there are constraints: $[\mathrm{Tr}\, F \wedge F] = [\mathrm{Tr}\, R \wedge R]$ $\left(\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)\right)$

(0,2) mirror symmetry

is a conjectured generalization that exchanges pairs

$$(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds
and the $\mathcal{E}_i \to X_i$ are holomorphic vector bundles

Same (0,2) SCFT

Reduces to ordinary mirror symmetry when
$\mathcal{E}_i \cong TX_i$

(0,2) mirror symmetry

Instead of exchanging (p,q) forms,
(0,2) mirror symmetry exchanges bundle-valued differential forms (=``sheaf cohomology"):

$$H^j(X_1, \Lambda^i \mathcal{E}_1) \leftrightarrow H^j(X_2, (\Lambda^i \mathcal{E}_2)^*)$$

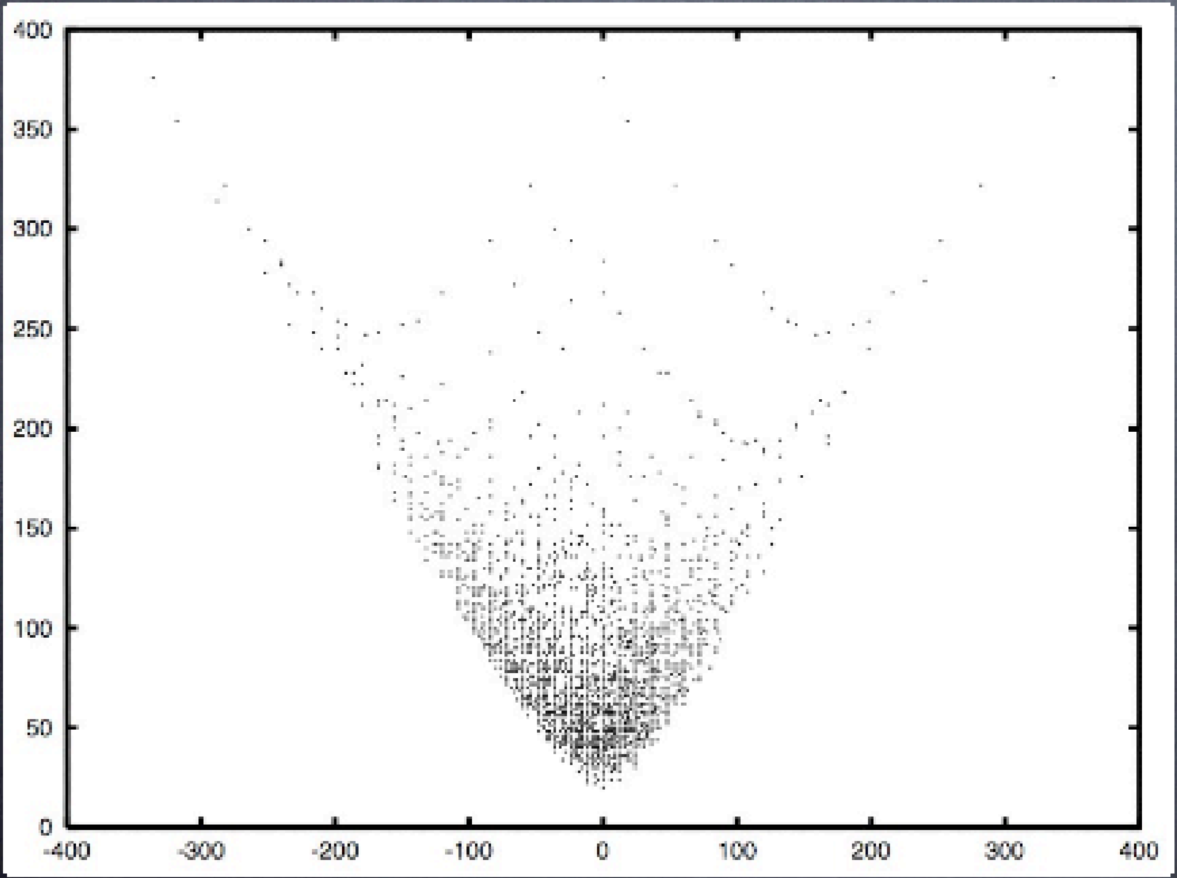
Note when $\mathcal{E}_i \cong TX_i$ this reduces to

$$H^{n-1,1}(X_1) \leftrightarrow H^{1,1}(X_2)$$

(for X_i Calabi-Yau)

(0,2) mirror symmetry

Not as much is known about (0,2) mirror symmetry, though basics are known, and more quickly developing.



Ex: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

(0,2) mirror symmetry

A few highlights:

* an analogue of the Greene-Plesser construction (quotients by finite groups) is known

(Blumenhagen, Sethi, NPB 491 ('97) 263-278)

* an analogue of Hori-Vafa (Adams, Basu, Sethi, hepth/0309226)

* analogue of quantum cohomology known since '04

(ES, Katz, Adams, Distler, Ernebjerg, Guffin, Melnikov, McOrist, ...)

* for def's of the tangent bundle, there now exists a (0,2) monomial-divisor mirror map

(Melnikov, Plesser, 1003.1303 & Strings 2010)

(0,2) mirrors are starting to heat up!

The rest of my talk today will focus on
the (0,2) mirrors analogue of quantum cohomology,
known as quantum sheaf cohomology,
which computes nonpert' corrections in (0,2) theories,
generalizing quantum cohomology.

(hep-th 0406226)

[Initially developed in '04 by S Katz, ES,
and later work by A Adams, J Distler, R Donagi,
M Ernebjerg, J Guffin, J McOrist, I Melnikov,
S Sethi, ...]

Aside on lingo:

The worldsheet theory for a heterotic string with the
``standard embedding''
(gauge bundle $\mathcal{E}$ = tangent bundle TX)
has (2,2) susy in 2d,
hence ``(2,2) model''

The worldsheet theory for a heterotic string with a more general gauge connection has (0,2) susy,
hence ``(0,2) model''

In a heterotic compactification on a (2,2) theory, the worldsheet instanton corrections responsible for quantum cohomology, generate corrections to charged-matter couplings.

Ex: If we compactify on a Calabi-Yau 3-fold, then, have 4d E_6 gauge symmetry, and these are corrections to $(27^*)^3$ couplings appearing in the spacetime superpotential.

For (2,2) compactification, computed by A model TFT, which we shall review next.

For non-standard embedding, (0,2) theory, need (0,2) version of the A model (= `A/2'), which we shall describe later.

A model:

This is a 2d TFT. 2d TFT's are generated by changing worldsheet fermions: worldsheet spinors become worldsheet scalars & (1-component chiral) vectors.

Concretely, if start with the NLSM

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ig_{i\bar{\jmath}}\psi_-^{\bar{\jmath}}D_z\psi_-^i + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\overline{z}}\psi_+^i + R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}$$

then deform the $D\psi$'s by changing the spin connection term. Since $J \sim \overline{\psi}\psi$, this is same as

$$L \mapsto L \pm \frac{1}{2}\omega J \;\;\Leftrightarrow\;\; T \mapsto T \pm \frac{1}{2}\partial J$$

A model:

More formally:

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ig_{i\bar{\jmath}}\psi_-^{\bar{\jmath}}D_z\psi_-^i + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}$$

$$\psi_-^i(\equiv \chi^i) \in \overline{\Gamma}((\phi^* T^{0,1}X)^*) \qquad \psi_+^i(\equiv \psi_z^i) \in \Gamma(K \otimes \phi^* T^{1,0}X)$$

$$\psi_-^{\bar{\imath}}(\equiv \psi_{\bar{z}}^{\bar{\imath}}) \in \overline{\Gamma}(\overline{K} \otimes \phi^* T^{0,1}X) \qquad \psi_+^{\bar{\imath}}(\equiv \chi^{\bar{\imath}}) \in \Gamma((\phi^* T^{1,0}X)^*)$$

Action has the same form,
but worldsheet spinors now scalars, vectors.

Ordinary A model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ig_{i\bar{\jmath}}\psi_-^{\bar{\jmath}}D_z\psi_-^i + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}$$

Fermions:

$$\psi_-^i(\equiv \chi^i) \in \overline{\Gamma}((\phi^* T^{0,1}X)^*) \qquad \psi_+^i(\equiv \psi_z^i) \in \Gamma(K \otimes \phi^* T^{1,0}X)$$

$$\psi_-^{\bar{\imath}}(\equiv \psi_{\bar{z}}^{\bar{\imath}}) \in \overline{\Gamma}(\overline{K} \otimes \phi^* T^{0,1}X) \qquad \psi_+^{\bar{\imath}}(\equiv \chi^{\bar{\imath}}) \in \Gamma((\phi^* T^{1,0}X)^*)$$

Under the scalar supercharge,

$$\delta\phi^i \propto \chi^i, \quad \delta\phi^{\bar{\imath}} \propto \chi^{\bar{\imath}}$$
$$\delta\chi^i = 0, \quad \delta\chi^{\bar{\imath}} = 0$$
$$\delta\psi_z^i \neq 0, \quad \delta\psi_{\bar{z}}^{\bar{\imath}} \neq 0$$

so the states are

$$\mathcal{O} \sim b_{i_1\cdots i_p\bar{\imath}_1\cdots\bar{\imath}_q}\chi^{\bar{\imath}_1}\cdots\chi^{\bar{\imath}_q}\chi^{i_1}\cdots\chi^{i_p} \quad \leftrightarrow \quad H^{p,q}(X)$$
$$Q \quad \leftrightarrow \quad d$$

Ordinary A model

The A model is, first and foremost, still a QFT.

But, if only consider correlation functions of Q-invariant states, then the corr' f'ns reduce to purely zero-mode computations -- (usually) no meaningful contribution from Feynman propagators or loops, and the correlators are independent of insertion positions.

4d analogue: corr' f'ns involving products of chiral operators are independent of insertion position (Cachazo-Douglas-Seiberg-Witten).
Idea: spacetime deriv's propto Q commutators, which vanish.

Ordinary A model

More gen'ly, TFT's are special QFT's that contain a `topological subsector' of correlators whose corr' f'ns reduce to zero mode computations.

As a result, can get exact answers, instead of asymptotic series expansions.

The A/2 model:

* (0,2) analogues of ((2,2)) A, B models

* computes `quantum sheaf cohomology'

* No longer strictly TFT, though becomes TFT on the (2,2) locus

* Nevertheless, some correlation functions still have a mathematical understanding

In more detail...

A/2 model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} + ih_{a\bar{b}}\lambda_-^{\bar{b}}D_z\lambda_-^a + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$$

Fermions:

$$\lambda_-^a \in \overline{\Gamma}((\phi^*\overline{\mathcal{E}})^*) \qquad \psi_+^i \in \Gamma(K\otimes\phi^*T^{1,0}X)$$

$$\lambda_-^{\bar{b}} \in \overline{\Gamma}(\overline{K}\otimes\phi^*\overline{\mathcal{E}}) \qquad \psi_+^{\bar{\imath}} \in \Gamma((\phi^*T^{1,0}X)^*)$$

Constraints: $\Lambda^{top}\mathcal{E}^* \cong K_X, \quad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

States:

$$\mathcal{O} \sim b_{\bar{\imath}_1\cdots\bar{\imath}_n a_1\cdots a_p}\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\lambda_-^{a_1}\cdots\lambda_-^{a_p} \leftrightarrow H^n(X,\Lambda^p\mathcal{E}^*)$$

When $\mathcal{E} = TX$, reduces to the A model,
since $H^q(X,\Lambda^p(TX)^*) = H^{p,q}(X)$

A model classical correlation functions

For X compact, have n $\chi^i, \chi^{\bar{\imath}}$ zero modes, plus bosonic zero modes $\sim X$, so

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int_X H^{p_1,q_1}(X) \wedge \cdots \wedge H^{p_m,q_m}(X)$$

Selection rule from left, right $U(1)_R$'s:

$$\sum_i p_i = \sum_i q_i = n$$

Thus:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X (\text{top-form})$$

A/2 model classical correlation functions

For X compact, we have n $\psi_+^{\bar{\imath}}$ zero modes and r λ^a zero modes:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int_X H^{q_1}(X, \Lambda^{p_1} \mathcal{E}^*) \wedge \cdots \wedge H^{q_m}(X, \Lambda^{p_m} \mathcal{E}^*)$$

Selection rule: $\sum_i q_i = n, \ \ \sum_i p_i = r$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X H^{top}(X, \Lambda^{top} \mathcal{E}^*)$$

The constraint $\Lambda^{top} \mathcal{E}^* \cong K_X$ makes the integrand a top-form.

A model -- worldsheet instantons

Moduli space of bosonic zero modes
= moduli space of worldsheet instantons, $\mathcal{M}$

If no $\psi^i_z, \psi^{\bar{\imath}}_{\bar{z}}$ zero modes, then

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{p_1,q_1}(\mathcal{M}) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M})$$

More generally,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{p_1,q_1}(\mathcal{M}) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M}) \wedge c_{top}(\mathrm{Obs})$$

In all cases: $\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} (\text{top form})$

A/2 model -- worldsheet instantons

The bundle $\mathcal{E}$ on X induces
a bundle $\mathcal{F}$ (of λ zero modes) on $\mathcal{M}$: $\quad \mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$

where $\pi : \Sigma \times \mathcal{M} \to \mathcal{M}, \quad \alpha : \Sigma \times \mathcal{M} \to X$

On the (2,2) locus, where $\mathcal{E} = TX$, have $\mathcal{F} = T\mathcal{M}$

When no `excess' zero modes,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{F}^*)$$

Apply anomaly constraints:

$$\left. \begin{array}{c} \Lambda^{top} \mathcal{E}^* \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{GRR}{\Longrightarrow} \Lambda^{top} \mathcal{F}^* \cong K_{\mathcal{M}}$$

so again integrand is a top-form.
(general case similar)

To do any computations, we need explicit expressions for the space $\mathcal{M}$ and bundle $\mathcal{F}$.

So, review linear sigma model (LSM) moduli spaces....

Gauged linear sigma models are 2d gauge theories, generalizations of the $\mathbf{CP}^N$ model, that RG flow in IR to NLSM's.

`Linear sigma model moduli spaces' are therefore moduli spaces of 2d gauge instantons.

The 2d gauge instantons of the UV gauge theory = worldsheet instantons in IR NLSM

In general, build $\mathcal{M}$ by expanding homogeneous coord's in a basis of zero modes on $\mathbf{P}^1$

Example: $\mathbf{CP}^{N-1}$

Have N chiral superfields $x_1, \cdots, x_N$, each charge 1

For degree d maps, expand:

$$x_i = x_{i0} u^d + x_{i1} u^{d-1} v + \cdots + x_{id} v^d$$

where u, v are homog' coord's on worldsheet = $\mathbf{P}^1$

Take (x_{ij}) to be homogeneous coord's on $\mathcal{M}$, then

$$\mathcal{M}_{\mathrm{LSM}} = \mathbf{P}^{N(d+1)-1}$$

What about induced bundles $\mathcal{F} \to \mathcal{M}$?

All bundles in GLSM are built from short exact sequences of bosons, fermions, corresponding to line bundles.

Physics:
Expand worldsheet fermions in a basis of zero modes, and identify each basis element with a line bundle of same U(1) weights as the original line bundle.

Math:
Idea: lift each such line bundle to a natural line bundle on $\mathbf{P}^1 \times \mathcal{M}$,
then pushforward to $\mathcal{M}$.

Induced bundles $\mathcal{F}$ for projective spaces:

Example: completely reducible bundle

$$\mathcal{E} = \oplus_a \mathcal{O}(n_a)$$

We expand worldsheet fermions in a basis of zero modes, and identify each basis element with a line bundle of same U(1) weights as the original line bundle.

Result:

$$\mathcal{F} = \oplus_a H^0\left(\mathbf{P}^1, \mathcal{O}(n_a d)\right) \otimes_{\mathbf{C}} \mathcal{O}(n_a)$$

Because of the construction, this works for short exact sequences in the way you'd expect....

From

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{E} \longrightarrow 0$$

we get

$$0 \longrightarrow \oplus_1^k H^0(\mathcal{O}) \otimes \mathcal{O} \longrightarrow \oplus_i H^0(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F}$$
$$\longrightarrow \oplus_1^k H^1(\mathcal{O}) \otimes \mathcal{O} \longrightarrow \oplus_i H^1(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

which simplifies:

$$0 \longrightarrow \oplus_1^k \mathcal{O} \longrightarrow \oplus_i H^0(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_i H^1(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i)$$

Check of (2,2) locus

The tangent bundle of a (cpt, smooth) toric variety can be expressed as

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0$$

Applying previous ansatz,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_i H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i)$$

This $\mathcal{F}$ is precisely $T\mathcal{M}_{\mathrm{LSM}}$, and $\mathcal{F}_1$ is the obs' sheaf. ✓

Quantum cohomology

... is an OPE ring. For $\mathbf{CP}^{N-1}$, correl'n f'ns:

$$\langle x^k \rangle = \begin{cases} q^m & \text{if } k = mN + N - 1 \\ 0 & \text{else} \end{cases}$$

Ordinarily need (2,2) susy, but:

* Adams-Basu-Sethi ('03') conjectured (0,2) exs
* Katz-E.S. ('04) computed matching corr'n f'ns
* Adams-Distler-Ernebjerg ('05): gen'l argument
* Guffin, Melnikov, McOrist, Sethi, etc

(cont'd)

Quantum sheaf cohomology

= (0,2) quantum cohomology

Example:

Consider a (0,2) theory describing $\mathbf{P}^1 \times \mathbf{P}^1$
with gauge bundle $\mathcal{E}$ = def' of tangent bundle,
expressible as a cokernel:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$$A, B, C, D \quad 2 \times 2 \text{ matrices}, \; x = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right], \; \tilde{x} = \left[\begin{array}{c} \tilde{x}_1 \\ \tilde{x}_2 \end{array} \right]$$

Example, cont'd

For $\mathbf{P}^1 \times \mathbf{P}^1$ with bundle

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

one finds (& we will show, later) that OPE ring is

$$\det\left(A\psi + B\tilde{\psi} \right) = q, \; \det\left(C\psi + D\tilde{\psi} \right) = \tilde{q}$$

where $\psi, \tilde{\psi}$ are operators generating chiral ring.

Consistency check:

$$\det\left(A\psi + B\tilde{\psi}\right) = q_1$$

$$\det\left(C\psi + D\tilde{\psi}\right) = q_2$$

In the special case $\mathcal{E} = T\mathbf{P}^1 \times \mathbf{P}^1$, one should recover the standard quantum cohomology ring.

That case corresponds to

$$A = D = I_{2\times 2},\ B = C = 0$$

and the above becomes $\psi^2 = q_1, \quad \tilde{\psi}^2 = q_2$

Matches ✓

Quantum sheaf cohomology

More results known:

For any toric variety, and any def′ of tangent bundle,

$$0 \longrightarrow \mathcal{O}^{\oplus r} \stackrel{E}{\longrightarrow} \oplus \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{E} \longrightarrow 0$$

the chiral ring is

$$\prod_{\alpha} \left(\det M_\alpha \right)^{Q^a_\alpha} = q_a$$

where M′s are matrices of chiral operators built from E′s.

(McOrist-Melnikov 0712.3272;
R Donagi, J Guffin, S Katz, ES, 1110.3751, 1110.3752)

Quantum sheaf cohomology

Next, I'll outline some of the mathematical details of the computations that go into these rings.

The rest of the talk will, unavoidably, be somewhat technical, but in principle, I'm just describing a computation of nonperturbative corrections to some correlation functions in 2d QFT's.

Quantum sheaf cohomology

Set up notation:

1st, write tangent bundle of toric variety X as

$$0 \longrightarrow W^* \otimes \mathcal{O} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0$$

where W is a vector space.

Write a deformation $\mathcal{E}$ of TX as

$$0 \longrightarrow W^* \otimes \mathcal{O} \stackrel{*}{\longrightarrow} Z^* \longrightarrow \mathcal{E} \longrightarrow 0$$

where $Z^* \equiv \oplus_i \mathcal{O}(\vec{q}_i)$

Quantum sheaf cohomology

Handy to dualize:

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z \longrightarrow W \otimes \mathcal{O} \longrightarrow 0$$

Correlators are elements of $H^1(\mathcal{E}^*)$

Compute:

$$H^0(Z) \longrightarrow H^0(W \otimes \mathcal{O}) \longrightarrow H^1(\mathcal{E}^*) \longrightarrow H^1(Z)$$

Can show $H^1(Z) = H^0(Z) = 0$

thus,

Correlators are elements of $H^1(\mathcal{E}^*) = H^0(W \otimes \mathcal{O})$

Quantum sheaf cohomology

On an n-dim'l toric variety X,
correlation functions $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ are maps

$$\mathrm{Sym}^n H^1(\mathcal{E}^*) \longrightarrow H^n(\Lambda^n \mathcal{E}^*) \cong \mathbf{C}$$

but because $H^1(\mathcal{E}^*) = H^0(W \otimes \mathcal{O}) = W$

we can think of correlation functions as maps

$$H^0\left(\mathrm{Sym}^n W \otimes \mathcal{O}\right) \left(= \mathrm{Sym}^n H^0(W \otimes \mathcal{O}), \mathrm{Sym}^n W\right)$$
$$\longrightarrow H^n(\Lambda^n \mathcal{E}^*) \cong \mathbf{C}$$

and it's this latter form that will be useful.

Quantum sheaf cohomology

So far:

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z \longrightarrow W \otimes \mathcal{O} \longrightarrow 0$$

and correlation functions are maps

$$H^0\left(\mathrm{Sym}^n W \otimes \mathcal{O}\right) \longrightarrow H^n(\Lambda^n \mathcal{E}^*) \cong \mathbf{C}$$

How to compute? Use the `Koszul resolution'

$$0 \longrightarrow \Lambda^n \mathcal{E}^* \longrightarrow \Lambda^n Z \longrightarrow \Lambda^{n-1} Z \otimes W$$
$$\longrightarrow \cdots \longrightarrow \mathrm{Sym}^n W \otimes \mathcal{O} \longrightarrow 0$$

which relates $\Lambda^n \mathcal{E}^*$ and $\mathrm{Sym}^n W \otimes \mathcal{O}$.

Quantum sheaf cohomology

So far:

Plan to compute correlation functions

$$H^0\left(\mathrm{Sym}^n W \otimes \mathcal{O}\right) \longrightarrow H^n(\Lambda^n \mathcal{E}^*) \cong \mathbf{C}$$

using the Koszul resolution of $\Lambda^n \mathcal{E}^*$.

In fact, instead of computing the entire map,
it suffices to compute just the kernel of that map,
which is what we do.

Here's a sample of how that works....

Quantum sheaf cohomology

Example:

Consider a (0,2) theory describing **P**[1]x**P**[1] with gauge bundle $\mathcal{E}$ = def' of tangent bundle, expressible as a cokernel:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$$A, B, C, D \quad 2 \times 2 \text{ matrices}, \ x = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right], \ \tilde{x} = \left[\begin{array}{c} \tilde{x}_1 \\ \tilde{x}_2 \end{array} \right]$$

Dualize:

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \underbrace{V \otimes \mathcal{O}(-1,0) \oplus \tilde{V} \otimes \mathcal{O}(0,-1)}_{Z} \stackrel{*}{\longrightarrow} W \otimes \mathcal{O} \longrightarrow 0$$

Classical correlation functions are a map

$$\mathrm{Sym}^2 W = H^0(\mathrm{Sym}^2 W \otimes \mathcal{O}) \longrightarrow H^2(\Lambda^2 \mathcal{E}^*) = \mathbf{C}$$

To build this map, we begin with

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z \xrightarrow{\ *\ } W \otimes \mathcal{O} \longrightarrow 0$$

and take the Koszul resolution

$$0 \longrightarrow \Lambda^2 \mathcal{E}^* \longrightarrow \Lambda^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

which will determine a map between cohomology groups above.

Let's build the map between cohomology groups.

Take the long exact sequence

$$0 \longrightarrow \Lambda^2\mathcal{E}^* \longrightarrow \Lambda^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

and break it up into short exacts:

$$0 \longrightarrow \Lambda^2\mathcal{E}^* \longrightarrow \Lambda^2 Z \longrightarrow Q \longrightarrow 0$$

$$0 \longrightarrow Q \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

Second gives a map $H^0\left(\mathrm{Sym}^2 W \otimes \mathcal{O}\right) \longrightarrow H^1(Q)$

First gives a map $H^1(Q) \longrightarrow H^2(\Lambda^2\mathcal{E}^*)$

& the composition computes corr' functions.

Let's work out those maps.

Take

$$0 \longrightarrow Q \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

The associated long exact sequence gives

$$H^0(Z \otimes W) \longrightarrow H^0(\mathrm{Sym}^2 W \otimes \mathcal{O}) \longrightarrow H^1(Q) \longrightarrow H^1(Z \otimes W)$$

but since Z is a sum of $\mathcal{O}(-1,0), \mathcal{O}(0,-1)$'s,

$$H^0(Z \otimes W) = 0 = H^1(Z \otimes W)$$

so we see that

$$H^0(\mathrm{Sym}^2 W \otimes \mathcal{O}) \xrightarrow{\sim} H^1(Q)$$

Next, take

$$0 \longrightarrow \Lambda^2\mathcal{E}^* \longrightarrow \Lambda^2 Z \longrightarrow Q \longrightarrow 0$$

The associated long exact sequence gives

$$H^1(\Lambda^2 Z) \longrightarrow H^1(Q) \longrightarrow H^2(\Lambda^2\mathcal{E}^*) \longrightarrow H^2(\Lambda^2 Z)$$

Here, $H^2(\Lambda^2 Z) = 0$

but $H^1(\Lambda^2 Z) = H^1\left(\mathbf{P}^1, \Lambda^2 V \otimes \mathcal{O}(-2,0) \oplus \Lambda^2\tilde{V} \otimes \mathcal{O}(0,-2)\right)$
$= \Lambda^2 V \oplus \Lambda^2 \tilde{V} = \mathbf{C} \oplus \mathbf{C}$

and so the map $H^1(Q) \longrightarrow H^2(\Lambda^2\mathcal{E}^*)$

has a two-dim'l kernel.

So far, we have computed the 2 pieces of classical correlation functions:

$$\mathrm{Sym}^2 W \;=\; H^0(\mathrm{Sym}^2 W \otimes \mathcal{O}) \;\xrightarrow{\sim}\; H^1(Q) \;\longrightarrow\; H^2(\Lambda^2 \mathcal{E}^*)$$

What we really want are the relations,
the kernel of the map above.

Since the first map is an isomorphism,
the kernel is determined by the second map.

So, let's compute the kernel of the second map in greater detail....

Compute kernel of $H^1(Q) \longrightarrow H^2(\Lambda^2 \mathcal{E}^*)$:

The original map $*: Z \longrightarrow W \otimes \mathcal{O}$ sends

$$V \otimes \mathcal{O}(-1,0) \oplus \tilde{V} \otimes \mathcal{O}(0,-1) \mapsto \left(\psi A + \tilde{\psi} B\right) V + \left(\psi C + \tilde{\psi} D\right) \tilde{V}$$

(by def'n of *), where $\psi, \tilde{\psi}$ are a basis for W.

Therefore, * induces a map $\Lambda^2 Z \longrightarrow Q$:

$$\Lambda^2 V \otimes \mathcal{O}(-2,0) \mapsto \det\left(\psi A + \tilde{\psi} B\right) \Lambda^2 V$$
$$\Lambda^2 \tilde{V} \otimes \mathcal{O}(0,-2) \mapsto \det\left(\psi C + \tilde{\psi} D\right) \Lambda^2 \tilde{V}$$

Classical ring rel'ns:

$$\det\left(\psi A + \tilde{\psi} B\right) = 0 = \det\left(\psi C + \tilde{\psi} D\right)$$

Quantum sheaf cohomology

What about nonperturbative sectors?

We can do exactly the same thing.

$\mathcal{M}$ = moduli space of instantons

$\mathcal{F}$ = induced bundle on the moduli space

If $\mathcal{E}$ is a deformation of TX,
then $\mathcal{F}$ is a deformation of $T\mathcal{M}$.

So: apply the same analysis as the classical case.

Quantum sheaf cohomology

Example: def' of T $\mathbf{P}^1 \times \mathbf{P}^1$

$$0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

Work in degree (d,e). $\mathcal{M} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2e+1}$

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \bigoplus_1^{2d+2} \mathcal{O}(1,0) \oplus \bigoplus_1^{2e+2} \mathcal{O}(0,1) \longrightarrow \mathcal{F} \longrightarrow 0$$

which we shall write as

$$0 \longrightarrow W \otimes \mathcal{O} \longrightarrow Z^* \longrightarrow \mathcal{F} \longrightarrow 0$$

(defining W, Z appropriately)

(cont'd)

Correlation functions are linear maps

$$\mathrm{Sym}^{2d+2e+2}\left(H^1(\mathcal{F}^*)\right)\left(=\mathrm{Sym}^{2d+2e+2}W\right) \longrightarrow H^{2d+2e+2}(\Lambda^{\mathrm{top}}\mathcal{F}^*) = \mathbf{C}$$

We compute using the Koszul resolution of $\Lambda^{\mathrm{top}}\mathcal{F}^*$:

$$0 \to \Lambda^{\mathrm{top}}\mathcal{F}^* \to \Lambda^{2d+2e+2}Z \to \Lambda^{2d+2e+1}Z \otimes W \to \Lambda^{2d+2e}Z \otimes \mathrm{Sym}^2 W$$

$$\cdots \to Z \otimes \mathrm{Sym}^{2d+2e+1}W \to \mathrm{Sym}^{2d+2e+2}W \otimes \mathcal{O}_{\mathcal{M}} \to 0$$

(cont'd)

(cont'd)

Briefly, the (long exact) Koszul resolution factors into a sequence of short exact sequences of the form

$$0 \longrightarrow S_i \longrightarrow \Lambda^i Z \otimes \mathrm{Sym}^{2d+2e+2-i} W \longrightarrow S_{i-1} \longrightarrow 0$$

and the coboundary maps $\delta : H^i(S_i) \longrightarrow H^{i+1}(S_{i+1})$ factor the map determining the correlation functions:

$$H^0\left(\mathrm{Sym}^{2d+2e+2} W \otimes \mathcal{O}_{\mathcal{M}}\right) \to H^1(S_1) \xrightarrow{\delta} H^2(S_2) \xrightarrow{\delta}$$

$$\cdots \xrightarrow{\delta} H^{2d+2e+1}\left(S_{2d+2e+1}\right) \xrightarrow{\delta} H^{2d+2e+2}\left(\Lambda^{\mathrm{top}} \mathcal{F}^*\right)$$

So, to evaluate corr' f'n, compute coboundary maps.

(cont'd)

(cont'd)

Need to compute coboundary maps.

Recall def'n

$$0 \longrightarrow S_i \longrightarrow \Lambda^i Z \otimes \mathrm{Sym}^{2d+2e+2-i} W \longrightarrow S_{i-1} \longrightarrow 0$$

Can show the $\Lambda^i Z$ only have nonzero cohomology in degrees 2d+2, 2e+2

Thus, the coboundary maps $\delta : H^i(S_i) \longrightarrow H^{i+1}(S_{i+1})$ are mostly isomorphisms; the rest have computable kernels.

(cont'd)

(cont'd)

Summary so far:

The correlation function factorizes:

$$H^0\left(\mathrm{Sym}^{2d+2e+2}W \otimes \mathcal{O}\right) \xrightarrow{\delta} H^1(S_1) \xrightarrow{\delta} H^2(S_2) \longrightarrow$$
$$\cdots \longrightarrow H^{2d+2e+2}\left(\Lambda^{\mathrm{top}}\mathcal{F}^\vee\right)$$

and one can read off the kernel.

Result:

For fixed (d,e), sheaf cohomology lives in

$$\mathrm{Sym}^* W / (Q^d, \tilde{Q}^e)$$

where
$$Q = \det(A\psi + B\tilde{\psi})$$
$$\tilde{Q} = \det(C\psi + D\tilde{\psi})$$

Quantum sheaf cohomology

Example: def' of T $\mathbf{P}^1 \times \mathbf{P}^1$

$$0 \longrightarrow W^* \otimes \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

Consider d=(1,0) maps. $\mathcal{M} = \mathbf{P}^3 \times \mathbf{P}^1$

$$0 \longrightarrow W^* \otimes \mathcal{O} \stackrel{*'}{\longrightarrow} \mathcal{O}(1,0)^4 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{F} \longrightarrow 0$$

$$*' = \left[\begin{array}{cc} \left[\begin{array}{cc} A & 0 \\ 0 & A \end{array} \right] y & \left[\begin{array}{cc} B & 0 \\ 0 & B \end{array} \right] y \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

Kernel generated by

$$\det \left(\psi \left[\begin{array}{cc} A & 0 \\ 0 & A \end{array} \right] + \tilde{\psi} \left[\begin{array}{cc} B & 0 \\ 0 & B \end{array} \right] \right) = \det(\psi A + \tilde{\psi} B)^2, \quad \det \left(\psi C + \tilde{\psi} D \right)$$

Quantum sheaf cohomology

So far I've discussed corr' f'ns in sectors of fixed instanton number $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\vec{d}}$ as maps

$$H^0\left(\mathrm{Sym}^n W \otimes \mathcal{O}\right) \longrightarrow H^n(\Lambda^n \mathcal{F}^*) \cong \mathbf{C}$$

whose kernels are computable.

Where do OPE's come from?

OPE's emerge when we consider the relations between *different* instanton sectors.

Quantum sheaf cohomology

Example: def' of T $\mathbf{P}^1 \times \mathbf{P}^1$

Define
$$Q = \det(\psi A + \tilde{\psi} B)$$
$$\tilde{Q} = \det(\psi C + \tilde{\psi} D)$$

I have stated $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 \in \mathrm{Sym}^n W / (Q, \tilde{Q})$

& more gen'ly,
$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{(a,b)} \in \mathrm{Sym}^n W / (Q^{a+1}, \tilde{Q}^{b+1})$$

OPE's relate corr' f'ns in different instanton degrees, and so, should map ideals to ideals.

Quantum sheaf cohomology

Existence of OPE's implies rel'ns of form

$$\langle \mathcal{O} \rangle_{a,b} \propto \langle \mathcal{O} R_{a,b,a',b'} \rangle_{a',b'}$$

for some $R_{a,b,a',b'}$ which must map kernels -> kernels.

We're calling the R's ``exchange rates,''
and they determine OPE's.

Quantum sheaf cohomology

Derive OPE ring for $\mathbf{P}^1 \times \mathbf{P}^1$ example:

Existence of OPE's implies rel'ns of form

$$\langle \mathcal{O} \rangle_{a,b} \propto \langle \mathcal{O} R_{a,b,a',b'} \rangle_{a',b'}$$

"exchange rate"

In order to be compatible with kernels, need

$$\langle \mathcal{O} \rangle_{a,b} \propto \langle \mathcal{O} Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

Assume proportionality constant is

$$\langle \mathcal{O} \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle \mathcal{O} Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

then have OPE's: $Q = q, \quad \tilde{Q} = \tilde{q}$

Summary of $\mathbf{P}^1 \times \mathbf{P}^1$ example:

$$Q = \det\left(A\psi + B\tilde{\psi}\right) = q$$
$$\tilde{Q} = \det\left(C\psi + D\tilde{\psi}\right) = \tilde{q}$$

* This is the result of our math analysis.

* Also was derived from GLSM's by McOrist-Melnikov (along with lin' def's in other GLSM's)

Quantum sheaf cohomology

Program so far:

* For each fixed instanton degree,
compute the kernels of corr' f'ns in that degree.

* To derive OPE's,
compute ``exchange rates" relating corr' f'ns
of different instanton degrees.
Required to map kernels --> kernels.

What about 4-fermi terms?

Quantum sheaf cohomology

In (0,2) theories, 4-fermi terms are of the form

$$F_{i\bar{\jmath}a\bar{b}}\psi_+^i \psi_+^{\bar{\jmath}} \lambda^a \lambda^{\bar{b}}$$

& can be used to soak up `excess' zero modes, ie, zero modes of worldsheet vectors.

Formally, each 4-fermi insertion ought to be identified with an insertion of

$$H^1(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes \mathrm{Obs}^\vee)$$

On (2,2) locus, this becomes Atiyah class of Obs, and reproduces old Aspinwall-Morrison story.

Quantum sheaf cohomology

4-fermi terms:

Unfortunately, we do not yet have a complete derivation from first-principles of the effects of 4-fermi terms in our computations.

However, the GLSM suggests an ansatz:

write $\ast : Z \longrightarrow W \otimes \mathcal{O}$ as $A_i^a \psi_a$

where (ψ_a) a basis for W,

then insert $\prod_c \left(\det_{i,j} \partial_i A_j^a \psi_a \right)^{n_c}$ in corr' f'ns.

(c runs over lin' equiv' classes.)

-- can show result is ind' of nonlinear def's !

Final result for quantum sheaf cohomology:

for deformations of tangent bundles of toric varieties,

$$\prod_c \left(\det_{i,j} (\partial_i A^a_j \psi_a) \right)^{Q^a_c} \; = \; q_a$$

generalizing Batyrev's ring $\prod_i \left(\sum_b Q^b_i \psi_b \right)^{Q^a_i} \; = \; q_a$

Linear case: McOrist-Melnikov 0712.3272

Here: generalized to all deformations,
trivially: does *not* depend on nonlinear def's.

(See papers for details.)

QSC for Grassmannians

Next, let us outline the fundamentals of q.s.c. for Grassmannians G(k,n) of k-planes in $\mathbf{C}^n$.

The GLSM is a U(k) gauge theory with n fundamentals.

Now, bundles are no longer defined by line bundles, but rather by vector bundles associated to products of rep's of U(k).

All the heterotic bundles will be built from (co)kernels of short exact sequences in which all the other elements are bundles defined by reps of U(k).

Ex:

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus^{n} \mathcal{O}(\mathbf{k}) \bigoplus^{k+1} \mathrm{Alt}^2 \mathcal{O}(\mathbf{k}) \longrightarrow \bigoplus^{k-1} \mathrm{Sym}^2 \mathcal{O}(\mathbf{k}) \longrightarrow 0$$

$\mathcal{O}(\mathbf{k})$ is bundle associated to fund' rep' of U(k)

What is the LSM moduli space $\mathcal{M}$?

As usual, expand GLSM fields in zero modes,
but now, end up quotienting by nonreductive gps.

Result is a "Quot scheme."

Specifically, the LSM moduli space of degree d maps
$\mathbf{P}^1$ --> G(k,n) = G(n-k,n)
is the Quot scheme

$$\mathrm{Quot}_{\mathbf{P}^1}\left(\mathcal{O}^n, k, -d\right)$$

of rk n-k subsheaves of $\mathcal{O}^{\oplus n}$ of degree d, over $\mathbf{P}^1$.

What about induced bundles $\mathcal{F} \to \mathcal{M}$?

The program is as before:

Given a short exact sequence of (fermions in GLSM),
lift to natural sheaves on $\mathbf{P}^1 \times \mathcal{M}$,
then pushforward to $\mathcal{M}$.
(just as for toric varieties)

Lift to nat'l sheaves on $\mathbf{P}^1 \times \mathcal{M}$,
pushforward to $\mathcal{M}$.

(& similarly for toric varieties)

Corresponding to $\mathcal{O}(\overline{\mathbf{k}})$ is a
rk k `universal subbundle' $\mathcal{S}$ on $\mathbf{P}^1 \times \mathcal{M}$.

Lift all others so as to be a U(k)-rep' homomorphism

Ex:

$$\mathcal{O}(\mathbf{k}) \mapsto \mathcal{S}^*$$

$$\mathcal{O}(\mathbf{k}) \otimes \mathcal{O}(\overline{\mathbf{k}}) \mapsto \mathcal{S}^* \otimes \mathcal{S}$$

$$\mathrm{Alt}^m \mathcal{O}(\mathbf{k}) \mapsto \mathrm{Alt}^m \mathcal{S}^*$$

Then pushforward to LSM moduli space, and compute.

Ex:

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus^{n} \mathcal{O}(\mathbf{k}) \bigoplus^{k+1} \mathrm{Alt}^2 \mathcal{O}(\mathbf{k}) \longrightarrow \bigoplus^{k-1} \mathrm{Sym}^2 \mathcal{O}(\mathbf{k}) \longrightarrow 0$$

$\mathcal{O}(\mathbf{k})$ is bundle associated to fund' rep' of U(k)

The bundle above, over G(k,n), naturally lifts to

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \oplus^n \mathcal{S}^* \oplus^{k+1} \mathrm{Alt}^2 \mathcal{S}^* \longrightarrow \oplus^{k-1} \mathrm{Sym}^2 \mathcal{S}^* \longrightarrow 0$$

over $\mathbf{P}^1 \times \mathcal{M}$

Pushforward to $\mathcal{M}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow R^0\pi_* \left(\oplus^n \mathcal{S}^* \oplus^{k+1} \mathrm{Alt}^2 \mathcal{S}^* \right) \longrightarrow R^0\pi_* \left(\oplus^{k-1} \mathrm{Sym}^2 \mathcal{S}^* \right)$$

$$\longrightarrow \mathcal{F}_1 \longrightarrow R^1\pi_* \left(\oplus^n \mathcal{S}^* \oplus^{k+1} \mathrm{Alt}^2 \mathcal{S}^* \right) \longrightarrow R^1\pi_* \left(\oplus^{k-1} \mathrm{Sym}^2 \mathcal{S}^* \right) \longrightarrow 0$$

& then compute! Details will appear elsewhere....

Summary:

-- overview of progress towards (0,2) mirrors; starting to heat up!

-- outline of quantum sheaf cohomology (part of (0,2) mirrors story)