# Application of decomposition to anomaly resolution

## Harvard/MIT Quantum Matter seminar
## October 29, 2021

Eric Sharpe
Virginia Tech

An overview of hep-th/0502027, 0502044, 0502053, 0606034, 0709.3855, 1012.5999, 1307.2269, 1404.3986, … (many …), 2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423

My talk today concerns the application of **decomposition**,
a new notion in quantum field theory (QFT),
to resolution of anomalies as proposed in Wang-Wen-Witten.

Briefly, decomposition is the observation that some QFTs
are secretly equivalent to
sums of other QFTs, known as 'universes.'

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al '06)

## 1) Existence of projection operators

The theory contains topological operators $\Pi_i$ such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

Math analogue:

If a space $X$ has $m$ connected components, then $\dim H^0(X) = m$

— multiple degree-zero elements of cohomology

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al '06)

## 2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i Z_i = \sum_i \sum \exp(-\beta H_i)$$

(on a connected spacetime)

Now, given a nontrivial structure, expect a symmetry....

## 3) In (n+1) spacetime dimensions, has a (possibly noninvertible) n-form symmetry.

I'll explain what that is in a few minutes....

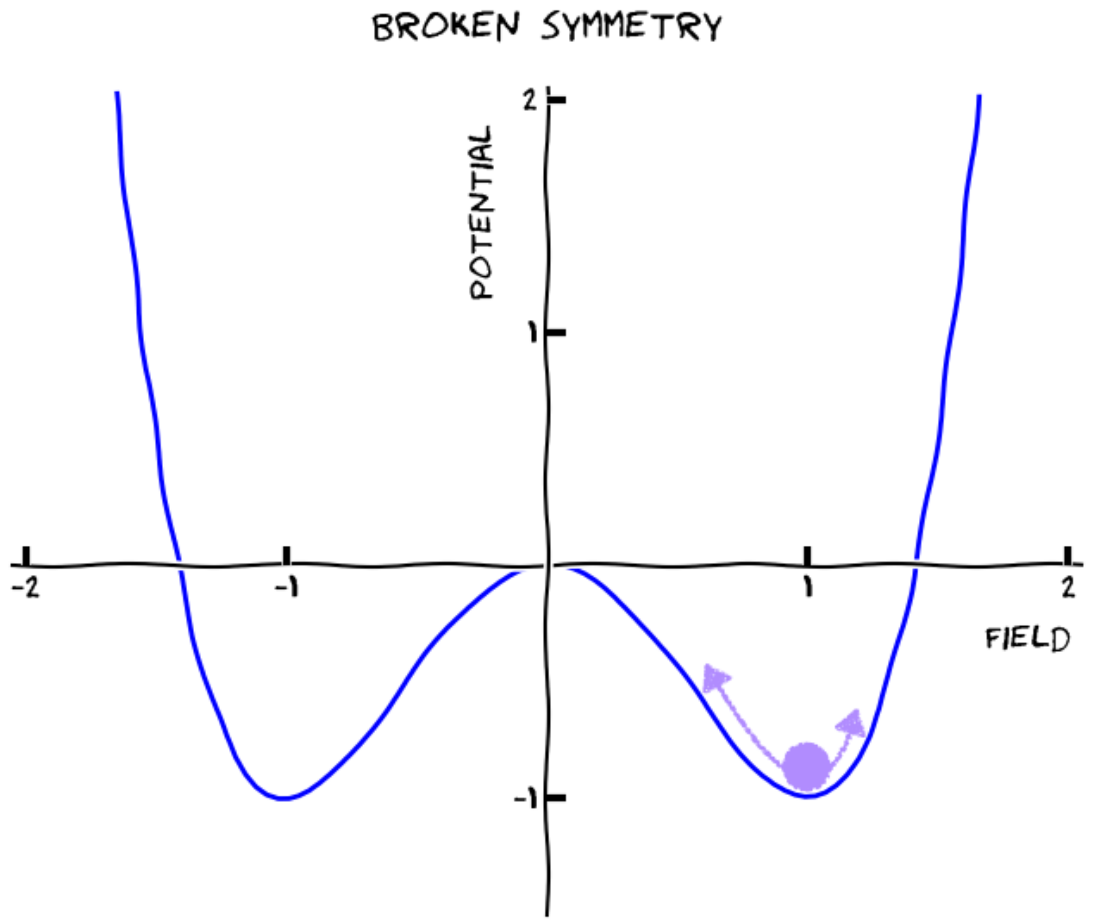
# Decomposition $\neq$ spontaneous symmetry breaking

## SSB:

**Superselection sectors:**

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

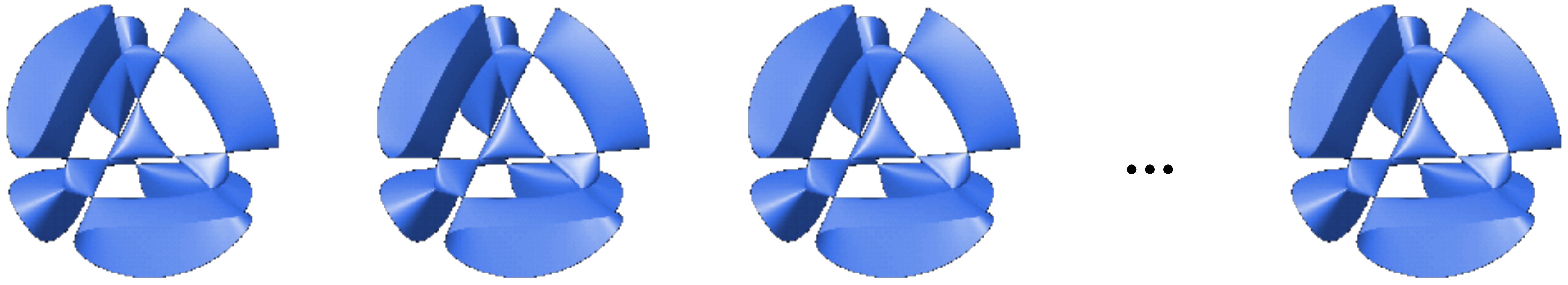



## Decomposition:

**Universes:**

- separated by nondynamical domain walls
- disjoint at all energy scales
- multiple different QFTs present

Prototype:

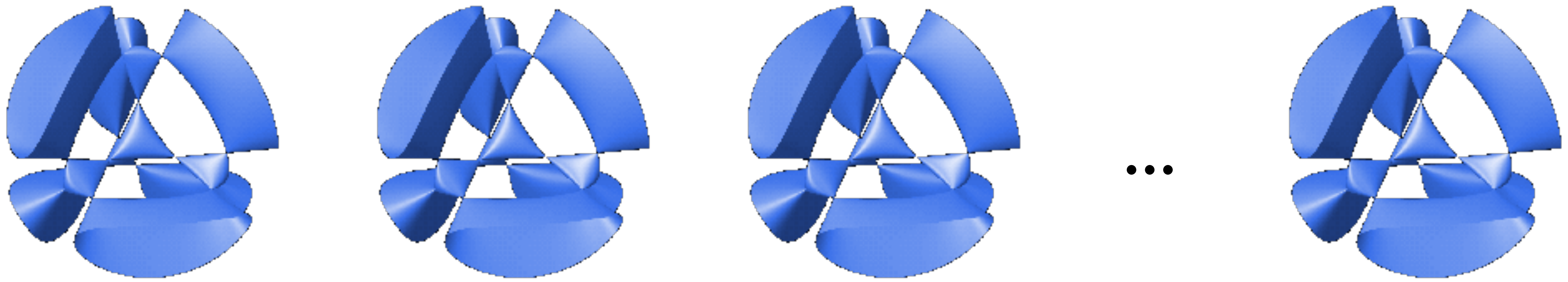


(see e.g. Tanizaki-Unsal 1912.01033)

## Decomposition $\neq$ spontaneous symmetry breaking

Note that they both have an order parameter, so be careful when distinguishing.

Ex: sigma model on disjoint union of n spaces ('universes')



Have topological projectors $\Pi_i$ $\quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1$

Have order parameter $X$ $\quad X = \sum_{i=0}^{n-1} \xi^i \Pi_i, \quad \xi = \exp(2\pi i/n)$

Vev in $i$th universe: $\quad \langle \Pi_i X \rangle = \langle \xi^i \Pi_i \rangle = \xi^i$

So, could be described as spontaneously broken phase
— but that clearly does **not** capture the physics.

I mentioned higher-form symmetries. What's a one-form symmetry?....

What is a one-form symmetry?

For this talk, *intuitively*, this will be a `group' that exchanges nonperturbative sectors.

Example: $G$ gauge theory in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume $K$ abelian,
but decompositions exist more generally.)

Then, at least for $K$ central, nonperturbative sectors are invariant under

$$(G-\text{bundle}) \mapsto (G-\text{bundle}) \otimes (K-\text{bundle})$$

$$A \mapsto A + A'$$

At least when $K$ central, this is the action of the `group' of $K$-bundles.

(Technically, is a 2-group, only weakly associative.)

That group is denoted $BK$ or $K^{(1)}$

One-form symmetries can also be seen in algebra of topological local operators,
as we'll see later.

What sort of QFTs admit a decomposition ? ....

The QFTs I'm interested in, which have a decomposition, are (1+1)-dimensional theories with "global 1-form symmetries," and can be described in several ways, such as

(Pantev, ES '05; Hellerman et al '06)

- Gauge theory w/ trivially-acting subgroup
- Theory w/ restriction on instantons
- Sigma models on gerbes
  = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$

We'll see in this talk how decomposition (into 'universes') relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\mathrm{point}/G]$

# Decomposition in (1+1)-d gauge theories

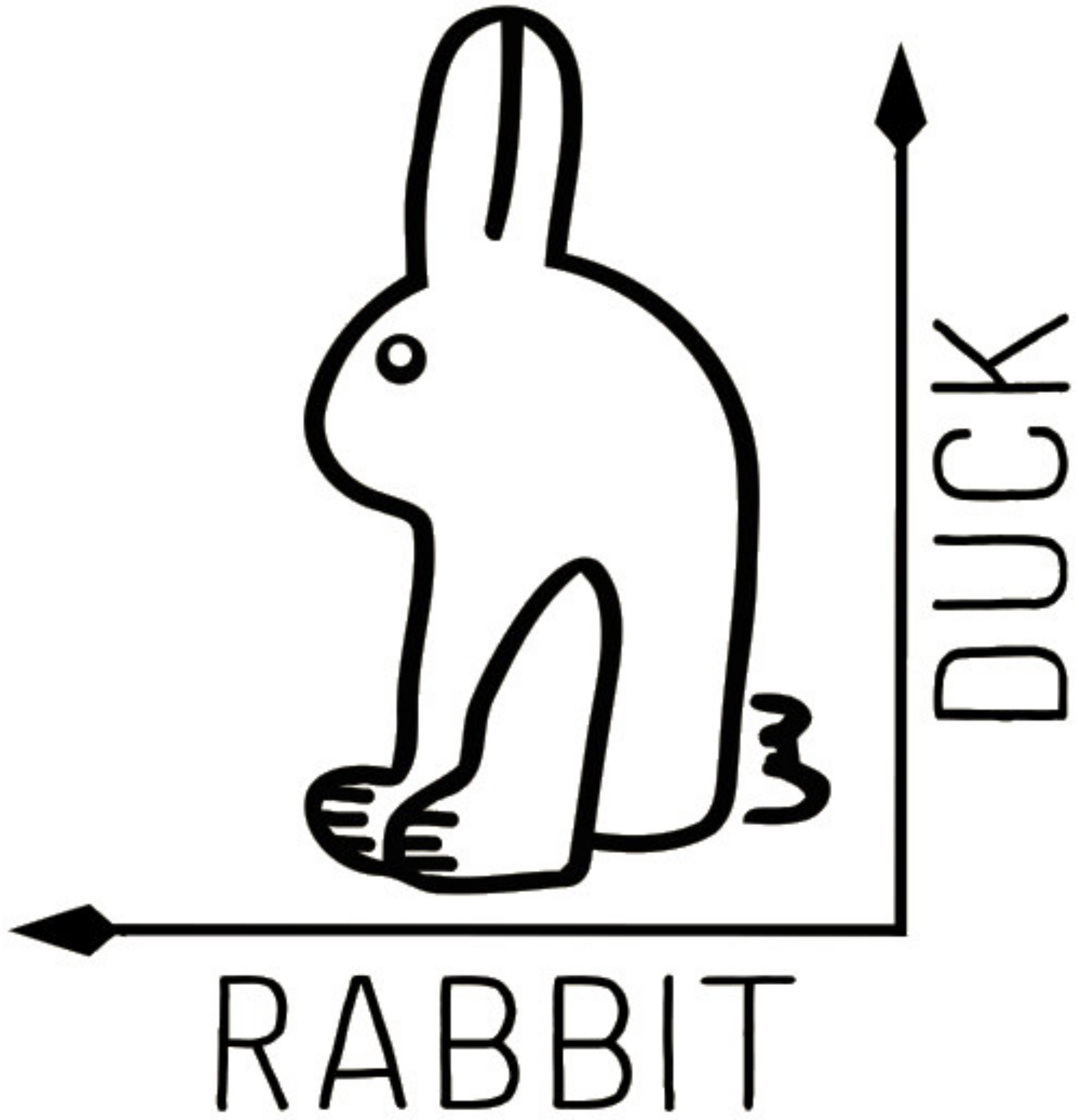
(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, $G$ semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume $K$ is in the center. Has $BK$ 1-form symmetry.

So far, this sounds like just one QFT.




However, I'll outline how, from another perspective, QFTs of this form are also each a disjoint union of other QFTs; they "decompose."

# Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, $G$ semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume $K$ is in the center.

Has $BK$ 1-form symmetry.

Claim this theory decomposes.
Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn's theorem.

Universes $\longleftrightarrow$ Irreducible representations of $K$

Partition functions & relation of decomp' to restrictions on instantons....

# Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, $G$ semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume $K$ is in the center. Has $BK$ 1-form symmetry.

Statement of decomposition:

$$\mathrm{QFT}(G-\text{gauge theory}) \;=\; \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles ($w_2$)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

# Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

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Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles ($w_2$)

$SU(2)$ instantons (bundles) $\subset SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

# Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, $G$ semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume $K$ is in the center. Has $BK$ 1-form symmetry.

Statement of decomposition:

$$\mathrm{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left( \sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

("multiverse interference" cancels out some sectors)

# Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

$$Z = \underbrace{\sum_{\theta\in\hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left( \overbrace{\sum_{\theta\in\hat{K}} \exp\left[\theta \int \omega_2(A)\right]}^{\text{projection operator}} \right)$$

# Decomposition in (1+1)-d gauge theories

(Hellerman et al ’06)

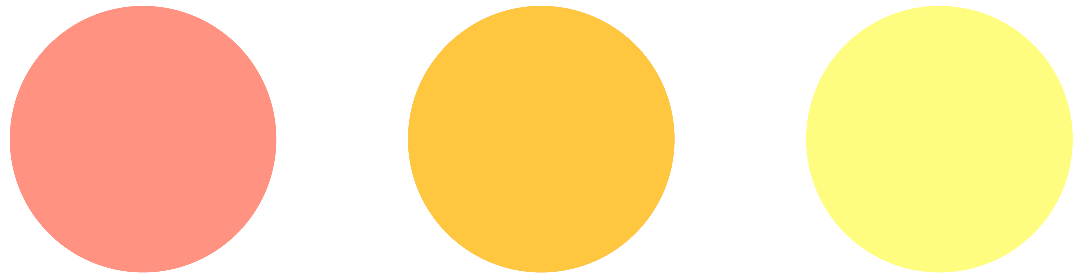
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left( \sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union of several QFTs / universes = \`One’ QFT with a restriction on nonperturbative sectors = \`multiverse interference’

Schematically,
two theories combine to form a distinct third:

universe
$(SO(3)_+)$



universe
$(SO(3)_-)$

multiverse interference effect
$(SU(2))$

# Decomposition in (1+1)-d gauge theories

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint $QCD_2$ (Komargodski et al '20)
- Numerical checks (Honda et al '21)
- Plus version for (3+1)d theories w/ 3-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

Applications include:

- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ...)
- Elliptic genera (Eager et al '20)
- Anomalies (Robbins et al '21)

After review, we'll look at application to anomalies....

# Decomposition in (1+1)-d gauge theories

My goal today is to apply decomposition to an anomaly resolution procedure in finite gauge theories (Wang-Wen-Witten '17), of which my go-to examples are "orbifolds."

An orbifold $[X/G]$ is a $G$-gauge theory for a finite group $G$, specifically, a $G$-gauged sigma model into a (target) space $X$.

What does that mean?

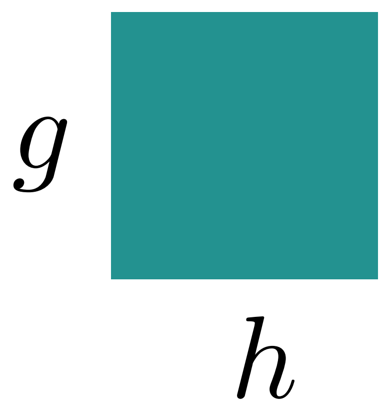
An ordinary sigma model is a path integral over maps into target space $X$:

$$\Sigma \longrightarrow X$$

When we gauge $G$, we identify field configurations related by $G$.

In the orbifold $[X/G]$, we allow for branch cuts defined by elements of $G$.

Example: $T^2 \longrightarrow X$




$\longrightarrow X$

# Decomposition in (1+1)-d gauge theories

My goal today is to apply decomposition to an anomaly resolution procedure
in finite gauge theories (Wang-Wen-Witten '17),
of which my go-to examples are "orbifolds."

An orbifold $[X/G]$ is a $G$-gauge theory for a finite group $G$,
specifically, a $G$-gauged sigma model into a (target) space $X$.

The details of $X$, and sigma models more generally,
aren't specifically relevant to either decomposition or anomaly resolution;
I'll use them simply to give this all a concrete underpinning.

If it's helpful, whenever I say "orbifold," just think, "finite gauge theory."

# Decomposition in (1+1)-d gauge theories

My goal today is to apply decomposition to an anomaly resolution procedure in finite gauge theories (Wang-Wen-Witten '17), of which my go-to examples are "orbifolds."

Briefly, the idea of WWW is that if a given orbifold $[X/G]$ is ill-defined because of an anomaly (which obstructs the gauging), then replace $G$ with a larger group $\Gamma$ whose action is anomaly-free.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

The larger group $\Gamma$ has a subgroup $K \subset \Gamma$ that acts trivially on X, and $G = \Gamma/K$.

However, orbifolds with trivially-acting subgroups are standard examples in which decomposition arises (in 1+1 dimensions), so one expects decomposition is relevant here.

(Hellerman et al '06)

Plan for the remainder of the talk:

- Describe decomposition in orbifolds with trivially-acting subgroups,
- Add a new modular invariant phase: "quantum symmetry," in $H^1(G, H^1(K, U(1)))$,
- Review the anomaly-resolution procedure of (Wang-Wen-Witten '17),
- and apply decomposition to that procedure.

What we'll find is that, in (1+1)-dimensions,

$$\mathrm{QFT}(\,[X/\Gamma]_B\,) \;=\; \mathrm{QFT}(\text{ copies and covers of } [X/(\text{nonanomalous subgp of } G]\,)$$

as a consequence of decomposition.
This gives a simple understanding of why the WWW procedure works,
as well as of the result.

# Decomposition in orbifolds in (1+1) dimensions

Let's begin by discussing ordinary orbifolds w/o extra phases.
(We'll need a more complicated version for anomaly resolution,
but let's start here, and build up.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ ($K$ need not be central) ($K, \Gamma, G$ finite)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of $K$

$G$ acts on $\hat{K}$: $\rho(k) \mapsto \rho(hkh^{-1})$ for $h \in \Gamma$ a lift of $g \in G$

$\hat{\omega}$ = phases called "discrete torsion" — we'll see more later.

# Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad (K \text{ need not be central})$$

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) \;=\; \mathrm{QFT}\left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of $K$

If $K$ is in the center of $\Gamma$, then the $G$ action on $\hat{K}$ is trivial, and decomposition specializes to

$$\mathrm{QFT}\left([X/\Gamma]\right) \;=\; \mathrm{QFT}\left( \coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

— a disjoint union, as many elements as $\hat{K}$

More gen'ly, get both copies and covers of $[X/G]$, as we shall see.

# Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ ($K$ need not be central)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Helllerman et al '06)

$\hat{K}$ = set of iso classes of irreps of $K$

Universes (summands of decomposition)
correspond to orbits of $G$ action on $\hat{K}$.

We'll see explicit formulas for projectors and $\hat{\omega}$ later....

# Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ ($K$ need not be central)

(Hellerman et al '06)

Boundaries also decompose.

The boundary of that theory in (1+1) dims can have e.g. fermions on which $\Gamma$ acts.

Although $K \subset \Gamma$ acts trivially on the bulk d.o.f.,
it can act *non*trivially on boundary d.o.f.

To compute which universe a given boundary lies in,
restrict the $\Gamma$ action to $K$,
at which point it becomes a representation of $K$.
Then, compare orbits — universes correspond to orbits in $\hat{K}$.

# Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ ($K$ need not be central)

(Hellerman et al '06)

Boundaries also decompose.

A quick note — boundaries can be understood in terms of K theory, and this boundary decomposition reflects math of K theory.

Technically, $[X/\Gamma]$ is an example of a "gerbe" on $[X/G]$, essentially, a fiber bundle in which the fibers are `groups' $BK$ of 1-form symmetries. ($BK = [\text{point}/K]$, which is why have triv'ly acting $K$.)

Fun math fact: K theory of a gerbe = disjoint union of K theory of underlying spaces/orbifolds, in the same fashion as we have described.

(Decomposition gives a physical explanation for this facet of K theory.)

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

**Example:** Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

(T Pantev, ES '05)

— has $B\mathbb{Z}_2$ (1-form) symmetry

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

(consequence of a general formula)

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \;=\; \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \;=\; 1.$

Using that relation, we form projection operators:

$$\Pi_\pm \;=\; \frac{1}{2}(1 \pm \hat{z})$$

$$\Pi_\pm^2 = \Pi_\pm \qquad\qquad \Pi_\pm\Pi_\mp = 0$$

Next: compare partition functions....

## Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where $z$ generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be $T^2$.

The partition function of any orbifold $[X/\Gamma]$ on $T^2$ is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left( g \underset{h}{\blacksquare} \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to $X$ with branch cuts $g$, $h$.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm d.t.}\right)$$

## Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

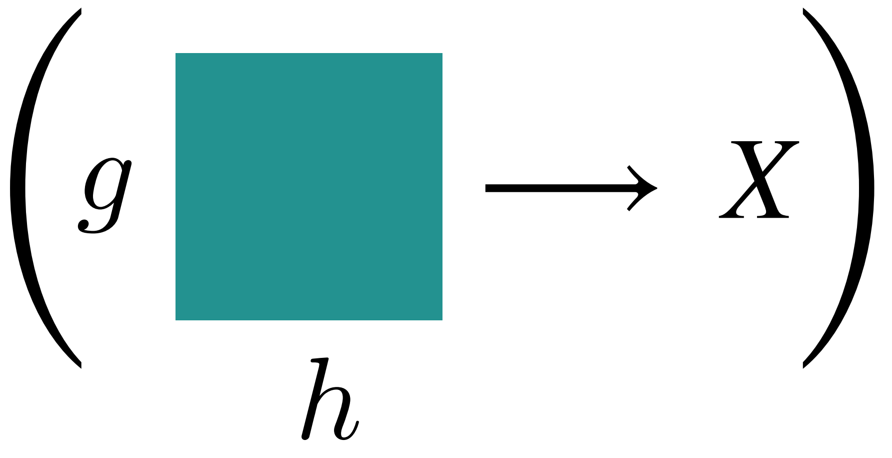
where $z$ generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

where $\overline{a} = \{a, az\}$ etc

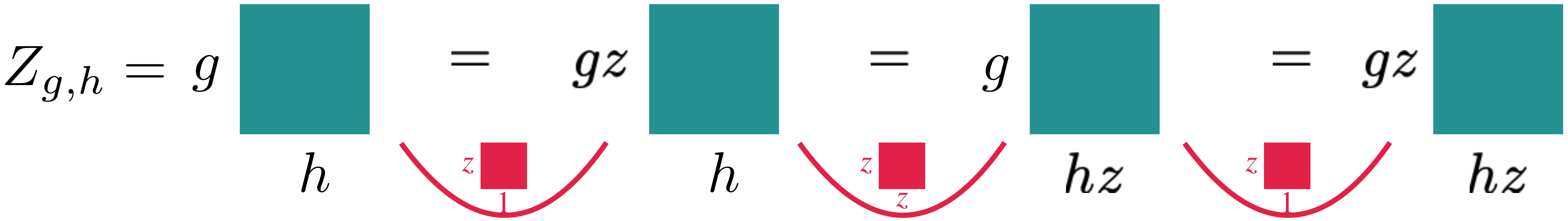
$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{gh \in D_4, gh=hg} Z_{g,h}$$

where $Z_{g,h} =$




Since $z$ acts trivially,

$Z_{g,h}$ is symmetric under multiplication by $z$




This is the $B\mathbb{Z}_2$ 1-form symmetry.

# Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where $z$ generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{1}{|D_4|} \sum_{gh \in D_4, gh=hg} Z_{g,h}$$ where $Z_{g,h} \;=\; \left( g\, \underset{h}{\blacksquare} \longrightarrow X \right)$

Each $D_4$ twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

**except** for the sectors $\overline{a}\, \underset{\overline{b}}{\blacksquare}$ $\quad \overline{a}\, \underset{\overline{ab}}{\blacksquare}$ $\quad \overline{b}\, \underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

## Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$=\; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

Different theory than $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

Fact: given any one partition function $\quad Z_{T^2}([X/G]) \;=\; \frac{1}{|G|}\sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' \;=\; \frac{1}{|G|}\sum_{gh=hg}\epsilon(g,h)\,Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G,U(1))$

This is called "discrete torsion."

## Example, cont'd

Compute the partition function of $[X/D_4]$

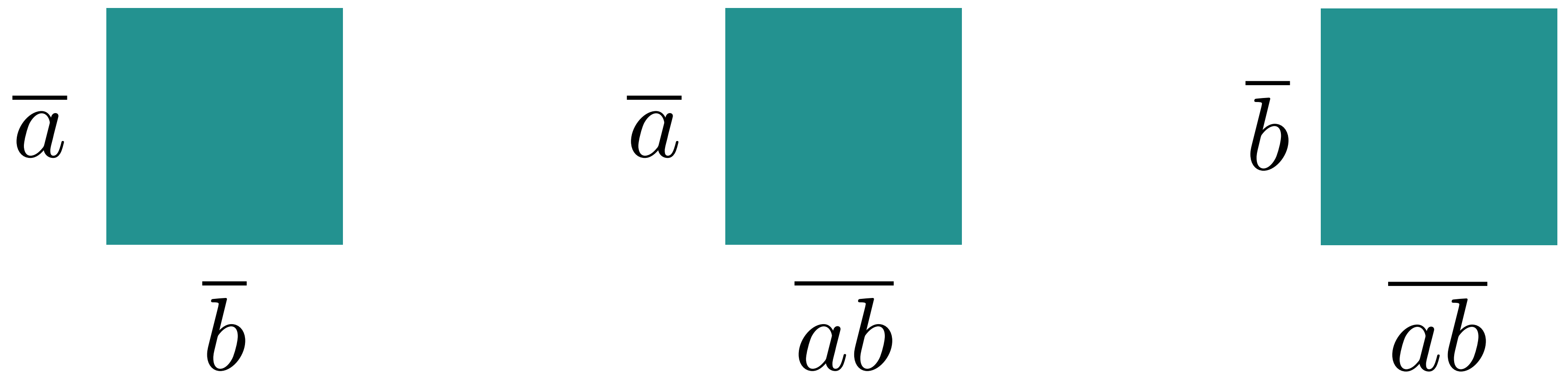
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, discrete torsion $\in \; H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \;=\; \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors




which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

## Example, cont'd

Compute the partition function of $[X/D_4]$

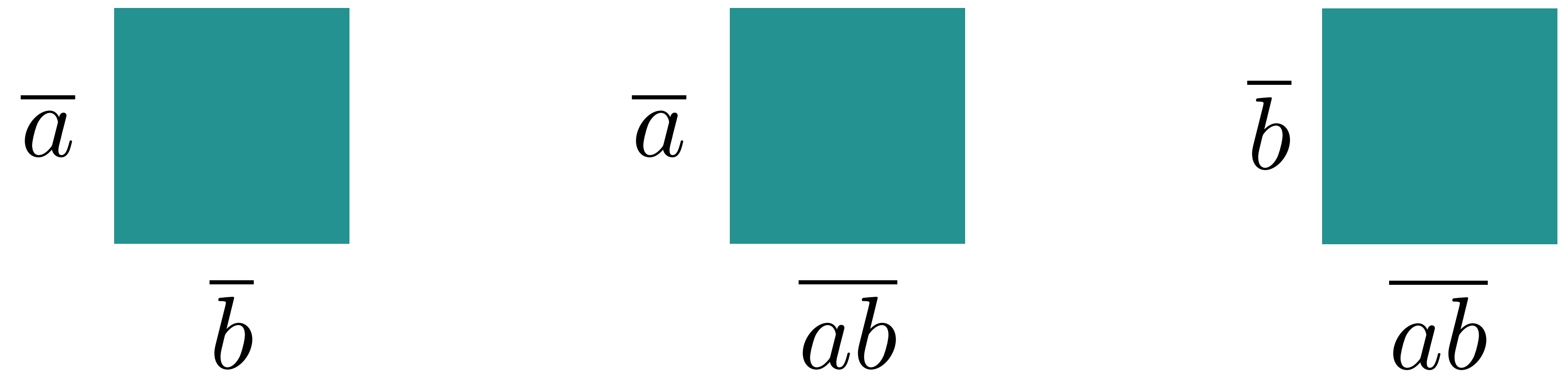
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left( Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; \left(\text{some twisted sectors}\right)\right)$$

$$= \; 2 \left( Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; \left(\text{some twisted sectors}\right)\right)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ ,

and acts as a sign on the twisted sectors




which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on $T^2$
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

I'll come back to dilaton shifts later in discussing example of
orbifold of a point ( = (1+1)-d Dijkgraaf-Witten theory).

# Example, cont'd

Quick aside on symmetries:

I mentioned that the one-form symmetry $B\mathbb{Z}_2$ shows up in permutations of nonperturbative sectors.

It also shows up in the dimension-zero twist field: $\hat{z}^2 = 1$

We'll come back to this later.

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts (Hellerman et al '06)

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

Write $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$

Dimension-zero twist fields: $1, \quad \sigma_{-1}, \quad \sigma_{[i]}$

obeying $\sigma_{-1}^2 = 1, \; \sigma_{-1}\sigma_{[i]} = \sigma_{[i]}, \; \sigma_{[i]}^2 = (1/2)\left(1 + \sigma_{-1}\right)$

Projectors:

$$\Pi_{\pm} = \frac{1}{4}\left(1 + \sigma_{-1} \pm 2\sigma_{[i]}\right), \quad \Pi_2 = \frac{1}{2}\left(1 - \sigma_{-1}\right)$$

(project onto $[X/\mathbb{Z}_2]$) (projects onto $X$)

which are easily checked to be idempotents. Partition functions….

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) \; = \; \mathrm{QFT}\left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

Write $\mathbb{H} \; = \; \{\pm 1, \pm i, \pm j, \pm k\}$

Partition function on $T^2$: Denote generator of $\mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ by $\xi$

$$Z_{T^2}\left([X/\mathbb{H}]\right) \; = \; \frac{1}{|\mathbb{H}|} \sum_{gh=hg} Z_{g,h} \; = \; \frac{1}{|\mathbb{H}|} \left( (16) \; {}^{1}\underset{1}{\blacksquare} + (8) \; {}^{1}\underset{\xi}{\blacksquare} + (8) \; {}^{\xi}\underset{\xi}{\blacksquare} \right)$$

$$= \; 2 Z_{T^2}\left([X/\mathbb{Z}_2]\right) \; + \; Z_{T^2}(X)$$

Works!

Higher genus partition functions also work (w/ dilaton shifts), see hep-th/0606034 sect 5.4.

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

One-form symmetries:

Recall this theory has dimension-zero twist fields: $1, \quad \sigma_{-1}, \quad \sigma_{[i]}$

obeying $\sigma_{-1}^2 = 1, \; \sigma_{-1}\sigma_{[i]} = \sigma_{[i]}, \; \sigma_{[i]}^2 = (1/2)\left(1 + \sigma_{-1}\right)$

This describes a noninvertible one-form symmetry,
which includes a $B\mathbb{Z}_2$ as a subset: $\sigma_{-1}^2 = 1$.

Now that we've seen some concrete examples,
let's relate this back to the general picture of decomposition given earlier.

(Then, later, we'll apply this to anomaly resolution.  Bear with me....)

# Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

where $\hat{K}$ = irreps of $K$

$\hat{\omega}$ = discrete torsion on universes

Example: $[X/D_4]$

Here, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially on $\hat{K} = \mathbb{Z}_2$ so RHS = 2 copies of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$

Example: $[X/\mathbb{H}]$

Here, $G = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

$$\text{so RHS } = X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2]$$

# Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

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Decomposition predicts

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where $\hat{K}$ = irreps of $K$

$\hat{\omega}$ = discrete torsion on universes

Projectors:

For $R = \oplus_i R_i$, $R_i \in \hat{K}$ related by the action of $G$, we have

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

# Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

where $\hat{K}$ = irreps of $K$
$\hat{\omega}$ = discrete torsion on universes

What about the $\hat{\omega}$ ? Where did that come from?

Its discussion is more technical & abstract, so I wanted to delay until after examples, but now that we've seen some examples, I'll outline it....

# Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Origin of $\hat{\omega}$: (apologies for the math, I'll try to be quick)

Let $\{\rho_a\}$ be a collection of irreps in $\hat{K}$ chosen s.t. $[\rho_a]$ represent orbits of G on $\hat{K}$.

For each $a$, let $H_a \subset G$ be the stabilizer of $[\rho_a] \subset \hat{K}$ (ie, the subgp that leaves it inv't).

$$\Delta_a = \pi^* H_a \subset \Gamma, \quad s_a : H_a \to \Delta_a \text{ a section}$$

Write $\rho_a : K \to \mathrm{End}(V_a)$, then there are intertwiners $f_a : H_a \to \mathrm{End}(V_a)$,

$$\begin{tikzcd} V_a \arrow[r, "\rho_a(k)"] \arrow[d, "f_a(h)"'] & V_a \arrow[d, "f_a(h)"] \\ V_a \arrow[r, "\rho_a(s_a(h)^{-1} k s_a(h))"] & V_a \end{tikzcd}$$

Define a projective rep' $\tilde{\rho}_a$ of $\Delta_a$ by $\tilde{\rho}_a(\,\iota(k)\, s_a(h)\,) = \rho_a(k) f_a(h)^{-1}$

Can show $\tilde{\rho}_a(g_1)\,\tilde{\rho}_a(g_2) = \hat{\omega}(g_1, g_2)\,\tilde{\rho}_a(g_1 g_2)$ where $\hat{\omega}$ is a cocycle & pullback from $H_a$

# Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Origin of $\hat{\omega}$: (apologies for the math, I'll try to be quick)

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For each $a$, let $H_a \subset G$ be the stabilizer of $[\rho_a] \subset \hat{K}$ (ie, the subgp that leaves it inv't).

$$\Delta_a = \pi^* H_a \subset \Gamma, \quad s_a : H_a \to \Delta_a \text{ a section}$$

Explicitly,

$$\hat{\omega}(h_1, h_2)\, I = f_a(h_1)^{-1} f_a(h_2)^{-1} f_a(h_1 h_2)\, \rho_a \left( s_a(h_1) s_a(h_2) s_a(h_1 h_2)^{-1} \right)^{-1} \quad \text{where } h_1, h_2 \in H_a$$

If $K$ is in the center of $\Gamma$, then $G$ acts triv'ly on $\hat{K}$, all irreps are 1d,

and then $\hat{\omega}$ is the (inverse of the) image of the characteristic class of the $K$-gerbe:

$$H^2([X/G], K) \xrightarrow{\rho_a} H^2([X/G], U(1))$$

You won't need this level of detail for this talk, but I wanted to stress that it exists.

So far I've outlined how decomposition works in orbifolds $[X/\Gamma]$,
with trivially-acting $K \subset \Gamma$,
and no discrete torsion or other phase modifications.

However, in order to apply this to anomaly resolution,
we're going to need to understand decomposition in orbifolds
modified by (modular-invariant) phases.

Next: decomposition in orbifolds $[X/\Gamma]_\omega$ with discrete torsion $\omega \in H^2(\Gamma, U(1))$....

# Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) = \mathrm{Hom}(G, \hat{K})$$

Cases:

1) If $\iota^*\omega \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{G}\right]_{\hat{\omega}}\right)$$

2) If $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, \beta(\omega)}}{\mathrm{Ker}\, \beta(\omega)}\right]_{\hat{\omega}}\right)$$

3) If $\iota^*\omega = 0$ and $\beta(\omega) = 0$, then $\omega = \pi^*\overline{\omega}$ for $\overline{\omega} \in H^2(G, U(1))$ and

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\overline{\omega}+\hat{\omega}}\right)$$

Projectors....

# Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1)))$$

Projectors:

For $R = \oplus_i R_i$, $R_i \in \hat{K}$ related by the action of $G$, we have

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \frac{\chi_{R_i}(k^{-1})}{\omega(k, k^{-1})} \tau_k$$

Suffice it to say,
in orbifolds in (1+1) dimensions,
we've got a pretty good handle on how
decomposition works.
(We should, we've been developing it since 2006.)

Let me mention one more family of examples,
related in one corner to orbifolds,
to illustrate other features of decomposition,
before turning to anomaly resolution.

# Decomposition in orbifolds in (1+1)-dims with discrete torsion

An important special case: $[\mathrm{point}/G]_\omega$

Decomposition implies $$\mathrm{QFT}\left([\mathrm{point}/G]_\omega\right) = \coprod \mathrm{QFT}(\mathrm{point})$$

(as many copies as $\omega$-proj' irreps of $G$)

up to overall dilaton shifts.

In math, this is a gen'l property of the center of the (twisted) group algebra $\mathbb{C}[G]_\omega$:
it has a basis corresponding to twist fields,
and another basis of projectors.

QFT(point) is an example of an \`invertible' field theory.

This is also two-dimensional Dijkgraaf-Witten theory, a 2d TQFT....

# Decomposition in orbifolds in (1+1)-dims with discrete torsion

An important special case: $[\mathrm{point}/G]_\omega$ = (1+1)d Dijkgraaf-Witten TQFT

Decomposition implies $\mathrm{QFT}\left([\mathrm{point}/G]_\omega\right) = \coprod \mathrm{QFT}(\mathrm{point})$

(as many copies as $\omega$-proj' irreps of $G$)

As a consistency check, consider the partition function.

On a genus $g$ Riemann surface,

$$Z = \frac{1}{|G|^g} \sum_{a_i,b_i} \delta\left(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \cdots a_g^{-1} b_g^{-1}\right) \epsilon_g(a_i, b_i)$$

$$= \sum_R \left(\frac{\dim R}{\sqrt{|G|}}\right)^{2-2g}$$

= theory of as many points as ($\omega$-proj') irreps,

each with dilaton = $\ln(\dim R/\sqrt{|G|})$

Works!

More generally, all 2d unitary TQFTs decompose....

# (1+1)d unitary topological & near-topological field theories

These are all the same as (decompose into) disjoint unions of invertible field theories (= QFT(point) w/ dilaton shifts).

Formal reason: semisimplicity of the Frobenius algebra, which implies not only that projectors exist, but that all local operators are linear comb's of projectors.

Ex: 2d Dijkgraaf-Witten

$$\text{2d DW} = [\text{point}/G]_\omega = \coprod_R \text{point} \quad \text{(with dilaton shifts)}$$

Ex: Abelian $BF$ at level $k$ (Hellerman, ES, 1012.5999)

Ex: $G/G$ model (Komargodski et al 2008.07567)

Ex: 2d pure Yang-Mills (Nguyen, Tanizaki, Unsal 2104.01824)

Wilson lines = defects joining universes

$$\text{All cases: (1+1)d unitary TQFT} = \coprod_R \text{Inv}(\ln(\dim R)) \text{ (in top' limit)}$$

# (1+1)d unitary topological & near-topological field theories

Ex: Abelian $BF$ at level $k$ (Hellerman, ES 1012.5999)

U(1) gauge theory, Action: $S = k \int BF_{z\bar{z}}$ $\quad B \sim B + 2\pi$ a scalar

$F$ the gauge field curvature

Local operators: $\mathcal{O}_p = : \exp(ipB(x)) :$, independent of $x$ $\quad p \sim p \mod k$

Wilson lines: $W_q = : \exp\left( iq \oint A \right) :$ $\quad q \sim q \mod k$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q O_p$ $\quad \xi = \exp(2\pi i/k)$

Projectors: $\Pi_m = \frac{1}{k} \sum_{n=0}^{k-1} \xi^{nm} \mathcal{O}_n$ $\quad$ which obey $\Pi_m \Pi_n = \delta_{mn} \Pi_m, \quad \sum_m \Pi_m = 1$

# (1+1)d unitary topological & near-topological field theories

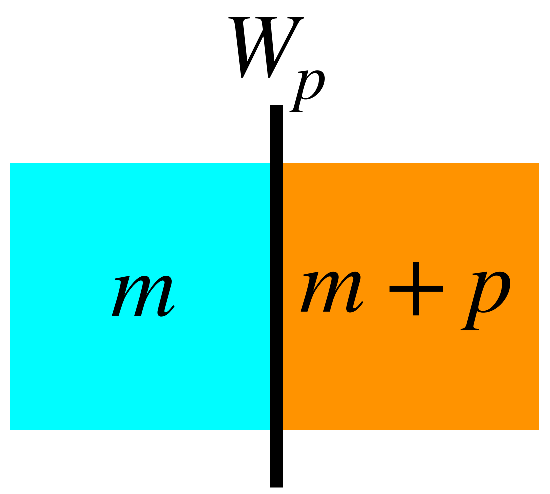
Ex: Abelian $BF$ at level $k$ (Hellerman, ES 1012.5999)

Clock-shift commutation relations: $\mathcal{O}_p W_q \; = \; \xi^{pq} \, W_q O_p$ $\qquad \xi = \exp(2\pi i / k)$

Projectors: $\quad \Pi_m \; = \; \frac{1}{k} \sum_{n=0}^{k-1} \xi^{nm} \mathcal{O}_n$ $\qquad$ which obey $\Pi_m \Pi_n \; = \; \delta_{mn} \Pi_m, \quad \sum_m \Pi_m \; = \; 1$

The clock-shift commutation relations imply

$$\Pi_m \, W_p \; = \; W_p \, \Pi_{m+p \mod k}$$




Interpretation: *The Wilson lines act as defects connecting different universes.*

That's a general feature of decomposition.

(1+1)d unitary topological & near-topological field theories

These are all the same as (decompose into) disjoint unions of invertible field theories
(= QFT(point) w/ dilaton shifts).

Formal reason: semisimplicity of the Frobenius algebra,
which implies not only that projectors exist,
but that all local operators are linear comb's of projectors.

Another reason: they all possess (noninvertible) 1-form symmetries,
defined by their (topological) operators and their OPE algebra.

Hence, as theories in (1+1)-dimensions w/ 1-form symmetry,
they decompose.

Let's get back on track.

My goal today is to talk about anomaly resolution in 1+1 dimensions.

Decomposition will play a vital role in understanding how the anomalies are resolved.

Recall the idea of www is that given an anomalous (ill-defined) $[X/G]$, replace $G$ by a larger finite group $\Gamma$ obeying certain properties,

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

and add phases.

Because $\Gamma$ has a subgroup $K$ that acts trivially, orbifolds $[X/\Gamma]$ will decompose, into copies & covers of $[X/G]$.

However, just getting copies of $[X/G]$ won't help. We also need to add certain new phases, which I will describe next....

# New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup $K$ acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$.

It acts on twisted sector states by phases. Schematically:

$$gz \, \underset{h}{\blacksquare} \;=\; B(\pi(h), z) \left( g \, \underset{h}{\blacksquare} \right)$$

where

$z \in K \qquad g, h \in \Gamma$

$B \in H^1(G, H^1(K, U(1)))$

## New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup $K$ acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \ = \ \mathrm{Hom}(G, \hat{K})$.

*Sometimes*, these quantum symmetries are equivalent to discrete torsion:

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

(Hochschild '77)

Specifically, $\beta(\omega) \in H^1(G, H^1(K, U(1)))$ for $\omega \in H^2(\Gamma, U(1))$ s.t. $\omega|_K = 0$.

Example: old-fashioned quantum symmetry in orbifolds

Start with $[X/\mathbb{Z}_n]$. Old story: This admits a $\mathbb{Z}_n$ symmetry that acts on twist fields, with the property that $\mathrm{QFT}([[X/\mathbb{Z}_n]/\mathbb{Z}_n]) \ = \ \mathrm{QFT}([X/\mathbb{Z}_n \times \mathbb{Z}_n]_B) = \mathrm{QFT}(X)$

However, the phases are determined by discrete torsion; $B = \beta(\omega)$

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup $K$ acts trivially.
Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

*Sometimes,* these quantum symmetries are equivalent to discrete torsion:

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

(Hochschild '77)

For purposes of resolving anomalies, we need $B \in H^1(G, H^1(K, U(1)))$ such that $d_2 B \neq 0$.

These cases are *not* determined by discrete torsion $\omega \in H^2(\Gamma, U(1))$.

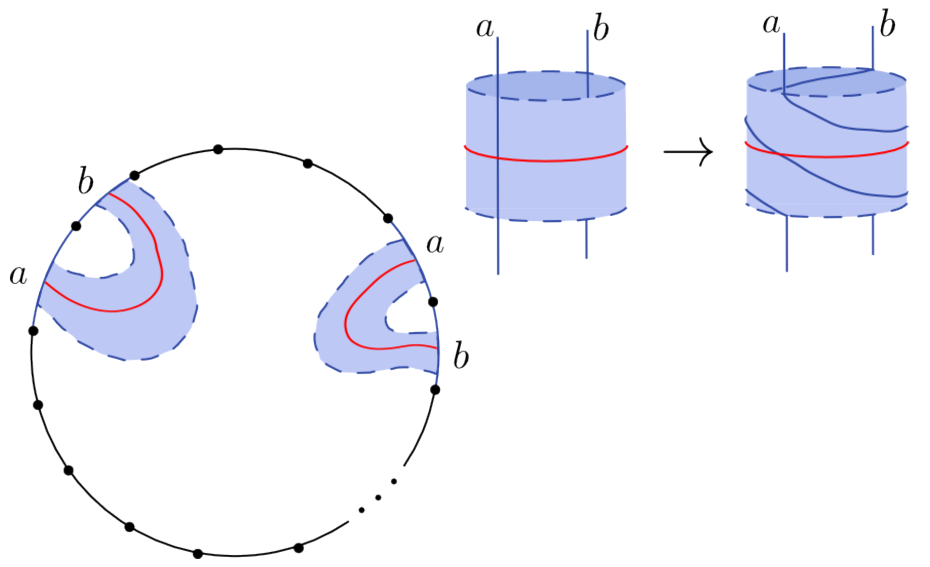
They're also of independent interest, beyond anomaly resolution.

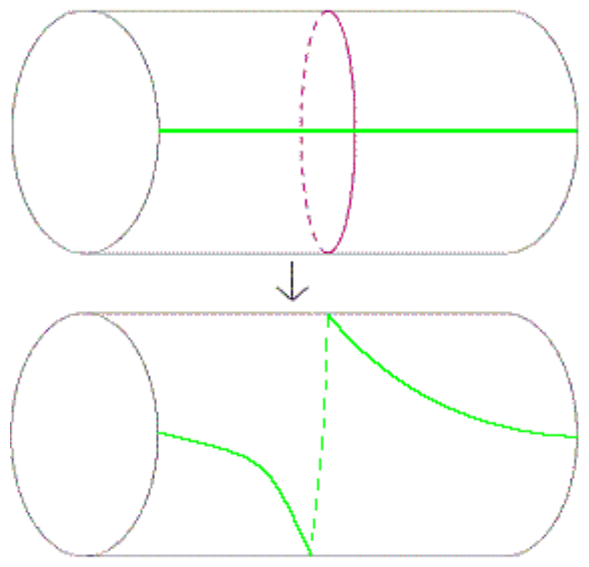
# New modular invariant phases: quantum symmetries

These are modular invariant — analogous to (but different from) discrete torsion.

How does that work?

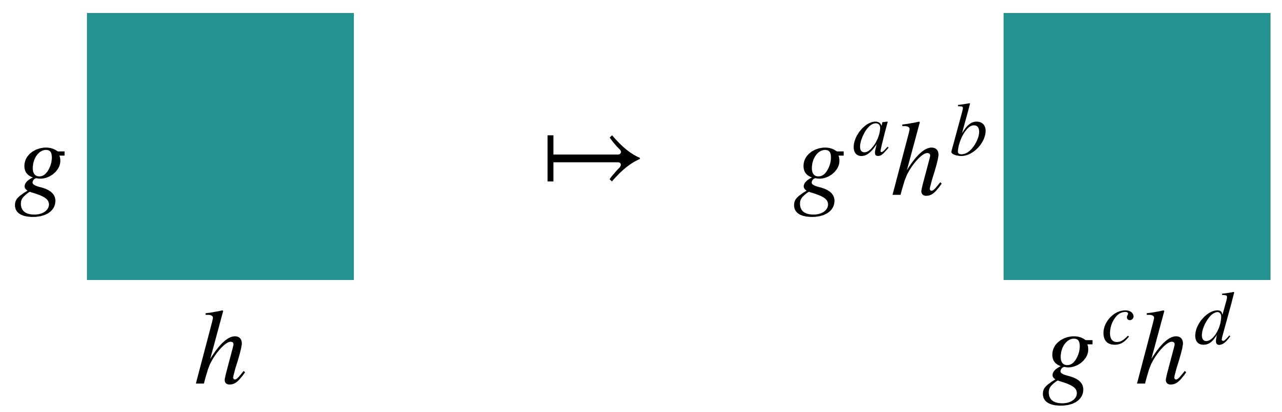
Work on $T^2$. Geometrically, this admits `Dehn twists'





which on $T^2$ are classified by elements of $SL(2,\mathbb{Z})$.

Under such a twist,



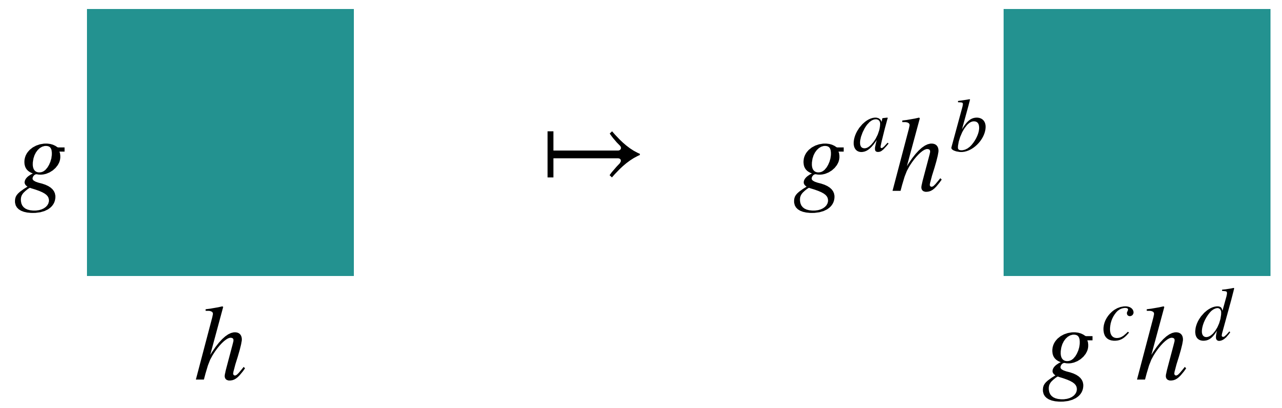

for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$

# New modular invariant phases: quantum symmetries

These are modular invariant — analogous to (but different from) discrete torsion.

Work on $T^2$. Geometrically, this admits `Dehn twists'

Under such a twist,




for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$

Discrete torsion: $\epsilon(g^a h^b, g^c h^d) \;=\; \epsilon(g,h)$

Quantum symmetry: $\displaystyle\sum_{k_1,k_2\in K} \epsilon(g^a k_1^a h^b k_2^b, g^c k_1^c h^d k_2^d) \;=\; \sum_{k_1,k_2\in K} \epsilon(g k_1, h k_2)$

How does decomposition work with such phases?....

# Decomposition in the presence of a quantum symmetry

Basic case:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

$$\text{where } B \;\in\; H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$$

This is more or less uniquely determined by consistency with previous results.

Recall for discrete torsion $\omega \in \mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))$, with $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right)$$

The result at top needs to include this as a special case, and it does.

# Decomposition in the presence of a quantum symmetry

Basic case:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B}\right]_{\hat{\omega}}\right)$$

Example: $\Gamma = \mathbb{Z}_4, \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Pick nontrivial $B \in H^1(G, H^1(K, U(1))) = H^1(\mathbb{Z}_2, \hat{\mathbb{Z}}_2) = \mathbb{Z}_2.$

$\mathrm{Ker}\, B = 0, \;\; \mathrm{Coker}\, B = 0$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\,(X)$

Check in partition function....

# Decomposition in the presence of a quantum symmetry

Basic case:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X\times\widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

Example: $\Gamma = \mathbb{Z}_4, \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$

Predict: $\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\,(X)$

Check $T^2$ partition function:

$$Z_{ij} \;=\; (-)^i Z_{i,j-2} \;=\; (-)^j Z_{i-2,j}$$

$$Z\left([X/\mathbb{Z}_4]_B\right) \;=\; \frac{1}{|\mathbb{Z}_4|}\sum_{i,j=0}^{4} Z_{ij} \;=\; \frac{1}{4}\left(Z_{00}+Z_{02}+Z_{20}+Z_{22}\right) \;=\; Z_{00} \;=\; Z(X)$$

Works!

# Decomposition in the presence of a quantum symmetry

If there is also discrete torsion $\omega \in H^2(\Gamma, U(1))$:

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1$$

Assume for simplicity $\iota^* \omega = 0$.

$$\left( \mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1)) \right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))$$

Cases:

1) Suppose $\beta(\omega) \neq 0$:

$$\mathrm{QFT}\left( [X/\Gamma]_{B,\omega} \right) = \mathrm{QFT}\left( \left[ \frac{X \times \widehat{\mathrm{Coker}\,(B/\beta(\omega))}}{\mathrm{Ker}\,(B/\beta(\omega))} \right]_{\hat{\omega}} \right)$$

2) Suppose $\omega = \pi^* \overline{\omega}$, $\overline{\omega} \in H^2(G, U(1))$:

$$\mathrm{QFT}\left( [X/\Gamma]_{B,\omega} \right) = \mathrm{QFT}\left( \left[ \frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\overline{\omega} + \hat{\omega}} \right)$$

All checked in examples;
I'll spare you....

Now, finally, we have the tools to start applying to anomalies.

For the purposes of this talk, anomalies in a finite $G$ gauge theory in $(n+1)$ dimensions will be classified by $H^{n+2}(G, U(1))$.

We'll begin with a simple model of anomalies,
to hopefully help explain how group cohomology arises in this context,

then study how anomaly resolution in (1+1) dimensions
can be understood via decomposition.

## Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions

Suppose a (finite) group $G$ acts on the states of a QM system: For any $|\psi\rangle$, get $\rho(g)|\psi\rangle$.

For an honest group action, require $\rho(g)\rho(h) = \rho(gh)$

However, b/c we only care about states up to phases, we might instead have

$\rho(g)\rho(h) = \omega(g,h)\,\rho(gh)$ for some $\omega(g,h) \in U(1)$

Associativity $\Rightarrow$ $\omega(g_2, g_3)\,\omega(g_1, g_2 g_3) = \omega(g_1 g_2, g_3)\,\omega(g_1, g_2)$ (coclosed)

Multiply $\rho$ by phase $\epsilon(g) \Rightarrow$ $\omega(g,h) \mapsto \omega(g,h)\dfrac{\epsilon(gh)}{\epsilon(g)\epsilon(h)}$ (coboundaries)

Thus, the obstructions $\omega$ are classified by $H^2(G, U(1))$

Anomaly
in 0+1 dims

States are all in $\omega$-projective representations of $G$.

# Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions

So far, have obstruction to honest action of $G$ encoded in anomaly $\omega \in H^2(G, U(1))$

Fix: extend $G$ to larger group $\Gamma$ for which states are in an honest representation.

1) Pick extension $\Gamma$, $\quad 1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$

such that $\quad \pi^* \omega = 0 \in H^2(\Gamma, U(1))$

2) Describe action of $\Gamma$, by picking $A \in H^0(G, H^1(K, U(1)))$

such that $\quad A(s_1 s_2 s_{12}^{-1}) = \omega(g_1, g_2) \quad$ for $s_i = s(g_i)$, $s : G \to \Gamma$ a section.

Then, define $\quad \tilde{\rho}(s(g)k) \equiv A(k)\rho(g)$

and one can show that $\tilde{\rho}$ defines an honest representation of $\Gamma$.

Anomaly fixed!

# Application to anomalies

Warmup: quantum-mechanical analogue, 0+1 dimensions

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1) Pick extension $\Gamma$, $\quad 1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$

such that $\quad \pi^*\omega = 0 \in H^2(\Gamma, U(1))$

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such that $\quad A(s_1 s_2 s_{12}^{-1}) = \omega(g_1, g_2) \quad$ for $s_i = s(g_i)$, $s : G \to \Gamma$ a section.

That was just QM, but the same pattern applies in higher dimensions.
In 1+1 dimensions, we'll see how decomposition gives a very
explicit understanding of how anomaly resolution works.

# Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make $G$ bigger: replace $G$ by $\Gamma$, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (I'll assume central)

where $\Gamma$ is chosen so that $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

The idea is then to replace $[X/G]$ with $[X/\Gamma]$,
but, need to describe how $\Gamma$ acts on X.

If $K$ acts triv'ly on X, and we do nothing else,
then we have accomplished nothing:

decomposition $\Rightarrow$ $\mathrm{QFT}([X/\Gamma]) = \coprod_{\hat{K}} \mathrm{QFT}([X/G])$ — still anomalous

Fix by adding quantum symmetry....

# Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make $G$ bigger: replace $G$ by $\Gamma$, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (assumed central)

2) Turn on quantum symmetry $B \in H^1(G, H^1(K, U(1)))$
chosen so that $d_2 B = \alpha$. This implies $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

$K$ acts trivially on $X$, but nontrivially on twisted sector states via $B$

These two together — extension $\Gamma$ plus $B$ — resolve anomaly.

Decomposition explains how....

# Application to anomaly resolution

Procedure: replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$

where $d_2 B \; = \; \alpha \in H^3(G, U(1))$, the anomaly of the $G$ orbifold.

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \; = \; \mathrm{QFT}\left( \left[ \frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\hat{\omega}} \right)$$

— using earlier results for decomp' in orb' w/ quantum symmetry

Note that since $d_2 B = \alpha, \quad \alpha|_{\mathrm{Ker}\, B} = 0$

So, $\mathrm{Ker}\, B \subset G$ is automatically anomaly-free!

Summary: $[X/\Gamma]_B =$ copies of orbifold by anomaly-free subgroup.

**Example:** Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 1: Define $\Gamma = D_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry $B$ determined by image on $\{a, b\}$

Results:

| $B(a)$ | $B(b)$ | d_2(B) (anomaly) | w/o d.t. in D4 | w/ d.t. in D4 |
|---|---|---|---|---|
| 1 | 1 | — | $[X/G] \coprod [X/G]_{\rm dt}$ | $[X/\langle b \rangle]$ |
| -1 | 1 | — | $[X/\langle b \rangle]$ | $[X/G] \coprod [X/G]_{\rm dt}$ |
| 1 | -1 | $\langle b \rangle$ | $[X/\langle a \rangle]$ | $[X/\langle ab \rangle]$ |
| -1 | -1 | $\langle b \rangle$ | $[X/\langle ab \rangle]$ | $[X/\langle a \rangle]$ |

Get only anomaly-free subgroups, varying w/ $B$.

Works!

**Example:** Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 2: Define $\Gamma = \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry $B$ determined by image on $\{a, b\}$

Results:

| $B(a)$ | $B(b)$ | d_2(B) (anomaly) | Result |
|---|---|---|---|
| 1 | 1 | — | $[X/G] \coprod [X/G]_{\mathrm{dt}}$ |
| -1 | 1 | $\langle a \rangle, \langle ab \rangle$ | $[X/\langle b \rangle]$ |
| 1 | -1 | $\langle b \rangle, \langle ab \rangle$ | $[X/\langle a \rangle]$ |
| -1 | -1 | $\langle a \rangle, \langle b \rangle$ | $[X/\langle ab \rangle]$ |

Get only anomaly-free subgroups, varying w/ $B$.

Works!

**Example:** Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 3: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry $B$ determined by image on $\{a, b\}$

Results:

| $B(a)$ | $B(b)$ | **d_2(B) (anomaly)** | **w/o d.t. in Z2 x Z4** | **w/ d.t. in Z2 x Z4** |
|---|---|---|---|---|
| 1 | 1 | — | $[X/G] \coprod [X/G]$ | $[X/G]_{\mathrm{dt}} \coprod [X/G]_{\mathrm{dt}}$ |
| -1 | 1 | $\langle ab \rangle$ | $[X/\langle b \rangle]$ | $[X/\langle b \rangle]$ |
| 1 | -1 | $\langle b \rangle, \langle ab \rangle$ | $[X/\langle a \rangle]$ | $[X/\langle a \rangle]$ |
| -1 | -1 | $\langle b \rangle$ | $[X/\langle ab \rangle]$ | $[X/\langle ab \rangle]$ |

Get only anomaly-free subgroups, varying w/ $B$.

Works!

**Example:** Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

In the examples so far, we picked a `minimal' resolution $\Gamma$.

If we pick larger $K$, we get copies.

Extension 4: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Results:

| $B(a)$ | $B(b)$ | **d_2(B) (anomaly)** | **Result** |
|---|---|---|---|
| 1 | 1 | – | $\coprod_2 \left( [X/G] \coprod [X/G]_{\mathrm{dt}} \right)$ |
| -1 | 1 | $\langle a \rangle, \langle ab \rangle$ | $\coprod_2 [X/\langle b \rangle]$ |
| 1 | -1 | $\langle b \rangle, \langle ab \rangle$ | $\coprod_2 [X/\langle a \rangle]$ |
| -1 | -1 | $\langle a \rangle, \langle b \rangle$ | $\coprod_2 [X/\langle ab \rangle]$ |

Get copies of orb's w/ anomaly-free subgroups.

Works!


Future directions

# Boundaries in orbifolds with quantum symmetries

We saw earlier that in orbifolds $[X/\Gamma]$ with triv'ly acting $K \subset \Gamma$,
the boundaries are naturally associated to universes of decomposition:

the boundary carries a (possibly projective) action of $\Gamma$,
so restrict to $K$,
that action descends to a (possibly projective) representation of K,
which tells us which universe(s) the boundary is associated to.

That works fine in cases in which $[X/\Gamma]$ has discrete torsion,
just projectivize. But what about quantum symmetries?

Specifically, quantum symmetries $B$ with $d_2 B \neq 0$ ?

Boundaries in orbifolds with quantum symmetries

Specifically, quantum symmetries $B$ with $d_2 B \neq 0$ ?

In this case, the associativity of the $\Gamma$ action is broken,
albeit weakly — the action is `homotopy associative.'

In principle, this structure should be understood formally in terms of a groupoid quotient.

WIP w/ Tony Pantev to give a careful description.

# Application to anomaly resolution

What about in 2+1 dimensions?

Mathematically, we can follow the same procedure:

Given $[X/G]$ with anomaly $\alpha \in H^4(G, U(1))$,

1) Extend $G$ to $\Gamma$, $\quad 1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1,$

2) Pick phases $\quad C \in H^2(G, H^1(K, U(1)))$

such that $d_2 C = \alpha,$

which then implies $\pi^* \alpha = 0 \in H^4(\Gamma, U(1))$

However, we do *not* expect a decomposition — would need a 2-form symmetry.

WIP w/ D Robbins, T Vandermeulen to see if anything else can be said.

## Summary

Decomposition: \`one' QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories
with $n-$form symmetries.

(I've focused on examples in 1+1d,
but examples exist in other dim's too.)

Can be used to understand anomaly-resolution procedure of www:

replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$,
but decomposition implies
$\mathrm{QFT}\left([X/\Gamma]_B\right) = \text{copies of } \mathrm{QFT}\left([X/\mathrm{Ker}\, B \subset G]\right)$,
which is explicitly non-anomalous.

Thank you for your time !