Nonperturbative effects from classical physics:
An introduction to mirror symmetry

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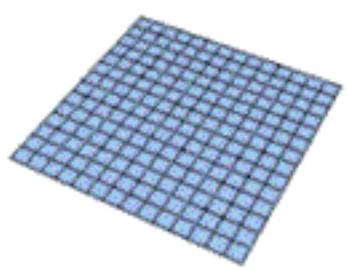
This will be a talk about string theory,
so let me discuss the motivation….

Twentieth-century physics saw two foundational advances:

General relativity
(special relativity)



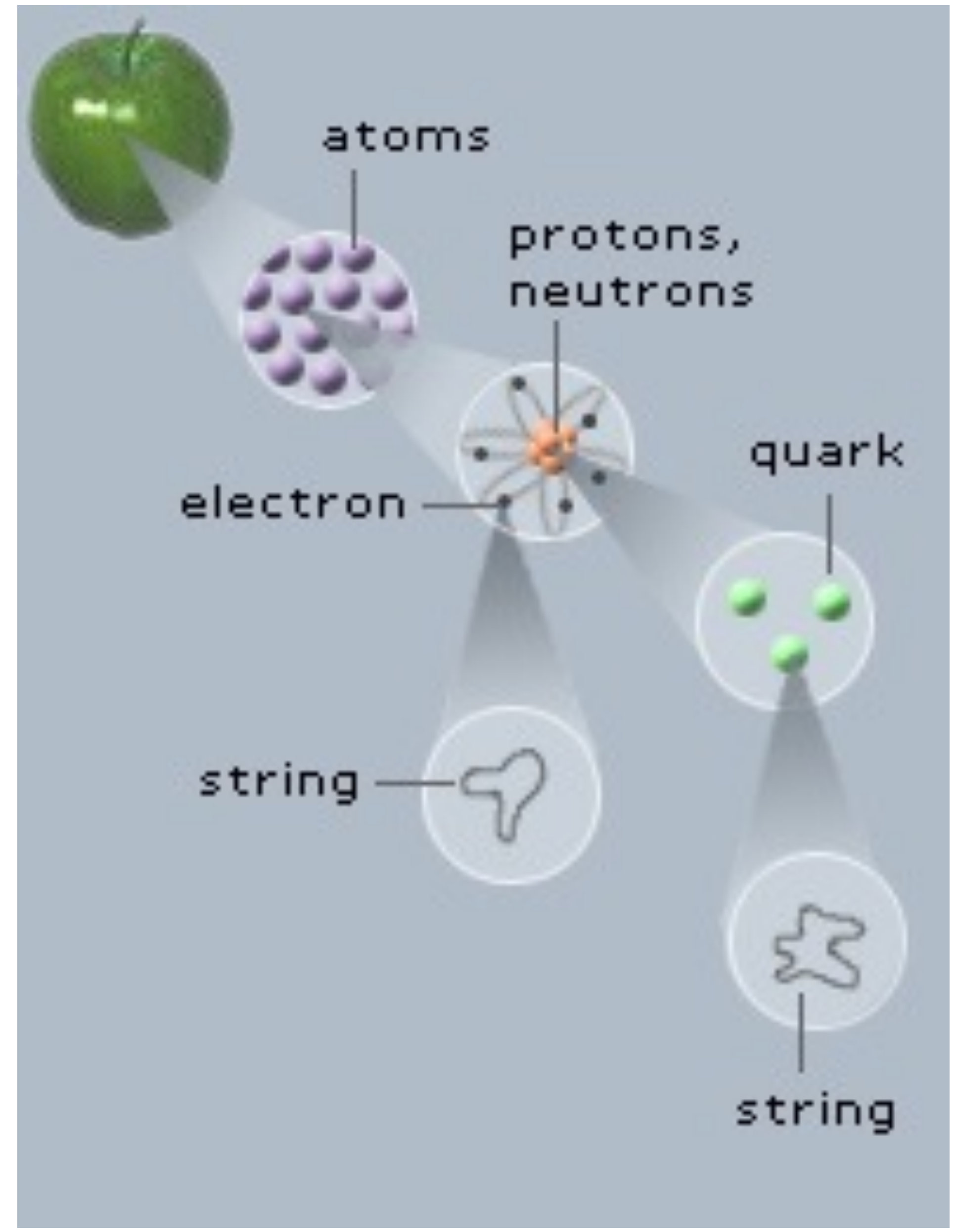
Quantum field theory
(quantum mechanics)

Problem: They contradict each other!
(in the sense that GR is only a low energy effective theory, useless below a certain scale)

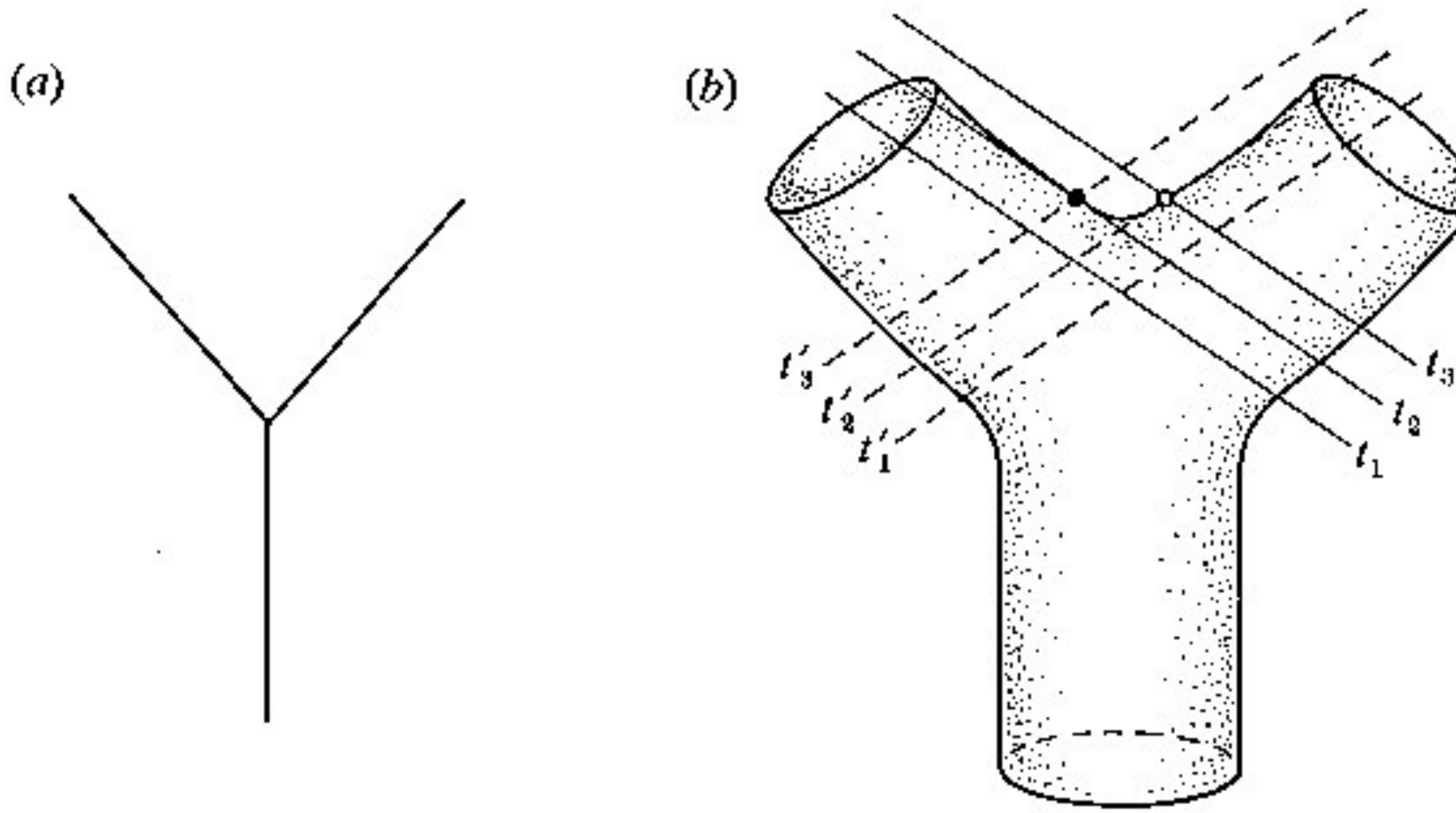
Something else is needed….

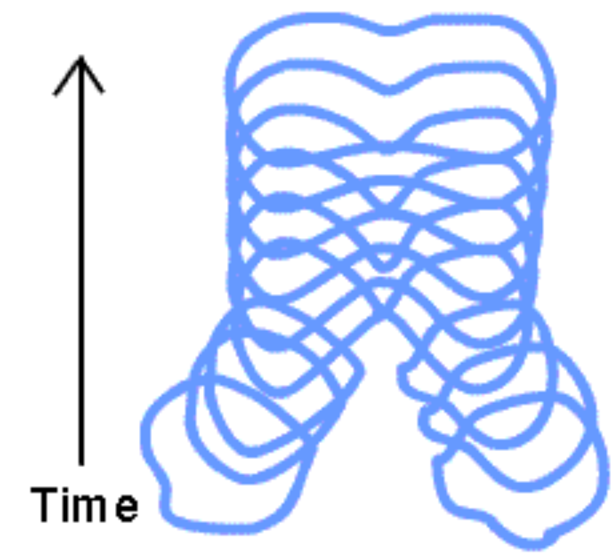
String theory…

… is a physical theory that reconciles GR & QFT, by replacing elementary particles by strings.



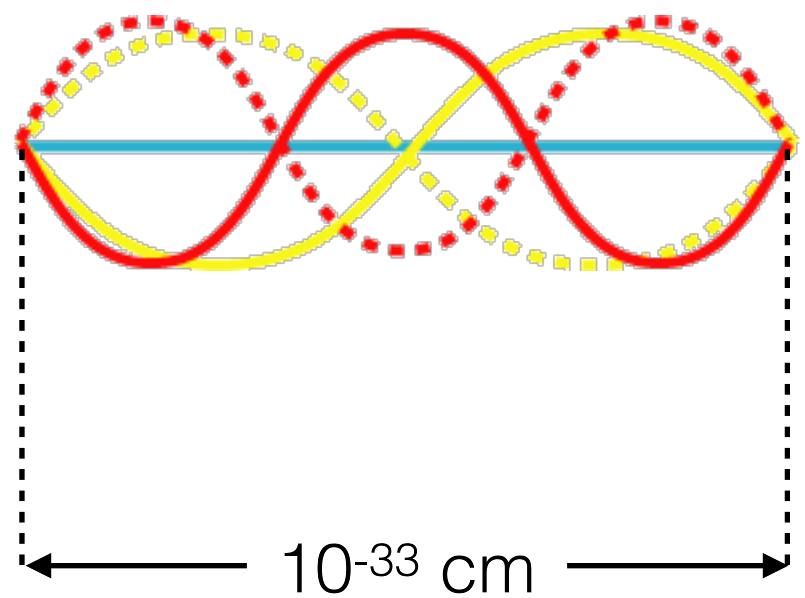
We fatten Feynman diagrams,
which removes QFT-like divergences.





In QFT, those divergences imply
scale-dependence of physics.

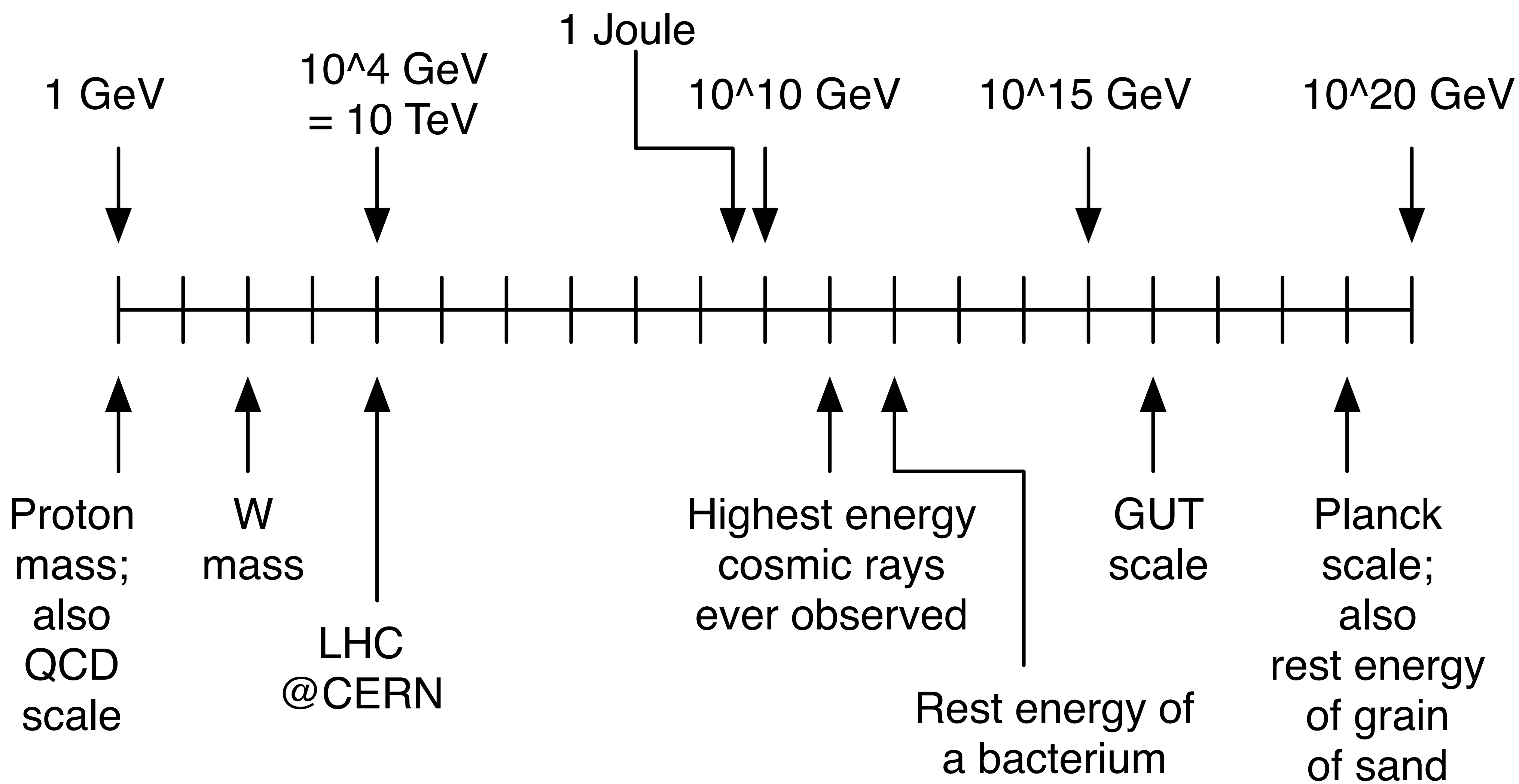
Do not expect scale-dependence in a fundamental theory,
hence removing divergences is desirable.



The typical sizes of the strings are very small — of order the Planck length. To everyday observers, the string appears to be a pointlike object.

From dim'l analysis, expected energy scale for strings is
Planck energy = $(h c^5 / G)^{1/2} \sim 10^{19}$ GeV

How big is that?



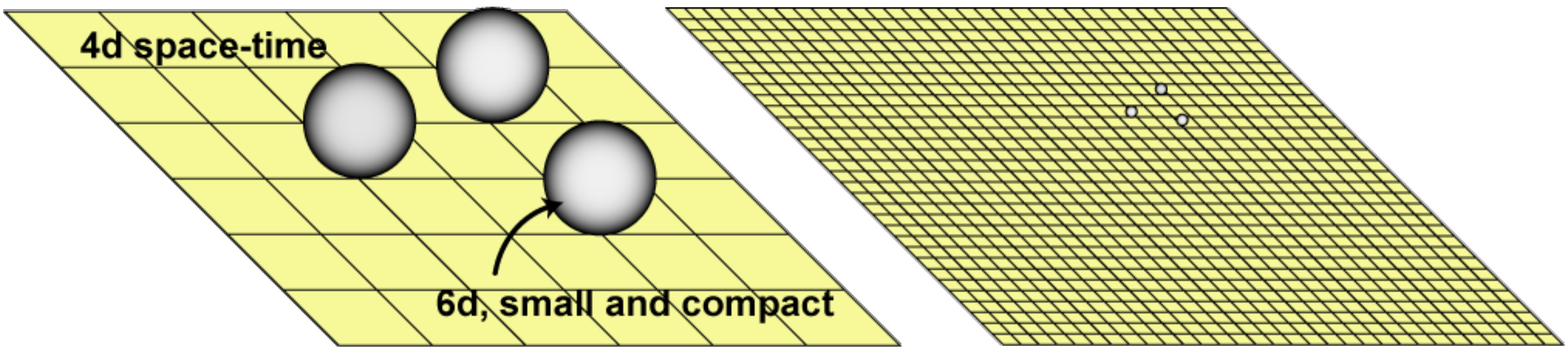
Perturbative (critical) string theory is consistent in 10 dimensions.

Yet, the real world is 4 dimensional (3 space, 1 time).

So, how can string theory describe the real world?

Compactification scenario

Assume 10d spacetime = $\mathbf{R}^4$ x M,
where M is some (small) (compact) 6d space.

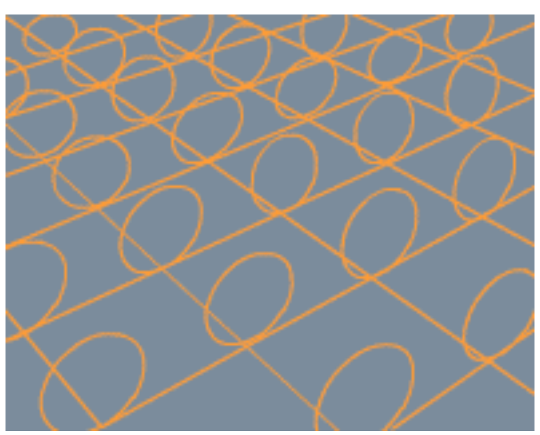


So long as you work at wavelengths much larger than the size of the compact space, you can't see the extra dimensions; spacetime looks like $\mathbf{R}^4$.

Kaluza-Klein theory

This picture of compactifying extra dimensions and getting extra fields in the 4d theory was historically first invented by Kaluza and Klein back in the 1920s.

They proposed a unification of gravity and electromagnetism, viewed as pure gravity in 5d, on $\mathbf{R}^4 \times S^1$.



$$g^{(5)}_{mn} = \begin{bmatrix} g^{(4)}_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix}$$

5d metric

4d metric

U(1) gauge field

scalar field

String compactifications are a natural generalization.

What sort of 6-dim’l space can you compactify on?

- Needs to satisfy Einstein’s equations for general relativity in vacuum

- To get a `supersymmetric’ low-energy four-dim’l effective theory, need some add’l properties

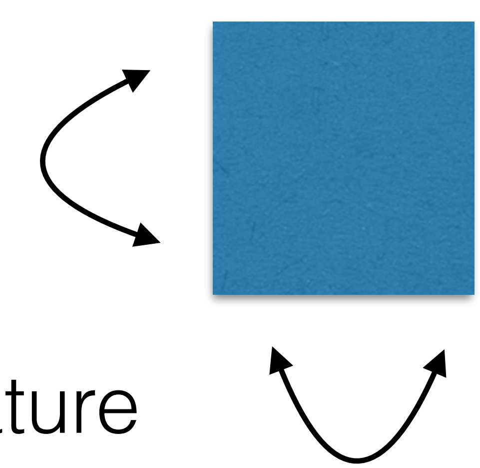
Result, in simplest cases, is that it must be Ricci-flat
plus a few other things;
known as a “**Calabi-Yau**” manifold

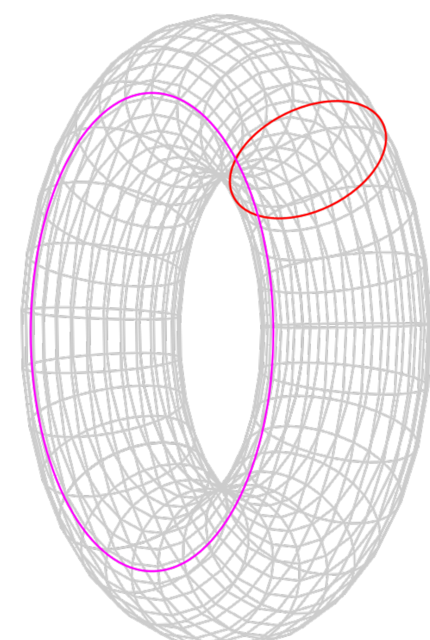
What's a Calabi-Yau ?

Technically: complex Kahler manifold with trivial canonical bundle

= a space that (among other things) satisfies Einstein's equations for a vacuum (meaning, it's Ricci-flat).

Example: T^2





no *intrinsic* curvature

But that's pretty boring;
more complicated examples also exist, in higher dimensions.

What difference can this make?

If that 6d space is too small to be observed,
what impact can it have on observable 4d physics?

Although you can't see the internal 6d space directly, the geometry of that space determines properties of the low-energy 4d theory.

Example: How to count massless spinors in 4d

Recall Dirac equ'n for spinors of mass m:

$$(i\not{D} - m)\psi = 0$$

Start with 10d massless spinors: $\not{D}_{10}\psi = 0$

We can decompose the 10d Dirac operator into

$$\not{D}_{10} = \not{D}_6 + \not{D}_4$$

and so we get 4d massless fermions from sol'ns of

$$\not{D}_6\psi = 0$$

The solutions of $\not{D}_6 \psi = 0$

(which determine 4d massless fermions)

can be characterized in terms of mathematical invariants of the 6-dim'l space, known as "cohomology groups"

For example, on a Calabi-Yau, there are numbers $h^{p,q}$ = dim's of certain (Dolbeault) cohomology groups.

Math: these are groups of closed complex differential forms

$$\omega_{a_1 \cdots a_p \overline{a}_1 \cdots \overline{a}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\overline{z}^{\overline{a}_1} \wedge \cdots \wedge d\overline{z}^{\overline{a}_q}$$

(mod exact forms).

In compactifications of type II strings, these count 4d fermions with charges p, q under a pair of U(1) symmetries.

More generally, not just the number of particles but also their couplings, etc, are determined by the geometry of the internal 6d space.

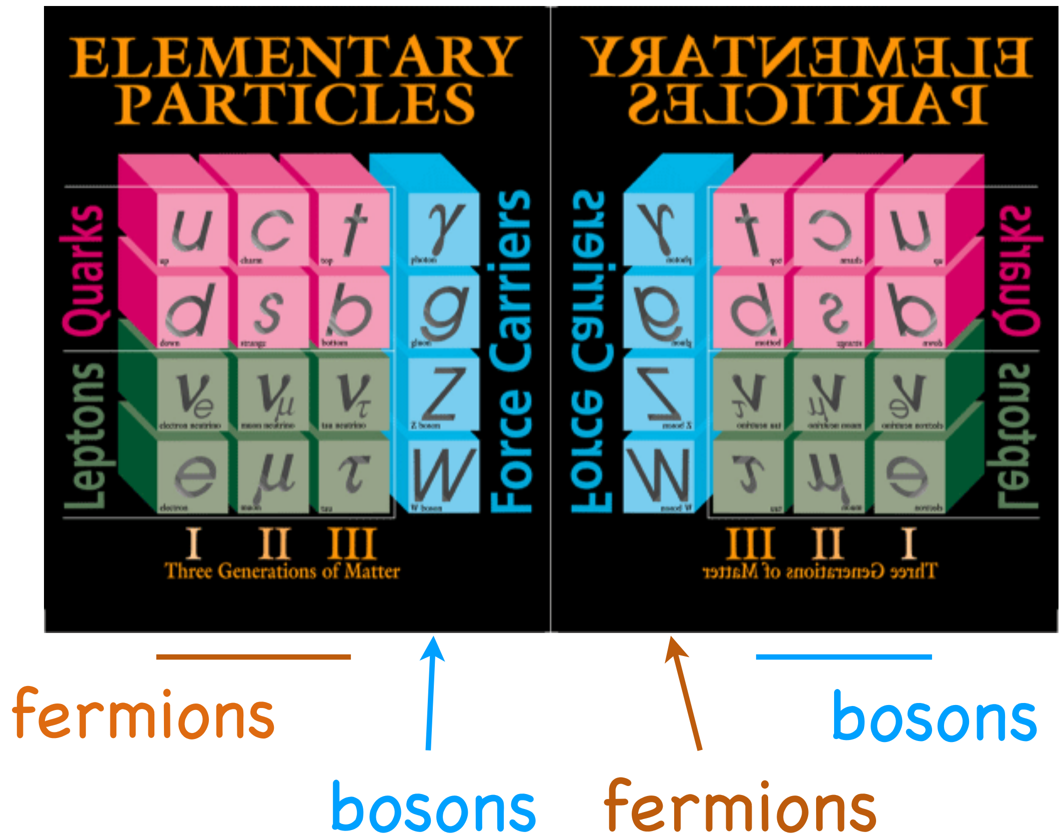
In short, learn about 4d physics by studying mathematical structure of the 6-manifold.

I've just told you why math is interesting to physicists,
but the reverse has also turned out to be true:

Thinking about the resulting physics has also led to new math
— nowadays, progress in each field stimulates the other,
and in so doing,
provides new families of self-consistency tests.

Before going on to talk about mirror symmetry, the focus of today's colloquium, I'll introduce the notion of `supersymmetry,' which will play a role today.

I'm going to discuss quantum field theories with **supersymmetry**.



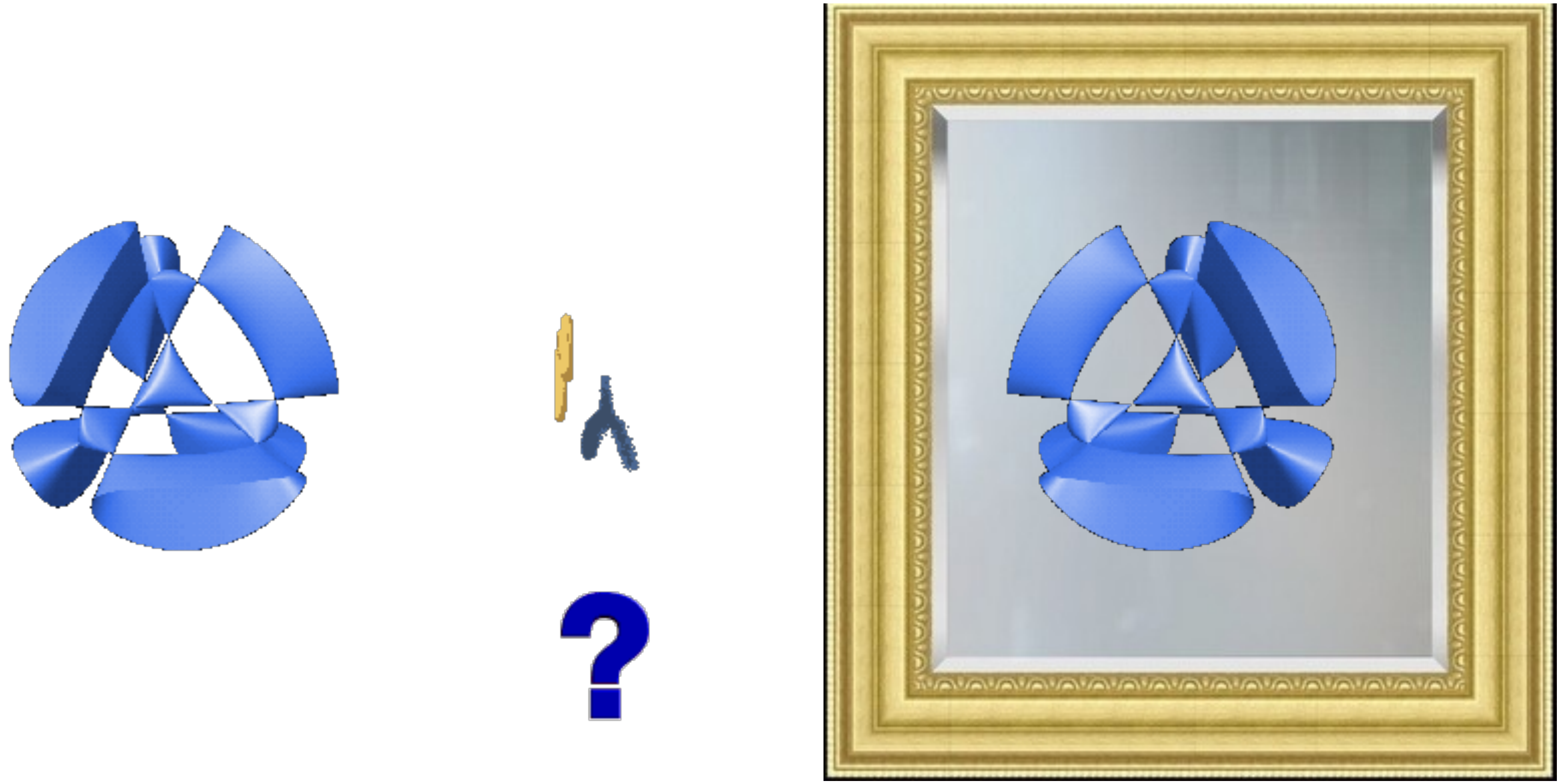
This is a symmetry between bosons and fermions; each boson has a fermionic partner with the same properties.

Supersymmetry simplifies QFTs:
many Feynman diagrams with loops cancel out,
making quantum corrections manageable.

Outline

- **Overview of mirror symmetry and curve-counting**
  - Mirrors to (susy) gauge theories & quantum cohomology
  - `Heterotic' generalizations: versions with less susy

Mirror symmetry



= a duality between 2d QFT's,
first worked out in early 1990s

Pairs of (usually topologically distinct) Calabi-Yau manifolds are described by **same** string theory — strings cannot distinguish.

Mirror symmetry

When two Calabi-Yau manifolds M, W are mirror,
they turn out to be very closely related
(but usually topologically distinct).

Example: dim M = dim W

After all, if strings are unable to distinguish one from the other,
then the compactified theory should be the same
— in particular, the dimension of the compactified theory had
better not change.

Mirror symmetry

Since the spectrum of light four-dimensional particles is determined by (Dolbeault) cohomology, we can conclude that

$$\sum \dim H^{*,*}(M) = \sum \dim H^{*,*}(W)$$

— total number of 4d particles should be unchanged.

Mirror symmetry

Calabi-Yau spaces are (incompletely) characterized by $h^{p,q}$'s (= dim's of Dolbeault cohomology groups), which compute the number of massless particles.

For example, for a 4-dim'l space, these are

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

"Hodge diamond"

Mirror symmetry acts as a rotation about the diagonal: if X is mirror to Y, then $h^{p,q}(X) = h^{n-p,q}(Y)$.

Example: T^2

T^2 is self-mirror topologically.

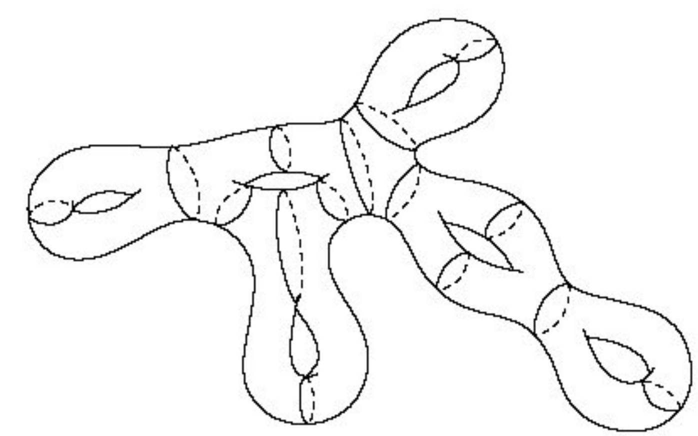
`Diamond' of $h^{p,q}$'s:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

— symmetric under rotation

This symmetry is specific to 2d manifolds with 1 handle;
for g handles:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$



Example: K3 manifolds

(Calabi-Yau of real dim 4)

K3 is self-mirror topologically;
complex, Kahler structures interchanged

$h^{1,1}$ ↗ (complex) $h^{1,1}$ ↖ (Kahler)

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Kummer surface

$$(x^2+y^2+z^2-aw^2)^2-\left(\tfrac{3a-1}{3-a}\right)pqts=0$$
$$p=w-z-\sqrt{2}x$$
$$q=w-z+\sqrt{2}x$$
$$t=w+z+\sqrt{2}y$$
$$s=w+z-\sqrt{2}y$$
$$a=1.5$$
$$w=1$$

Example: Quintic

The "quintic" (deg 5 hypersurface in $\mathbb{P}^4$) is a nontrivial Calabi-Yau 6-manifold.

Quintic

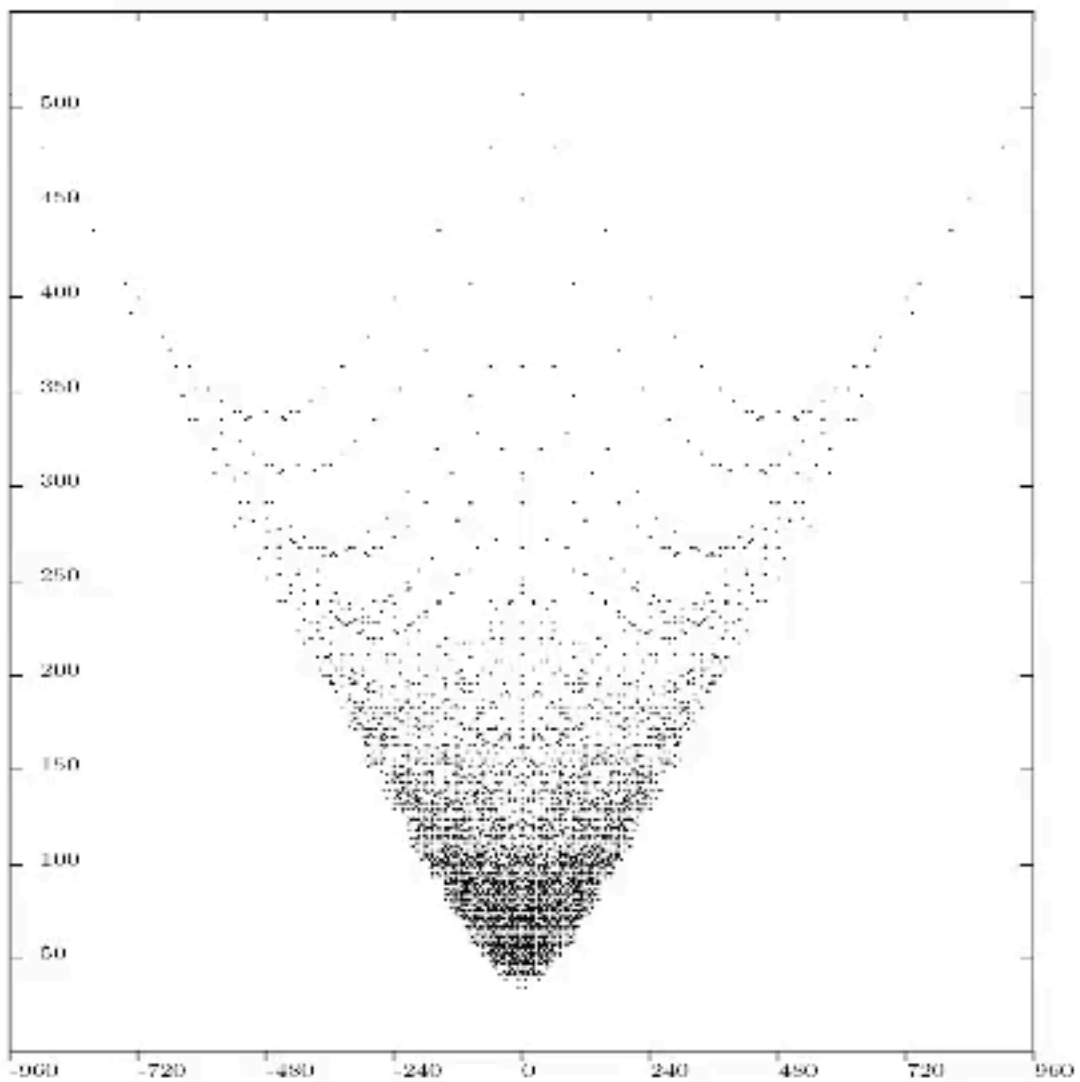
			1			
		0		0		
	0		1		0	
1		101		101		1
	0		1		0	
		0		0		
			1			

Mirror

			1			
		0		0		
	0		101		0	
1		1		1		1
	0		101		0	
		0		0		
			1			

Numerical checks of mirror symmetry

Plotted below are data for a large number of Calabi-Yau manifolds.



Vertical axis: $h^{1,1} + h^{2,1}$

Horizontal axis: $2(h^{1,1} - h^{2,1})$
= 2 (# Kahler - # cpx def's)

Mirror symmetry exchanges $h^{1,1} \longleftrightarrow h^{2,1}$

==> symm' across vert' axis

(Klemm, Schimmrigk, NPB 411 ('94) 559-583)

Mirror symmetry between spaces M ⟷ W exchanges:

classical computations on M

(perturbative in 2d QFT)

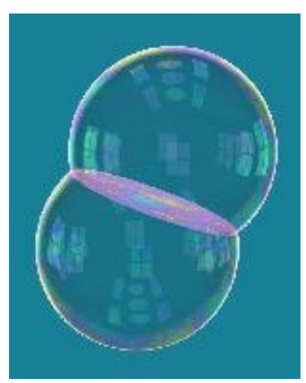
(Feynman diagrams)

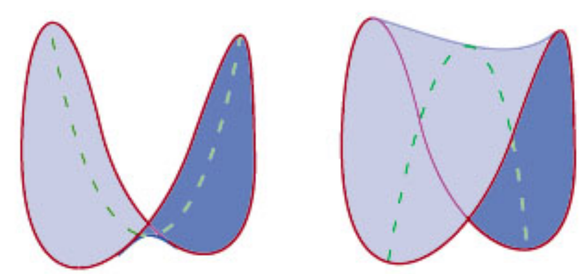
⟷

sums over minimal area (2d) surfaces on W

(nonperturbative in 2d QFT)

(2d instantons)





A curve and two possible saddle-shaped film surfaces

The fact that this duality exchanges (easy) perturbative effects & (hard) nonperturbative effects makes it very useful for computations!

Degree k	n_k	
1	2875	(Schubert, 1877)
2	609250	(Katz, 1986)
3	317206375	(Ellingsrud, Stromme, 1991)

Shown: numbers of minimal area S^2's in one particular Calabi-Yau (the "quintic"), of fixed degree.

These three degrees were the state-of-the-art in mathematics before mirror symmetry
(deg' 2 in '86, deg' 3 in '91)

Then, b/c of physics, mirror symmetry ~ '92….

Degree k	n_k	
1	2875	(Schubert, 1877)
2	609250	(Katz, 1986)
3	317206375	(Ellingsrud, Stromme, 1991)
4	24246753000	
5	229305888887625	
6	248249742118022000	
7	295091050570845659250	
8	375632160937476603550000	
9	503840510416985243645106250	
10	704288164978454686113488249750	
...	...	

Mirror symmetry

(Candelas et al, 1991, …)

This launched an army of algebraic geometers…
became 'Gromov-Witten theory.'

Mirror symmetry gave rise to an important example of math/ physics interactions, but there are many more.

Example: Seiberg-Witten theory in 4d susy gauge theories

— describes IR behavior of strongly-coupled gauge theory

— also revolutionized `Donaldson invariants' of 4-manifolds in math

Example: Jones knot polynomials as Wilson loops in Chern-Simons theories

Revolutions are rare, but nowadays there are many routine interactions, in which mathematicians and physicists work together to learn something of interest to both.

Ex: F theory (J Halverson, J M Esole, C Beasley, ….); also, B Nelson, P Nath, T Taylor….

More globally, such predictions for mathematics form important self-consistency tests of string theory.

We can't yet experimentally test string theory to determine if it is the quantum theory of gravity present in nature,
but,
we *can* and do frequently perform self-consistency tests, via mathematical predictions.

Outline

- Overview of mirror symmetry and curve-counting
- **Mirrors to (susy) gauge theories & quantum cohomology**
- `Heterotic' generalizations: versions with less susy

Versions of mirror symmetry exist for susy gauge theories.

Ex in 4d: Seiberg duality between susy gauge theories

(Technically, a `duality' rather than a `mirror.') (Seiberg, '94)

Prototype:

$SU(N_c)$ gauge theory with N_f flavors

dual or `mirror' to

$SU(N_f-N_c)$ gauge theory with N_f flavors plus meson field + potential

weak couplings <—> strong coupling

elementary fields <—> composite operators

Today we'll focus on lower dimensions….

Versions of mirror symmetry exist for susy gauge theories.

Ex in 3d: 3d mirror symmetry

(Intriligator, Seiberg, '96)

Prototype:

$U(1)^{n-1}$ gauge theory with n electrons

mirror to

U(1) gauge theory with n electrons

Higgs branch <—> Coulomb branch

Closely related to `symplectic duality' in mathematics.

Today we'll focus on lower dimensions….

Mirror symmetry for 2d gauge theories

Mirrors in 3d, 4d are known.

In 2d, there are known examples of dual theories,
but,
systematic construction of mirror to 2d nonabelian theory was an open problem.

Today: proposed construction of 2d nonabelian mirrors

Mirror symmetry for 2d gauge theories

Mirror to a gauge theory is a "Landau-Ginzburg model," a supersymmetric 2d Klein-Gordon theory w/ potential.

$$L = \sum_i \left(-|\partial_t \phi^i|^2 + |\partial_x \phi^i|^2\right) + \sum_i |\partial_i W(\phi)|^2$$

2d Klein-Gordon

potential

$$+ \sum_i \overline{\psi}^i \partial_\mu \gamma^\mu \psi^i + \psi_+^i \psi_-^j \partial_i \partial_j W(\phi) + \text{c.c.}$$

fermions

— determined by holomorphic function $W(\phi)$, called "superpotential"

LG model determined by fields + superpotential.

Proposed mirrors of 2d nonabelian gauge theories:

Consider a 2d susy G-gauge theory, with matter in representation ρ of dim n.

Mirror is a (Weyl-group orbifold of a) Landau-Ginzburg model:

(W Gu, ES, '18)

Fields $\sigma_a, \quad a = 1, \cdots, \operatorname{rank} G$

$Y_i, \quad i = 1, \cdots, n = \dim \rho$

$X_{\tilde{\mu}} \leftrightarrow$ nonzero roots of $\mathfrak{g}$

Slogan:
abelian duality in a max torus

Superpotential

$$W = \sum_a \sigma_a \left(\sum_i \rho_i^a Y_i - \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) + \sum_i \exp(-Y_i) + \sum_{\tilde{\mu}} X_{\tilde{\mu}}$$

$\rho_i^a =$ weights of ρ

$\alpha_{\tilde{\mu}}^a =$ roots of $\mathfrak{g}$

Example: 2d susy CP^n model

2d susy U(1) gauge theory with n+1 complex scalars, each of charge 1.

Taking into account interactions, semiclassical Higgs moduli space = CP^n.

Pertinent part of OPE ring: $\mathbb{C}[x]/(x^{n+1} - q)$

x is an operator in the 2d theory, ~ 2-form on CP^n, q is a nonperturbative correction.

This part of OPE ring is called *quantum cohomology*, as the operators ~ differential forms, and the product on differential forms on CP^n is encoded in

$$\mathbb{C}[x]/(x^{n+1} - 0)$$

Example: 2d susy CP^n model

Compute mirror: Superpotential

$$W = \sum_a \sigma_a \left(\sum_i \rho_i^a Y_i - \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) + \sum_i \exp(-Y_i) + \sum_{\tilde{\mu}} X_{\tilde{\mu}}$$

For the CP^n model, only one U(1), and all fields of charge 1:

(Hori-Vafa, '00)

$$W = \sigma \left(\sum_{i=1}^{n+1} Y_i - t \right) + \sum_{i=1}^{n+1} \exp(-Y_i)$$

Integrate out σ to get constraint: $\sum_{i=1}^{n+1} Y_i = t$

Eliminate Y_{n+1}: $Y_{n+1} = t - \sum_{i=1}^{n} Y_i$ and plug in:

$$W = \sum_{i=1}^{n+1} \exp(-Y_i) = \sum_{i=1}^{n} \exp(-Y_i) + q \prod_{i=1}^{n} \exp(+Y_i)$$

Analyze….

Example: 2d susy CP^n model

We will see the nonperturbative OPE ring of the original theory, as a **classical** computation in the mirror theory.

In the susy Lagrangian, the potential is $V = |\nabla W|^2$

so vacua are field configurations for which $\nabla W = 0$

"critical locus"

Here:

$$W = \exp(-Y_1) + \cdots + \exp(-Y_n) + q \prod_{i=1}^{n} \exp(+Y_i)$$

hence

$$\frac{\partial W}{\partial Y_i} = -\exp(-Y_i) + q \prod_{j=1}^{n} \exp(+Y_j)$$

so if we define $\sigma = \exp(-Y_i)$

then vacua are $\sigma = q\sigma^{-n} \Leftrightarrow \underline{\sigma^{n+1} = q}$

Matches OPE !

$\mathbb{C}[x]/(x^{n+1} - q)$

Example: U(k) gauge theory with n fundamentals

(= Grassmannian G(k,n), generalizes CP^{n-1})

Original gauge theory:

Coulomb branch is S_k orbifold of $\sigma_1, \cdots, \sigma_k$

Vacua: $\sigma_a \neq \sigma_b$ for $a \neq b$ "excluded locus"

$(\sigma_a)^n = (-)^{k-1} q$ "quantum cohomology"

Mirror:

$$W = \sum_a \sigma_a \left(\sum_i Y_{ia} + \sum_{\nu \neq a} \ln \left(\frac{X_{a\nu}}{X_{\nu a}} \right) - t \right) + \sum_{ia} \exp(-Y_{ia}) + \sum_{\mu \neq \nu} X_{\mu\nu}$$

Operator mirror map:

$$X_{\mu\nu} \leftrightarrow \sigma_\nu - \sigma_\mu \qquad \exp(-Y_{ia}) \leftrightarrow \sigma_a$$

Example: U(k) gauge theory with n fundamentals

Original gauge theory:

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Operator mirror map: $X_{\mu\nu} \leftrightarrow \sigma_\nu - \sigma_\mu$

We can already see the mirror of the excluded locus condition: the superpotential has poles at $\{X_{\mu\nu} = 0\}$, which are mirror to the excluded locus.

We'll see the quantum cohomology rel'n is mirror to the classical critical locus….

Example: U(k) gauge theory with n fundamentals

Mirror:

$$W = \sum_a \sigma_a \left(\sum_i Y_{ia} + \sum_{\nu \neq a} \ln\left(\frac{X_{a\nu}}{X_{\nu a}}\right) - t \right) + \sum_{ia} \exp\left(-Y_{ia}\right) + \sum_{\mu\neq\nu} X_{\mu\nu}$$

Eliminate the σ_a:

$$W = \sum_{i=1}^{n-1} \sum_a \exp\left(-Y_{ia}\right) + \sum_{\mu\neq\nu} X_{\mu\nu} + \sum_a \Pi_a$$

where $$\Pi_a = q \left(\prod_{i=1}^{n-1} \exp\left(+Y_{ia}\right)\right) \left(\prod_{\nu\neq a} \frac{X_{a\nu}}{X_{\nu a}} \right)$$

Critical loci:

$$\{dW = 0\} = \{\exp(-Y_{ia}) = \Pi_a, \quad X_{\mu\nu} = \Pi_\nu - \Pi_\mu\}$$

and plugging into the def'n of Π_a, we find

$$(\Pi_a)^n = (-)^{k-1} q$$

Example: U(k) gauge theory with n fundamentals

Mirror:

$$W = \sum_{i=1}^{n-1} \sum_{a} \exp\left(-Y_{ia}\right) + \sum_{\mu \neq \nu} X_{\mu\nu} + \sum_{a} \Pi_a$$

Critical loci: $$\left(\Pi_a\right)^n = (-)^{k-1} q$$

The operator mirror map relates $\Pi_a \leftrightarrow \sigma_a$

so this critical locus condition can be identified with the quantum cohomology relation

$$\left(\sigma_a\right)^n = (-)^{k-1} q$$

of the original gauge theory.

Mirror classically realizes quantum effects in gauge theory.

(One can match more, but for sake of time, I'll go on….)

Another example, briefly: SO(2k) gauge theory

A 2d susy SO(2k) gauge theory with n vectors.

This gauge theory was studied in (Hori, 2011):

Excluded locus: $\sigma_a \neq \pm\sigma_b, \quad \sigma_a \neq \pm\tilde{m}_i$

Coulomb branch / quantum cohomology relation:

$$\prod_{i=1}^{n} (\sigma_a - \tilde{m}_i) \; = \; q \prod_{i=1}^{n} (-\sigma_a - \tilde{m}_i)$$

where $q = \pm 1$ "discrete theta angle"

Mirror….

Another example, briefly: SO(2k) gauge theory (cont'd)

Following the same methods discussed earlier, the mirror is described by the superpotential

$$
\begin{aligned}
W \;=\; & \sum_{i=1}^{n-1}\sum_{a=1}^{k} \exp\left(-Y_{i,2a}\right) \;+\; \sum_{i=1}^{n}\sum_{a=1}^{k} \exp\left(-Y_{i,2a-1}\right) \;+\; \sum_{\mu<\nu} X_{\mu\nu} \;+\; \sum_{a=1}^{k} \Pi_a \\
& - \sum_{i=1}^{n-1}\sum_{a=1}^{k} \tilde{m}_i Y_{i,2a} \;-\; \sum_{i=1}^{n}\sum_{a=1}^{k} \tilde{m}_i Y_{i,2a-1} \\
& - \tilde{m}_n \sum_{a=1}^{k}\left(- \sum_{i=1}^{n-1} Y_{i,2a} \;+\; \sum_{i=1}^{n} Y_{i,2a-1} \;+\; \sum_{\mu<2a-1} \ln\left(\frac{X_{\mu,2a}}{X_{\mu,2a-1}}\right) \;+\; \sum_{\mu>2a} \ln\left(\frac{X_{2a,\mu}}{X_{2a-1,\mu}}\right)\right)
\end{aligned}
$$

where

$$
\Pi_a \;=\; q\left(\prod_{i=1}^{n-1} \exp\left(+Y_{i,2a}\right)\right)\left(\prod_{i=1}^{n} \exp\left(-Y_{i,2a-1}\right)\right)\left(\prod_{\mu<2a-1} \frac{X_{\mu,2a}}{X_{\mu,2a-1}}\right)\left(\prod_{\mu>2a} \frac{X_{2a,\mu}}{X_{2a-1,\mu}}\right)
$$

Granted, it has a lot of terms, but it's *classical*.

Another example, briefly: SO(2k) gauge theory (cont'd)

Critical loci:

$$\left.\begin{array}{c}\exp(-Y_{i,2a}) = \Pi_a - \tilde{m}_i + \tilde{m}_n \\ \exp(-Y_{i,2a-1}) = -\Pi_a - \tilde{m}_i - \tilde{m}_n\end{array}\right\} \neq 0 \implies \Pi_a + \tilde{m}_n \neq \pm \tilde{m}_i$$

$$\left.\begin{array}{c}X_{2a,2b} = \Pi_a + \Pi_b + 2\tilde{m}_n \\ X_{2a-1,2b-1} = -\Pi_a - \Pi_b - 2\tilde{m}_n \\ X_{2a,2b-1} = \Pi_a - \Pi_b \\ X_{2a-1,2b} = -\Pi_a + \Pi_b\end{array}\right\} \neq 0 \implies \Pi_a + \tilde{m}_n \neq \pm\left(\Pi_b + \tilde{m}_n\right)$$

Compare excluded locus $\begin{array}{c}\sigma_a \neq \pm \tilde{m}_i, \\ \sigma_a \neq \pm \sigma_b\end{array}$

and can show that on the critical locus,

$$\prod_{i=1}^{n} (\Pi_a + \tilde{m}_n - \tilde{m}_i) = q \prod_{i=1}^{n} (-\Pi_a - \tilde{m}_n - \tilde{m}_i)$$

Compare Coulomb branch rel'n $\displaystyle\prod_{i=1}^{n} (\sigma_a - \tilde{m}_i) \; = \; q \prod_{i=1}^{n} (-\sigma_a - \tilde{m}_i)$

Outline

- Overview of mirror symmetry and curve-counting
- Mirrors to (susy) gauge theories & quantum cohomology

`Heterotic' generalizations: versions with less susy

Heterotic mirror symmetry

is a conjectured generalization involving `heterotic' strings.

Ordinary mirror symmetry involves `type II' strings
which are specified by space + metric in 10d.

Heterotic strings are specified by
space + metric + nonabelian gauge field in 10d.

Thus, heterotic mirror symmetry involves not just spaces,
but also nonabelian gauge fields.

Also: less susy — specifically, (0,2) instead of (2,2)

Heterotic mirror symmetry

is a generalization that exchanges pairs

$$(X_1, \mathcal{E}_1) \longleftrightarrow (X_2, \mathcal{E}_2)$$

where the X_i are Calabi-Yau manifolds
and the $\mathcal{E}_i$ are bundles / nonabelian gauge fields over X_i .

Constraints: for each $\mathcal{E}$, X,

$$[\mathrm{tr}\, F \wedge F] \;=\; [\mathrm{tr}\, R \wedge R] \;+\; d(\cdots)$$

equivalently: $\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX)$

If nonabelian gauge field = spinor connection,
then $F = R$ $(\mathcal{E} = TX)$ & so satisfied trivially.

Heterotic mirror symmetry

The (2d) quantum field theories defining heterotic strings, include those of other ("type II") string theories as special cases.

Hence, heterotic mirror symmetry ought to reduce to ordinary mirror symmetry in a special case, & that turns out to be when $\mathcal{E}_i \cong T X_i \;\; (F_i = R_i)$.

Heterotic mirror symmetry

Much as in ordinary mirror symmetry,
dimensions and ranks are closely constrained:

If $(X_1, \mathcal{E}_1)$ is mirror to $(X_2, \mathcal{E}_2)$,
then

$$\dim X_1 \ = \ \dim X_2$$

$$\mathrm{rank}\, \mathcal{E}_1 \ = \ \mathrm{rank}\, \mathcal{E}_2$$

Heterotic mirror symmetry

Here, massless particles are computed by different cohomology groups: $H^q(X, \wedge^p \mathcal{E}^*)$.

— "bundle-valued differential forms"

Heterotic mirror symmetry exchanges

$$H^q(X_1, \wedge^p \mathcal{E}_1^*) \leftrightarrow H^q(X_2, \wedge^{r-p} \mathcal{E}_2^*)$$

just as ordinary mirror symmetry exchanges

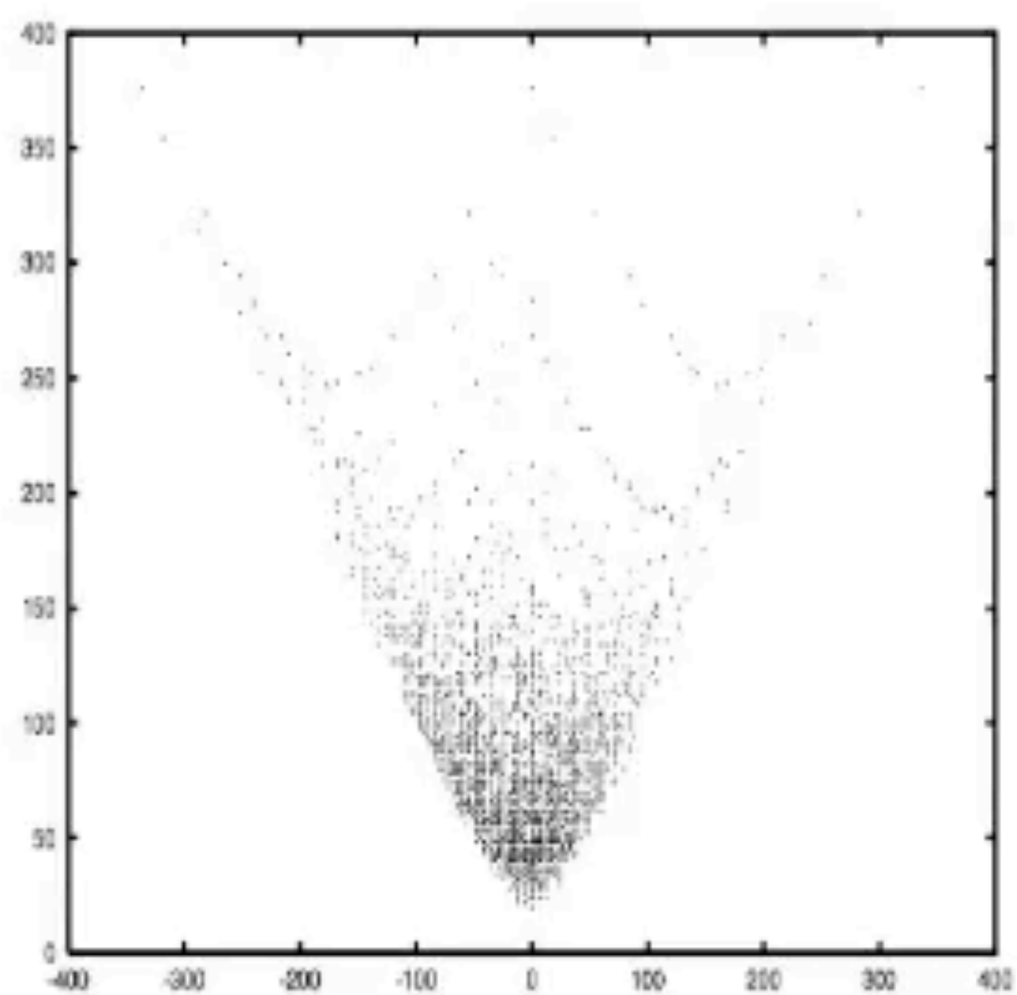
$$H^{p,q}(X_1) \leftrightarrow H^{n-p,q}(X_2)$$

When $\mathcal{E} = TX$, $H^{p,q}(X) = H^q(X, \wedge^p \mathcal{E}^*)$

& so we see ordinary mirrors as special cases.

Heterotic mirror symmetry

Less is known about the heterotic version,
though basics have been worked out.



(Blumenhagen, Schimmrigk, Wisskirchen,
NPB 486 ('97) 598-628)

Example: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

Heterotic mirror symmetry

Counting minimal area surfaces played
a crucial role in the original mirror symmetry,
and also arises in the heterotic version.

In the heterotic version, it's more complicated
(count minimal area surfaces + take into account the
nonabelian gauge field).

Result: *quantum sheaf cohomology*

generalizing quantum cohomology

(ES, Katz, Donagi, Distler, Melnikov, ….)

Currently known for toric varieties, Grassmannians, flag mflds,
with deformations of the tangent bundle.

Example of quantum sheaf cohomology:

$$\mathbb{P}^1 \times \mathbb{P}^1 = S^2 \times S^2$$

Nonabelian gauge field defined by `holomorphic bundle'

Bundle = deformation of tangent bundle, given as cokernel:

$$0 \longrightarrow \mathcal{O}^2 \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ Cy & Dy \end{array} \right]$$

Quantum sheaf cohomology ring

$$\mathbb{C}[\sigma, \tilde{\sigma}] / \left(\det(A\sigma + B\tilde{\sigma}) = q, \, \det(C\sigma + D\tilde{\sigma}) = \tilde{q} \right)$$

For tangent bundle, reduces to $\mathbb{C}[\sigma, \tilde{\sigma}] / \left(\sigma^2 = q, \, \tilde{\sigma}^2 = \tilde{q} \right)$

Can be derived from 2d gauge theory….

There exist 2d gauge theories with (0,2) susy,
half as much as in the theories discussed earlier.

Types of matter:

- Chiral multiplets (complex bosons + superpartners) Φ_i

- Fermi multiplets (chiral fermions + superpartners) Λ_a

$$\overline{D}_+\Lambda_a = E_a(\Phi) \qquad \text{(holomorphic)}$$

Mirrors to (0,2) susy gauge theories also exist.

These mirrors are “(0,2) susy Landau-Ginzburg models,” which are 2d QFTs defined by

- Chiral multiplets (complex bosons + superpartners) Φ_i

- Fermi multiplets (chiral fermions + superpartners) Λ_a

- Superpotential W, now Grassmann odd

Mirror to (0,2) gauge theory:

Fields:

σ_a, Υ_a corresponding to U(1)s in max torus

Y_i, F_i corresponding to weights of reprepresentation

$X_{\tilde{\mu}}, \Lambda_{\tilde{\mu}}$ corresponding to nonzero roots

Superpotential:

$$W \ = \ \sum_{a=1}^{\rm rank} \Upsilon_a \left(\sum_i \rho_i^a Y_i \ - \ \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} \ - \ t_a \right)$$

$$+ \ \sum_i F_i \left(E_i(\sigma) - \exp(-Y_i) \right) \ + \ \sum_{\tilde{\mu}} \Lambda_{\tilde{\mu}} \left(1 - \sum_{a=1}^{\rm rank} \sigma_a \frac{\alpha_{\tilde{\mu}}^a}{X_{\tilde{\mu}}} \right)$$

& can show this reproduces quantum sheaf cohomology, ….

Example:

For $\mathbb{P}^1 \times \mathbb{P}^1$, can simplify mirror to a LG model of the form

Fields: $\Phi, \tilde{\Phi}; F, \tilde{F}$ with superpotential

$$W = F\Phi^{-1}\left(\det(A\Phi + B\tilde{\Phi}) - q_1\right) + \tilde{F}\tilde{\Phi}^{-1}\left(\det(C\Phi + D\tilde{\Phi}) - q_2\right)$$

to reproduce quantum sheaf cohomology relations

$$\det(A\psi + B\tilde{\psi}) = q_1$$
$$\det(C\psi + D\tilde{\psi}) = q_2$$

as well as correlation functions etc

where A, B, C, D are 2x2 const' matrices encoding gauge field

— reproduces nonperturbative physics of original theory, via *classical* computation (critical locus) in mirror

Can also compute for toric varieties, Grassmannians, etc; result much more complicated, but, doable.

Main open problem in heterotic case:

We have worked out analogues of quantum cohomology rings
(quantum sheaf cohomology),
but,
no one has yet computed heterotic version of
Gromov-Witten invariants.

They arise as nonperturbative superpotential terms,
so we know they exist,
but not yet how to compute them.

Summary

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Thank you for your time!