Quantum sheaf cohomology

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C. Closset, W. Gu, B. Jia, ES, arXiv:1512.08058
J. Guo, V. Lu, ES, arXiv:1512.08586 & 1605.01410
Z. Chen, ES, R. Wu, arXiv: 1603.09634

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Let's review.

In 10d, a heterotic string describes metric & gauge field.

To compactify, must specify not only a space X,
but also a holomorphic vector bundle $\mathcal{E}$ on that space,
satisfying consistency conditions

$$[\mathrm{tr}\, F \wedge F] \ = \ [\mathrm{tr}\, R \wedge R] \qquad\qquad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$$

Described on worldsheet by 2d (0,2) susy theory.

Simplest case: $\mathcal{E} = TX$, corresponding to (2,2) susy.
"embed the spin connection in gauge connection"

Massless states/chiral primaries…

What are the massless states/chiral primaries?

(2,2) locus: NLSM, target X Kahler

Massless states are counted by ordinary cohomology

$$H^{\bullet,\bullet}(X)$$

plus moduli $= H^1(T) \oplus H^1(T^*) \oplus H^1(\mathrm{End}\, T)$

(0,2): X Kahler, $\mathcal{E} \to X$ hol' v.b.

Massless states are counted by sheaf cohomology

$$H^{\bullet}(X, \wedge^{\bullet}\mathcal{E}^*),\ H^{\bullet}(X, \wedge^{\bullet}\mathcal{E})$$

(Distler-Greene '88)

plus moduli $\subseteq H^1(T) \oplus H^1(T^*) \oplus H^1(\mathrm{End}\, \mathcal{E})$

Examples….

Examples of massless states in heterotic compactifications

Let X be Calabi-Yau 3-fold, for simplicity.

Let $\mathcal{E}$ denote the gauge bundle.

(Distler-Greene '88)

Rank 3 bundle: low-energy $E_6 \times E_8$

$\overline{\mathbf{27}} \sim H^1(X, \mathcal{E}^*)$ $\quad\quad$ $\mathbf{27} \sim H^1(X, \mathcal{E})$

(2,2): $\sim H^1(X, T^*X)$ Kähler $\quad\quad$ $\sim H^1(X, TX)$ complex

Rank 4 bundle: low-energy $\mathrm{Spin}(10) \times E_8$

$\mathbf{16} \sim H^1(X, \mathcal{E})$ $\quad\quad$ $\mathbf{10} \sim H^1(X, \wedge^2 \mathcal{E})$

Rank 5 bundle: low-energy $SU(5) \times E_8$

$\mathbf{10} \sim H^1(X, \mathcal{E})$ $\quad\quad$ $\overline{\mathbf{5}} \sim H^1(X, \wedge^2 \mathcal{E})$

What are the Yukawa couplings?

What are the Yukawa couplings?

Suppose bundle is rank 3, for simplicity,
so that we have low-energy $E_6 \times E_8$.

$$\overline{\mathbf{27}}^3 = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \mathcal{O}(q) \quad \text{where} \quad \omega_i \in H^1(X, \mathcal{E}^*)$$

No perturbative loop corrections, but there are nonperturbative corrections.
(Dine-Seiberg-Wen-Witten '86)

Example: (2,2) quintic

(Candelas, de la Ossa,
Green, Parkes, '91)

$$\overline{\mathbf{27}}^3 = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1-q^k} = 5 + 2875 q + 4876875\, q^2 + \cdots$$

intersection number
(Strominger '85)

nonperturbative
contributions

n_k = Gromov-Witten
invariants

The purpose of today's talk is to discuss (0,2) analogues.

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Schematically, Yukawa couplings have the form:

$$\overline{\mathbf{27}}^3 = \text{(classical cohomology product)} + \mathcal{O}(q)$$

(e.g. Blesneag, Buchbinder, Candelas, Lukas, 1512.05322)

nonperturbative contributions

where the classical cohomology product is of the form

$$\int \omega_1 \wedge \omega_2 \wedge \omega_3$$

How to compute nonperturbative corrections in (0,2) cases?

They're not merely Gromov-Witten invariants in general, so what to do?

How to compute nonperturbative corrections in (0,2) cases?

Historically, on the (2,2) locus,
used mirror symmetry.

For (0,2), would need a generalization called
(0,2) mirror symmetry.

Some results do exist — state of the art is a version of Batyrev's mirror map due to Melnikov-Plesser '10 — but we have not yet worked out analogue of flat coordinates or how to compute nonperturbative corrections using (0,2) mirrors alone.

We'll do this directly instead….

How to compute nonperturbative corrections in (0,2) cases?

It's convenient to work in an analogue of a TFT.

On (2,2) locus,

$$\overline{\mathbf{27}}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\mathrm{phys}} \;=\; \langle V^3 \rangle_{\mathrm{A\ TFT}}$$

$$\mathbf{27}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\mathrm{phys}} \;=\; \langle V^3 \rangle_{\mathrm{B\ TFT}}$$

& the TFT expressions are convenient for computations.

There are analogues for more general (0,2) theories;
these are A/2, B/2 pseudo-TFT's, which also have the property

$$\overline{\mathbf{27}}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\mathrm{phys}} \;=\; \langle V^3 \rangle_{\mathrm{A/2\ TFT}}$$

$$\mathbf{27}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\mathrm{phys}} \;=\; \langle V^3 \rangle_{\mathrm{B/2\ TFT}}$$

So: 4d Yukawa couplings are 3-pt functions in
A/2, B/2 theories,
(0,2) versions of the A, B model TFT's.

For the rest of today's talk,
I'm going to focus on the A/2 and B/2 theories,
and their correlation functions,
whose OPE's form a generalization of quantum cohomology,
called *quantum sheaf cohomology*.

First, let me remind everyone,
what are the A/2, B/2 theories ?

The A/2, B/2 pseudo-TFT's

These (0,2) NLSM's have two anomalous global U(1)'s:

- a right-moving $U(1)_R$
- a canonical left-moving U(1),
  rotating the phase of all left fermions,
  which becomes $U(1)_L$ on (2,2) locus

If $\det \mathcal{E}^{\pm 1} \cong K_X$, then a nonanomalous U(1) exists along which we can twist right & left moving fermions.

There are two distinct possibilities,
which on (2,2) locus become the A, B model TFT's,
and are called the A/2, B/2 models.

A little more explicitly:

(0,2) NLSM has Lagrangian density

$$\mathcal{L} \;=\; g_{i\bar{\jmath}} \partial \phi^{\bar{\jmath}} \overline{\partial} \phi^i \;+\; i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} D_- \psi_+^i \;+\; i h_{a\bar{b}} \lambda_-^{\bar{b}} D_+ \lambda_-^a \\ + F_{i\bar{\jmath}a\bar{b}} \psi_+^i \psi_+^{\jmath} \lambda_-^a \lambda_-^{\bar{b}}$$

$$\psi_+ \;\sim\; TX \qquad\qquad \lambda_- \;\sim\; \mathcal{E}$$

subject to Green-Schwarz condition: $\mathrm{ch}_2(TX) = \mathrm{ch}_2(\mathcal{E})$

A/2 twist: take $\psi_+^i, \lambda_-^{\overline{a}}$ to be scalars

B/2 twist: take $\psi_+^{\bar{\imath}}, \lambda_-^{\overline{a}}$ to be scalars

so we get a scalar half of susy — but this BRST operator is purely right-moving, so this not a standard TFT.

In order for this twist to be anomaly-free, there are constraints..

A/2 model: Exists when $(\det \mathcal{E})^{-1} \cong K_X$

(on (2,2) locus, always possible; reduces to A model)

States: $H^\bullet \left(X, \wedge^\bullet \mathcal{E}^* \right)$

B/2 model: Exists when $\det \mathcal{E} \cong K_X$

(on (2,2) locus, requires $K_X^{\otimes 2} \cong \mathcal{O}_X$; reduces to B model)

States: $H^\bullet \left(X, \wedge^\bullet \mathcal{E} \right)$

Exchanging $\mathcal{E} \leftrightarrow \mathcal{E}^*$ swaps the A/2, B/2 models.

(Physically, just a complex conjugation of left movers.)

What do the A, A/2 model correlation functions look like?

Classical contributions, schematically:

A model: Classical contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_X \omega_1 \wedge \cdots \wedge \omega_n \; = \; \int_X (\mathrm{top} - \mathrm{form})$$

$$\omega_i \in H^{p_i, q_i}(X)$$

A/2 model: Classical contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_X \omega_1 \wedge \cdots \wedge \omega_n \qquad \omega_i \in H^{q_i}\left(X, \wedge^{p_i} \mathcal{E}^*\right)$$

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\mathrm{top}}(X, \wedge^{\mathrm{top}} \mathcal{E}^*) = H^{\mathrm{top}}(X, K_X)$

using the anomaly constraint $\det \mathcal{E}^* \cong K_X$

Again, a top form, so get a number.

What do the A, A/2 model correlation functions look like?

Instanton sectors, schematically:

A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \; = \; \int_{\mathcal{M}} (\mathrm{top} - \mathrm{form})$$

$$\omega_i \in H^{p_i, q_i}(\mathcal{M})$$

where $\mathcal{M}$ is moduli space of worldsheet instantons.

A/2 model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \; = \; \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \qquad \omega_i \in H^{q_i}(\mathcal{M}, \wedge^{p_i} \mathcal{F}^*)$$

where $\mathcal{F}$ is sheaf on $\mathcal{M}$ induced by $\mathcal{E}$.

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\mathrm{top}}(\mathcal{M}, \wedge^{\mathrm{top}} \mathcal{F}^*)$

so need to explain how to get top-form etc….

What do the A, A/2 model correlation functions look like?

To actually define A model correlation functions,
need to compactify $\mathcal{M}$.

To actually define A/2 model correlation functions,
need to not only compactify $\mathcal{M}$,
but also extend $\mathcal{F}$ over compactification divisor,
consistent with symmetries.

Then, formally, get a top-form so long as no anomalies:

$$\left.\begin{array}{c} \wedge^{\rm top}\mathcal{E}^* \cong K_X \\ {\rm ch}_2(\mathcal{E}) = {\rm ch}_2(TX) \end{array}\right\} \stackrel{\rm GRR}{\Longrightarrow} \wedge^{\rm top}\mathcal{F}^* \cong K_{\mathcal{M}}$$

All of this has been done for toric var's. (Katz-ES hepth/0406226, …., Donagi-Guffin-Katz-ES 1110.3751, .3752)

Resulting corr' f'ns encoded in quantum (sheaf) cohomology

We'll be interested in computing correlation functions and OPE's = quantum sheaf cohomology in the A/2 model.

Recall ordinary quantum cohomology = OPE's in A model — (2,2) locus.

OPE's take form

$$\wedge : \; H^{j_1,i_1}(X) \otimes H^{j_2,i_2}(X) \longrightarrow H^{j_1+j_2,i_1+i_2}(X)$$

so ~ ordinary cohomology ring, but with modification to relations.

Example: $\mathbb{P}^n$

A model correlation functions:

$$\langle \psi^n \rangle = 1, \qquad \langle \psi^{2n+1} \rangle = q, \quad \langle \psi^{n+d(n+1)} \rangle = q^d$$

==> OPE (quantum cohomology rel'n) $\quad \psi^{n+1} = q$

Quantum sheaf cohomology is analogous. It refers to quantum-corrected product structure on

$$\oplus H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$$

where

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \quad \text{and} \quad \det \mathcal{E}^* \cong K_X$$

and the classical product structure is of form

$$H^{i_1}(X, \wedge^{j_1}\mathcal{E}^*) \otimes H^{i_2}(X, \wedge^{j_2}\mathcal{E}^*) \longrightarrow H^{i_1+i_2}(X, \wedge^{j_1+j_2}\mathcal{E}^*)$$

(OPE's in A/2 model)

Reduces to ordinary quantum cohomology when $\mathcal{E} = TX$

For example: $\oplus H^\bullet(X, \wedge^\bullet \mathcal{E}^*) \rightsquigarrow \oplus H^\bullet(X, \Omega^\bullet) = \oplus H^{\bullet,\bullet}(X)$

(In physics, charged matter states in the heterotic compactification are counted by these sheaf cohom' gps.)

One of the main challenges of quantum sheaf cohomology is that unlike ordinary quantum cohomology, where the classical cohomology rings are well-known, in q.s.c. even the classical rings need to be computed!

That, plus the fact that the technology developed for GW theory, largely no longer applies.

Examples of resulting mathematical structures:

$\mathbb{P}^1 \times \mathbb{P}^1$:

Ordinary quantum cohomology ring is

$$\mathbb{C}[\sigma, \tilde{\sigma}]/(\sigma^2 = q, \tilde{\sigma}^2 = \tilde{q})$$

For $\mathcal{E}$ a deformation of the tangent bdle defined by four 2x2 matrices A, B, C, D,

quantum sheaf cohomology ring is

$$\mathbb{C}[\sigma, \tilde{\sigma}]/(\det(A\sigma + B\tilde{\sigma}) = q, \det(C\sigma + D\tilde{\sigma}) = \tilde{q})$$

Recover tangent bundle & ordinary quantum cohomology when A=D=I, B=C=0.

We'll see details later.

G(k,n):

Ordinary quantum cohomology ring is

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(n-k)}] / \langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q \rangle =$$

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots] / \langle D_{k+1}, D_{k+2}, \cdots, \sigma_{(n-k+1)}, \cdots, \sigma_{(n-1)}, \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \rangle$$

where $D_m = \det \left(\sigma_{(1+j-i)} \right)_{1 \leq i,j \leq m}$

For a def' of tangent bundle, the quantum sheaf coh' ring is

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots,] / \langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)}, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \rangle$$

where $R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$ for I_i the coeff's of char' poly' of matrix defining deformation.

We'll see details later.

Methods to compute A/2 correlation functions

	toric varieties	Grassmannians
Mathematics		
• Direct Cech computations (Katz, ES '04; Guffin, Katz '07)	✓	✗
• Koszul resolutions (Donagi, Guffin, Katz, ES '11)	✓	½
Physics (GLSMs)		
• Coulomb branch 1-loop eff' twisted superpot's (McOrist, Melnikov '08; Guo, Lu, ES '15)	✓	½
• Supersymmetric localization (Closset, Gu, Jia, ES,'15; Guo, Lu, ES '15)	✓	✓

I'll describe only some of these….

Methods to compute A/2 correlation functions

Today I'll outline

	toric varieties	Grassmannians
Mathematics		
• Direct Cech computations (Katz, ES '04; Guffin, Katz '07)	✓	✗
• **Koszul resolutions** (Donagi, Guffin, Katz, ES '11)	✓	½
Physics (GLSMs)		
• Coulomb branch 1-loop eff' twisted superpot's (McOrist, Melnikov '08; Guo, Lu, ES '15)	✓	½
• **Supersymmetric localization** (Closset, Gu, Jia, ES,'15; Guo, Lu, ES '15)	✓	✓

Outline of the rest of this talk:

- Koszul resolution methods for computing A/2
- Susy localization in A/2 model for def's of (2,2) theories
- A/2 behaves same as true TFT at genus zero
- Examples: $\mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{F}_n, \quad G(k,n)$

  — new expressions for old results: JKG residues

  — new results: nonabelian GLSM's
- (0,2) Toda duals
- Analogous computations in dual B/2 theories

Quantum sheaf cohomology via Koszul resolutions

Let's consider the

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \to H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $\cancelto{0}{H^0(Z \otimes W)} \to H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \to \cancelto{0}{H^1(Z \otimes W)}$

Since Z is a sum of $O(-1,0)$'s, $O(0,-1)$'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$

0

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \underset{\delta}{\overset{\sim}{\to}} H^1(S_1) \underset{\delta}{\to} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\mathrm{top}} E^{*} \cong K_X \\ \mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX}) \end{array}\right\} \overset{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} F^{*} \cong K_M$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet} \mathrm{W} / (Q^{a+1}, \tilde{Q}^{b+1})$$

where $\quad Q = \det(A\psi + B\tilde{\psi}), \quad \tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => OPE's $Q = q, \; \tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Review of quantum sheaf cohomology

(Math: Donagi, Guffin, Katz, ES, '11)
(Physics: McOrist, Melnikov '08)

General result:

For any toric variety, and any def' E of its tangent bundle,

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \to \mathrm{E} \to 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} = \mathrm{q}_a$$

where the M's are matrices of chiral operators built from *.

Generalizes Batyrev's ring $\displaystyle \prod_i \left(\sum_b Q_i^b \psi_b \right)^{Q_i^a} = q_a$

So far I’ve outlined how to use Koszul resolutions to compute classical and quantum sheaf cohomology rings.

Now, let’s switch gears.

Next: susy localization & qsc.

Susy localization

I'll first discuss A/2 theories obtained by deforming off the (2,2) locus, generalizing A model susy localization described in

Benini-Zaffaroni 1504.03698 Closset-Cremonesi-Park 1504.06308

Corresponding (0,2) GLSM's will have a Coulomb branch, along which we shall work.

Schematically, correlation functions take general form

$$\langle f(\sigma) \rangle \; = \; \sum_{\mathfrak{m} \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res}_{\sigma=0} \left\{ Z^{1-\mathrm{loop}} q^{\mathfrak{m}} f(\sigma) \right\}$$

for $\quad Z^{1-\mathrm{loop}} \; = \; \dfrac{\det \mathcal{O}_{\mathrm{fermi}}}{\det \mathcal{O}_{\mathrm{bose}}}$

and $\quad \sigma$ = adjoint-valued scalar defining Coulomb branch

Susy localization

$$Z^{1-\text{loop}} \;=\; \frac{\det \mathcal{O}_{\text{fermi}}}{\det \mathcal{O}_{\text{bose}}}$$

For deformations off the (2,2) locus, in a GLSM, ψ_+, ψ_- have same gauge charges.

Fermi interactions: $\overline{\psi}_-^i \psi_+^j E_i^j \;+\; \overline{\psi}_+^j \psi_-^i (E_i^j)^*$

$$\mathcal{O}_{\text{fermi}} \;=\; \left[\begin{array}{ccccc} E_1^1 & D_+ & E_1^2 & 0 & \cdots \\ D_- & (E_1^1)^* & 0 & (E_2^1)^* & \cdots \\ E_2^1 & 0 & E_2^2 & D_+ & \cdots \\ 0 & (E_1^2)^* & D_- & (E_2^2)^* & \cdots \\ \vdots & & & & \ddots \end{array} \right]$$

$$\det \mathcal{O}_{\text{fermi}} \;=\; (S(\det E))^{|b+1|} \prod_{n \geq 1} \left[\sum_{k=0}^{N} t_n^{2k} \left(\sum_{i_1 < i_2 < \cdots i_k, j_1 < j_2 < \cdots j_k} \left| \tilde{E}_{i_1 \cdots i_k j_1 \cdots j_k} \right|^2 \right) \right]^{2n + |b+1|}$$

where $\quad t_n \;=\; n(n + |b+1|) \qquad\qquad b \;=\; Q(\mathfrak{m})$

Susy localization

$$Z^{1-\mathrm{loop}} = \frac{\det \mathcal{O}_{\mathrm{fermi}}}{\det \mathcal{O}_{\mathrm{bose}}}$$

Bosonic potential:

$$|E^i(\phi)|^2 = \sum_i \left(\sum_j |E^i_j|^2 |\phi^j|^2 \right) + \sum_{i \neq j} \left(\sum_k (E^k_i)^* E^k_j \right) \overline{\phi}^{\bar{\imath}} \phi^j$$

$$\mathcal{O}_{\mathrm{bose}} = \begin{bmatrix} -D^2 + |E^1_1|^2 + \cdots + |E^N_1|^2 & (E^1_1)^* E^1_2 + \cdots + (E^N_1)^* E^N_2 & \cdots \\ E^1_1 (E^1_2)^* + \cdots + E^N_1 (E^N_2)^* & -D^2 + |E^1_2|^2 + \cdots + |E^N_2|^2 & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

$$\det \mathcal{O}_{\mathrm{bose}} = \prod_{n \geq 0} \left[\sum_{k=0}^{N} t_n^{2k} \left(\sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{E}_{i_1 \cdots i_k j_1 \cdots j_k} \right|^2 \right) \right]^{2n+|b|+1}$$

where $\quad t_n = \frac{1}{2} \left(2n(n+1) + (2n+1)|b| - b \right)$

Susy localization

Putting this together, can show

$$Z^{1-\text{loop}} = \frac{\det \mathcal{O}_{\text{fermi}}}{\det \mathcal{O}_{\text{bose}}} = \left(\frac{1}{\det E}\right)^{Q(\mathfrak{m})+1}$$

so *schematically* correlation functions take form

$$\begin{aligned}\langle f(\sigma)\rangle &= \sum_{\mathfrak{m}\in\mathbb{Z}} \mathrm{JKG}-\mathrm{Res}_{\sigma=0}\left\{Z^{1-\text{loop}} q^{\mathfrak{m}} f(\sigma)\right\} \\ &= \sum_{\mathfrak{m}\in\mathbb{Z}} \mathrm{JKG}-\mathrm{Res}_{\sigma=0}\left\{\left(\frac{1}{\det E}\right)^{Q(\mathfrak{m})+1} q^{\mathfrak{m}} f(\sigma)\right\}\end{aligned}$$

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Build a (0,2) theory that deforms (2,2) model.

Math: $$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$x, \tilde{x}$ vectors of homogeneous coordinates,

A, B, C, D 2×2 matrices describing deformation

(2,2) locus: $A = D = I_{2\times 2}, \quad B = C = 0$

Physics….

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Build a (0,2) theory that deforms (2,2) model.

Physics:

$x^i, \tilde{x}^i$ chiral superfields charge (1,0), (0,1)

$\Lambda^i, \tilde{\Lambda}^i$ Fermi superfields charge (1,0), (0,1) s.t.

$$\overline{D}_+ \Lambda^i \;=\; A^i_j \sigma x^j + B^i_j \tilde{\sigma} x^j \qquad \overline{D}_+ \tilde{\Lambda}^i \;=\; C^i_j \sigma \tilde{x}^j + D^i_j \tilde{\sigma} \tilde{x}^j$$

σ neutral (adj-valued) chiral superfield

no superpotential

On (2,2) locus, x^i, Λ^i combine into (2,2) chiral superfields,
$\tilde{x}^i, \tilde{\Lambda}^i$ combine into (2,2) chiral superfields, and
σ part of (2,2) vector multiplet.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Localization computation: (genus zero)

$$\langle f(\sigma,\tilde{\sigma}) \rangle \;=\; \sum_{\mathfrak{m}_1,\mathfrak{m}_2 \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res}_{\sigma=\tilde{\sigma}=0} \left\{ \left(\frac{1}{\det E} \right)^{\mathfrak{m}_1+1} \left(\frac{1}{\det \tilde{E}} \right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1} \tilde{q}^{\mathfrak{m}_2} f(\sigma,\tilde{\sigma}) \right\}$$

for $E = A\sigma + B\tilde{\sigma}, \quad \tilde{E} = C\sigma + D\tilde{\sigma}$

Note: Looks like a TFT result — no propagators, no worldsheet position dependence — but this is not quite TFT.

"Non-topological TFT"

How can that be?

“Non-topological TFT”

The basic reason we’re getting a TFT-like structure, albeit not in an actual TFT, is that the OPE’s close on dim zero A/2 op’s.

(Adams-Distler-Ernebjerg ’05) argued that e.g. in an open patch on moduli space containing (2,2) locus, the OPE’s of the A/2 model operators close into other A/2 model operators.

For conformal cases, combination of

- right-moving N=2 algebra to bound dimensions
- worldsheet conformal invariance to relate left, right dim’s

to argue closure on patches.

Since operators have dim’ zero, & OPE’s close, no worldsheet dependence in correlation functions.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Let's take another look at the result:

$$\langle f(\sigma,\tilde{\sigma})\rangle \;=\; \sum_{\mathfrak{m}_1,\mathfrak{m}_2\in\mathbb{Z}} \mathrm{JKG}-\mathrm{Res}_{\sigma=\tilde{\sigma}=0}\left\{\left(\frac{1}{\det E}\right)^{\mathfrak{m}_1+1}\left(\frac{1}{\det \tilde{E}}\right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1}\tilde{q}^{\mathfrak{m}_2} f(\sigma,\tilde{\sigma})\right\}$$

for $\quad E = A\sigma + B\tilde{\sigma}, \quad \tilde{E} = C\sigma + D\tilde{\sigma}$

Inserting a factor of, say, $\det E$ in the correlation f'n is equivalent to shifting q.

Quantum sheaf cohomology ring rel'ns:

$$\det E \;=\; q, \quad \det \tilde{E} \;=\; \tilde{q}$$

This result already known (for all toric varieties w/ def's):

Physics: McOrist-Melnikov 0810.0012 Math: Donagi-Guffin-Katz-ES 1110.3751, .3752

but the derivation is new.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Compare

Quantum sheaf cohomology (q.s.c.) ring rel'ns:

$$\det(A\sigma + B\tilde{\sigma}) \; = \; q, \quad \det(C\sigma + D\tilde{\sigma}) \; = \; \tilde{q}$$

Ordinary quantum cohomology ring rel'ns:

$$\sigma^2 \; = \; q, \quad \tilde{\sigma}^2 \; = \; \tilde{q}$$

On the (2,2) locus, where $A = D = I_{2\times 2}, \quad B = C = 0$

quantum sheaf cohomology reduces to
ordinary quantum cohomology.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

2-pt correlation functions:

$$\langle \sigma\sigma \rangle \;=\; -\alpha^{-1}\Gamma_1 \qquad \langle \sigma\tilde{\sigma} \rangle \;=\; \alpha^{-1}\Delta \qquad \langle \tilde{\sigma}\tilde{\sigma} \rangle \;=\; -\alpha^{-1}\Gamma_2$$

where

$$\Gamma_1 \;=\; \gamma_{AB}\det D - \gamma_{CD}\det B \qquad \Gamma_2 \;=\; \gamma_{CD}\det A - \gamma_{AB}\det C$$

$$\gamma_{AB} \;=\; \det(A+B) - \det A - \det B$$

$$\gamma_{CD} \;=\; \det(C+D) - \det C - \det D$$

$$\Delta \;=\; \det A \det D - \det B \det C$$

$$\alpha \;=\; \Delta^2 - \Gamma_1\Gamma_2$$

$\{\alpha = 0\}$ = locus where bundle degenerates

JKG residue results match Cech cohomology computation.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

2-pt correlation functions:

$$\langle \sigma \sigma \rangle \; = \; -\alpha^{-1} \Gamma_1 \qquad \langle \sigma \tilde{\sigma} \rangle \; = \; \alpha^{-1} \Delta \qquad \langle \tilde{\sigma} \tilde{\sigma} \rangle \; = \; -\alpha^{-1} \Gamma_2$$

Can show these 2-pt functions obey

$$\langle \det(A\sigma + B\tilde{\sigma}) \rangle \; = \; 0 \qquad \langle \det(C\sigma + D\tilde{\sigma}) \rangle \; = \; 0$$

matching classical limit of q.s.c. relations.

Can also compute higher-pt functions.
They also match Cech computations, and obey suitable OPE's;
for brevity, let's move on.

Example: Hirzebruch surfaces $\mathbb{F}_n$

Build a (0,2) theory that deforms (2,2) model.

Math:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \begin{bmatrix} Ax & Bx \\ \gamma_1 w + s f_n(x_1,x_2) & \beta_1 w + s g_n(x_1,x_2) \\ \gamma_2 s & \beta_2 s \end{bmatrix}$$

Physics:

$$\overline{D}_+ \Lambda^i = A^i_j \sigma x^j + B^i_j \tilde{\sigma} x^j$$

$$\overline{D}_+ \Lambda_w = \sigma(\gamma_1 w + s f_n) + \tilde{\sigma}(\beta_1 w + s g_n)$$

$$\overline{D}_+ \Lambda_s = \sigma \gamma_2 s + \tilde{\sigma} \beta_2 s$$

— depends upon deg n polynomials f_n, g_n ;
however, they don't contribute to correlation functions:

Math: 1110.3751, .3752

Also follows from susy localization

Example: Hirzebruch surfaces $\mathbb{F}_n$

Localization computation:

$$\langle f(\sigma,\tilde{\sigma})\rangle \;=\; \sum_{\mathfrak{m}_1,\mathfrak{m}_2\in\mathbb{Z}} \mathrm{JKG}-\mathrm{Res}_{\sigma=\tilde{\sigma}=0}\left\{\left(\frac{1}{\det E}\right)^{\mathfrak{m}_1+1}\left(\frac{1}{Q_w}\right)^{n\mathfrak{m}_1+\mathfrak{m}_2+1}\left(\frac{1}{Q_s}\right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1}\tilde{q}^{\mathfrak{m}_2} f(\sigma,\tilde{\sigma})\right\}$$

$$E \;=\; A\sigma + B\tilde{\sigma}$$

$$Q_w \;=\; \gamma_1\sigma + \beta_1\tilde{\sigma}$$

$$Q_s \;=\; \gamma_2\sigma + \beta_2\tilde{\sigma}$$

Can read off quantum sheaf cohomology ring rel'ns:

$$(\det E)Q_w^n \;=\; q \qquad\qquad Q_s Q_w \;=\; \tilde{q}$$

— matches previous results of McOrist-Melnikov; Donagi-Guffin-Katz-ES

— reduce to ordinary quantum cohomology on (2,2) locus

Example: Hirzebruch surfaces $\mathbb{F}_n$

2-pt correlation functions:

$$\langle \sigma\sigma \rangle \;=\; \alpha^{-1}\left[\Delta - \beta_1\beta_2 \det(A+B) + (\gamma_1+\beta_1)(\gamma_2+\beta_2)\det B\right]$$
$$\langle \sigma\tilde{\sigma} \rangle \;=\; \alpha^{-1}\Delta$$
$$\langle \tilde{\sigma}\tilde{\sigma} \rangle \;=\; \alpha^{-1}\left[\Delta + \gamma_1\gamma_2 \det(A+B) - (\gamma_1+\beta_1)(\gamma_2+\beta_2)\det A\right]$$

where

$$\Delta \;=\; \beta_1\beta_2 \det A - \gamma_1\gamma_2 \det B$$

$$\alpha \;=\; \Phi_1\Phi_2$$

for

$$\Phi_i \;=\; \beta_i^2 \det A - \beta_i\gamma_i\gamma_{AB} + \gamma_i^2 \det B$$
$$\gamma_{AB} \;=\; \det(A+B) - \det A - \det B$$

JKG residue results match Cech cohomology computation.

For a general toric variety + deformation of tangent bundle,

$$\langle f(\vec{\sigma}) \rangle = \sum_{\mathfrak{m}_1, \cdots \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res}_{\vec{\sigma}=0} \left\{ \left[\prod_{a,\alpha} \left(\frac{1}{\det M_{(\alpha)}} \right)^{Q^a_\alpha(\mathfrak{m}_a)+1} q_a^{\mathfrak{m}_a} \right] f(\vec{\sigma}) \right\}$$

It can be shown that this matches result of McOrist-Melnikov '08:

$$\langle f(\sigma) \rangle = \sum_{\sigma | J=0} f(\sigma) \left((\det J_{a,b}) \prod_\alpha \det M_{(\alpha)} \right)^{-1}$$

where $$J_a = \ln \left(q_a^{-1} \prod_\alpha M_{(\alpha)}^{Q^a_\alpha} \right)$$

Now, getting new expressions for old results is nice,
but, what's even better is that we can also get new results…..

Nonabelian cases

So far, we have discussed the results of applying susy localization to A/2 theories describing toric varieties.

Next: Grassmannians

Understanding A/2 twists of Grassmannians has been an open problem for many years, as older GLSM techniques don't easily apply.

We'll see that susy localization allows us to quickly derive results not previously obtainable.

Basic example, (2,2): G(k,n) = Grassmannian of k planes in $\mathbb{C}^n$

Physics: U(k) gauge theory

n chiral multiplets in fundamental rep'

(0,2) deformation:

U(k) gauge theory

n chiral multiplets ϕ^i in fundamental rep'

n Fermi multiplets Λ^i in fundamental rep'

$$\overline{D}_+ \Lambda^i_a \;=\; \sigma^b_a \phi^i_b \;+\; B^i_j (\mathrm{Tr}\,\sigma) \phi^j_a$$

The B's define deformation off (2,2) locus.

Can show, total num' of deformations = $n^2 - 1$
(for $1 < k < n-1$)

— overall trace of B defines trivial deformation; rest interesting

General formula for A/2 correlation functions:

$$\langle f(\sigma_1, \cdots, \sigma_k) \rangle \; =$$

$$\frac{1}{k!} \sum_{\mathfrak{m}_1, \cdots, \mathfrak{m}_k \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res}_{\vec{\sigma}=0} \left\{ q^{\sum \mathfrak{m}_i} \left(\prod_{\alpha \neq \beta} (\sigma_\alpha - \sigma_\beta) \right) \prod_{\alpha=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_\alpha)} \right)^{\mathfrak{m}_\alpha + 1} f(\vec{\sigma}) \right\}$$

where $\tilde{E}^i_j(\sigma) \; = \; \sigma \delta^i_j \; + \; B^i_j \left(\sum_\alpha \sigma_\alpha \right)$

Q.s.c. relations: $\det \tilde{E}(\sigma_\alpha) \; = \; q$ for all α

We'll see more meaningful expressions shortly….

We'll give formulas for cases in which B is diagonal, for simplicity.

Example: Deformation of TG(2,4)

Classical correlation functions ($\mathfrak{m}_1 = \mathfrak{m}_2 = 0$)

$$\langle \sigma_1^4 \rangle = \Delta^{-1} \left(I_1 + 2I_1^2 + 4I_1 I_2 - 2I_3 + 2I_2^2 + 2I_1 I_3 - 4 \det B + 2I_2 I_3 - 2I_1 \det B \right)$$

$$\langle \sigma_1^3 \sigma_2 \rangle = \Delta^{-1} \left(-1 - 3I_1 - 2I_1^2 - 3I_2 - 4I_1 I_2 - 2I_2^2 - I_3 - 2I_1 I_3 + 4 \det B - 2I_2 I_3 + 2I_1 \det B \right)$$

$$\langle \sigma_1^2 \sigma_2^2 \rangle = \Delta^{-1} \left(2 + 4I_1 + 2I_1^2 + 4I_2 + 4I_1 I_2 + 2I_3 - 4 \det B + 2I_2^2 + 2I_1 I_3 + 2I_2 I_3 - 2I_1 \det B \right)$$

$$\langle \sigma_1 \sigma_2^3 \rangle \; = \; \langle \sigma_1^3 \sigma_2 \rangle \qquad \langle \sigma_2^4 \rangle \; = \; \langle \sigma_1^4 \rangle$$

$$\Delta \; = \; 2 \prod_{i<j} (1 + B_{ii} + B_{jj})$$

is the locus on which bundle degenerates.

$$I_1 = \sum_i B_{ii} = \operatorname{tr} B \qquad I_2 \; = \; \sum_{i<j} B_{ii} B_{jj} \qquad I_3 = \sum_{i<j} B_{ii} B_{jj} B_{kk}$$

are coefficients in the characteristic polynomial of B.

How can we interpret those correlation functions usefully?
How can we compare to ordinary cohomology, on (2,2) locus?

Gauge-invariant combinations naturally correspond to Young diagrams (via `Schur polynomials'):

$$\sigma_{\ydiagram{1}} = \sigma_1 + \sigma_2$$

$$\sigma_{\ydiagram{2}} = \sigma_1^2 + \sigma_2^2 + \sigma_1 \sigma_2$$

$$\sigma_{\ydiagram{1,1}} = \sigma_1 \sigma_2$$

$$\sigma_{\ydiagram{2,1}} = \sigma_1^2 \sigma_2 + \sigma_1 \sigma_2^2$$

$$\sigma_{\ydiagram{2,2}} = \sigma_1^2 \sigma_2^2$$

Cohomology of G(k,n) is naturally in 1-1 correspondence with Young diagrams inside k x (n-k) box.

G(2,4): $\yng(2,2)$

so on (2,2) locus, for example,

$$\sigma_{\yng(3)} = 0$$

$$\sigma_{\yng(3,1)} = 0$$

Classical correlation functions on (2,2) locus:

$$\langle \sigma_{\yng(2,2)} \rangle = +1 \qquad \langle \sigma_{\yng(4)} \rangle = 0 = \langle \sigma_{\yng(3,1)} \rangle$$

which imply OPE $\sigma_{\yng(3)} \cdot \sigma_{\yng(1)} = 0$

hence $\sigma_{\yng(3)} = 0$

Agree!

(2,2) locus: $\sigma_{\ydiagram{3}} = \sigma_{\ydiagram{4}} = \sigma_{\ydiagram{3,1}} = 0$ classically

$\sigma_{\ydiagram{4}} - \sigma_{\ydiagram{3,1}} = 2q$ nonpert'ly

(0,2):

(a) $(1 + I_1 + I_2 + I_3)\sigma_{\ydiagram{3}} + (I_3 + 2I_2 + 2I_1)\sigma_{\ydiagram{2,1}} = 0$

(b) $(1 + I_1 + I_2 + I_3)\sigma_{\ydiagram{4}} + (1 + 3I_1 + 3I_2 + 2I_3)\sigma_{\ydiagram{3,1}} + (I_3 + 2I_2 + 2I_1)\sigma_{\ydiagram{2,2}} = 0$

(c) $$(1 + I_3 + I_2 + I_1 + 2\det B)\sigma_{\ydiagram{4}} + (-1 + I_2 + 3I_1 + 6\det B)\sigma_{\ydiagram{3,1}} + (-I_3 + 2I_1 + 4\det B)\sigma_{\ydiagram{2,2}} = 2q$$

(derived from $\det E = q$)

More generally, for any deformation of given form of TG(k,n), we've recently argued that

classical sheaf cohomology ring =

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(k(n-k))}\right] / \left\langle D_{k+1}, \cdots, D_{k(n-k)}, R_{(n-k+1)}, \cdots, R_{(k(n-k))}\right\rangle$$

$$D_m = \det(\sigma_{1+j-i})_{1 \leq i,j \leq m}$$

$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$$

for $\sigma_{(1)} = \sigma_{\Box}$ and so forth, and

(I_i) the coefficients in the characteristic polynomial of B.

Exs: $I_0 = 1, \quad I_1 = \mathrm{Tr}\, B, \quad I_n = \det B$

More generally, for any deformation of given form of TG(k,n), we've recently argued that

classical sheaf cohomology ring =

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(k(n-k))}\right] / \left\langle D_{k+1}, \cdots, D_{k(n-k)}, R_{(n-k+1)}, \cdots, R_{(k(n-k))}\right\rangle$$

$$D_m = \det(\sigma_{1+j-i})_{1 \leq i,j \leq m}$$

$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$$

On (2,2) locus, $R_r = \sigma_{(r)}$ and the above simplifies to

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_n\right\rangle$$

a standard presentation of the ordinary cohomology of G(k,n).

So: matches (2,2) locus. Next: quantum case…

Structure of *quantum* sheaf cohomology ring for a generic deformation of T G(k,n)

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \rangle$$

where
$$D_m = \det \left(\sigma_{(1+j-i)} \right)_{1 \leq i,j \leq m}$$
$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$$

for I_i the char' poly's of B:
$$\det(tI + B) = \sum_{i=0}^{n} I_{n-i} t^i$$

Exs: $I_0 = 1, \quad I_1 = \mathrm{Tr}\, B, \quad I_n = \det B$

Quantum sheaf cohomology ring:

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\Big\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \Big\rangle$$

If we turn off the deformation (set B=0), then

$$R_{(n)} = \sigma_{(n)}$$

and with some work it can be shown that the ring above can be presented as

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(n-k)}]/\langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q\rangle$$

which is a standard presentation of the (ordinary) quantum cohomology ring of G(k,n).

(Buch, Kresch, Tamvakis, Bertram, Witten, Siebert, Tian, ….)

Example: G(1,3)

This has no nontrivial deformations, so any result should be equivalent to ordinary quantum cohomology ring of $\mathbb{P}^2$.

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\langle D_2, \cdots, R_{(3)} + q, R_{(4)} + q\sigma_{(1)}, \cdots \rangle$$

which $= \mathbb{C}[\sigma_{(1)}]/\langle R_{(3)} + q \rangle\rangle$

using $D_2 = \sigma_{(1)}^2 - \sigma_{(2)}, \cdots$ to eliminate $\sigma_{(m)}$ for m>1, and

the result $R_{(3+\ell)} + q\sigma_{(\ell)} = \sigma_{(\ell)}(R_{(3)} + q)$

Now, $R_{(3)} = \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma^i = \left(\sum_{i=0}^{3} I_i \right) \sigma^3 = (\det(I + B))\, \sigma^3$

so the qsc ring is $\mathbb{C}[\sigma]/\langle \det(I+B)\sigma^3 + q \rangle$

which is equivalent to std quantum cohomology ring.

So far we've used susy localization & other methods to

— find new residue-based formulas for toric results

— derive new results for nonabelian GLSM's
(Grassmannian quantum sheaf cohomology)

Next: Toda duals, then B/2 results

Toda duals

The mirror to the A model on $\mathbb{P}^n$ is a B-twisted Landau-Ginzburg model, defined by a superpotential

$$W \; = \; X_1 + \cdots + X_n + \frac{q}{X_1 \cdots X_n}$$

often referred to as the `Toda dual.'

Analogous statements are known for heterotic theories, which we'll describe, but first let's review how this works.

Toda duals (to $\mathbb{P}^n$)

$$W \; = \; X_1 + \cdots + X_n + \frac{q}{X_1 \cdots X_n}$$

Genus zero correlation functions:

$$\langle f(X_1, \cdots, X_n) \rangle \; = \; \sum_{d_{\ln X} W = 0} \frac{f(X_1, \cdots, X_n)}{\det\left(\partial^2_{\ln X} W\right)}$$

$$dW = 0 \Longrightarrow \quad X_1 = X_2 = \cdots = X_n \equiv X$$

& $\quad X = q X^{-n}$ or $\quad X^{n+1} = q \qquad$ (q.c. rel'n!)

Can show $\quad \det(\partial^2 W) = (n+1) X^n$

hence $\quad \langle X^m \rangle = \sum_{X^{n+1}=q} \frac{X^m}{(n+1) X^n}$

thus $\qquad \langle X^{n+d(n+1)} \rangle = q^d \qquad$ matching A model.

What's the heterotic analogue?

A heterotic Landau-Ginzburg model is defined by

- complex Kahler manifold X
- holomorphic vector bundle $\mathcal{E} \to X$
- holomorphic section $(J_a) \in \Gamma(\mathcal{E}^*)$

Recover ordinary Landau-Ginzburg models when

$$\mathcal{E} \,=\, TX, \quad J_a \,=\, \partial_a W$$

The mirror to the (A/2) theory on $\mathbb{P}^n \times \mathbb{P}^m$,

with def' of tangent bundle param'd by matrices A, B, C, D,

is a Landau-Ginzburg theory on $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^m$ with $\mathcal{E} = T$

$$J_i \; = \; a^{(1-n)/n} \left(a X_i + b \frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^n \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} - \frac{q_1}{X_1 \cdots X_n} \right)$$

$$\tilde{J}_k \; = \; d^{(1-m)/m} \left(d \tilde{X}_k + c \frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m \nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} - \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} \right)$$

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$$\det(Ax + By) \; = \; a x^{n+1} + b y^{n+1} + \sum_{i=1}^n \mu_i x^i y^{n+1-i},$$

$$\det(Cx + Dy) \; = \; c x^{m+1} + d y^{m+1} + \sum_{k=1}^n \nu_k x^k y^{n+1-k},$$

Let me outline correlation functions in these theories.

For heteronic LG models of the form just discussed,

at genus 0,

$$\langle f(X_i, \tilde{X}_k) \rangle = \sum_{J, \tilde{J}=0} \frac{f(X_i, \tilde{X}_k)}{\det(\partial(J, \tilde{J}))}$$

$$J, \tilde{J} = 0 \Longrightarrow \quad X_1 = \cdots = X_n \equiv X, \quad \tilde{X}_1 = \cdots = \tilde{X}_m \equiv \tilde{X}$$

& $\quad \det(AX + B\tilde{X}) = q, \quad \det(CX + D\tilde{X}) = \tilde{q}$

— the quantum sheaf cohomology ring rel'ns

Can show all (genus 0) correlation functions match those of the corresponding A/2 theory, which is how we've checked this proposal.

Analogues for the B/2 model

So far I've only discussed susy localization in the A/2 model, for deformations of (2,2) theories.

We can also apply the same ideas to B/2 twists of `dual' theories.

Which theories?

Recall mentioned earlier that

$$A/2(X, \mathcal{E}) \; = \; B/2(X, \mathcal{E}^*)$$

so we're going to be able to apply B/2 to spaces with deformations of cotangent bundles — no (2,2) locus.

Analogues for the B/2 model

Quick aside: how is this related to (0,2) mirror symmetry?

Suppose (0,2) NLSM's on $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ define same SCFT.

$$
\begin{tikzcd}
A/2(X, \mathcal{E}) \arrow[r, equal, "\text{(0,2) mirror}"] \arrow[d, equal] & B/2(Y, \mathcal{F}) \arrow[d, equal] \\
B/2(X, \mathcal{E}^*) \arrow[r, equal, "\text{(0,2) mirror}"] & A/2(Y, \mathcal{F}^*)
\end{tikzcd}
$$

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Math:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2 \xrightarrow{*} \mathcal{O}^2 \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$x, \tilde{x}$ vectors of homogeneous coordinates,

A, B, C, D 2×2 matrices describing deformation

No (2,2) locus; but cotangent bundle at

$$A = D = I_{2\times 2}, \quad B = C = 0$$

Physics….

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Physics:

$x^i, \tilde{x}^i$ chiral superfields charge (1,0), (0,1)

$p, \tilde{p}$ neutral chiral superfields

$\Lambda_i, \tilde{\Lambda}_i$ Fermi superfields charge (-1,0), (0,-1)

plus (0,2) superpotential

$$W = \Lambda_i F^i_j x^j + \tilde{\Lambda}_i \tilde{F}^i_j \tilde{x}^j$$

where
$$F^i_j = A^i_j p + B^i_j \tilde{p}$$
$$\tilde{F}^i_j = C^i_j p + D^i_j \tilde{p}$$

(Compare A/2 version: there, no superpotential, and charges matched.)

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Unlike the A/2 case, here there is no σ field — no adjoint-valued scalar that is part of vector multiplet on (2,2) locus.

So, no Coulomb branch along which to compute.

Instead, have p field, which plays a \`dual' role.

In effect, the Coulomb branch replaced by (part of) Higgs branch.

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Result:

Correlation functions are given by:

$$\langle f(p,\tilde{p})\rangle \; = \; \sum_{\mathfrak{m}_1,\mathfrak{m}_2\in\mathbb{Z}} \mathrm{JKG}-\mathrm{Res}_{p=\tilde{p}=0}\left\{ \left(\frac{1}{\det F}\right)^{\mathfrak{m}_1+1} \left(\frac{1}{\det \tilde{F}}\right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1}\tilde{q}^{\mathfrak{m}_2} f(p,\tilde{p}) \right\}$$

where $\quad F \; = \; Ap + B\tilde{p} \qquad \tilde{F} \; = \; Cp + D\tilde{p}$

— equivalent to results in dual A/2 model, as expected

Other cotangent bundle deformations similar.

Summary

- Outline of A/2, B/2 models
- Koszul resolution methods for computing A/2
- Susy localization in A/2 model for def's of (2,2) theories
- A/2 behaves same as true TFT at genus zero
- Examples: $\mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{F}_n, \quad G(k,n)$

  — new expressions for old results: JKG residues

  — new results: nonabelian GLSM's

- (0,2) Toda duals
- Analogous computations in dual B/2 theories

Next steps….

Next steps

- Mathematical derivation of qsc for def's of G(k,n),

  in terms of induced sheaves on moduli spaces = Quot schemes

  Already know physics derivation of qsc,
  and math derivation of classical ring structure.

- QSC for def's of flag manifolds (just started)

  Once that's accomplished,
  QSC for def's of quiver varieties will be next.

- Nonabelian Toda duals (just started)

More to do….

To Do

- Math derivation of McOrist-Melnikov results

Why is A/2 independent of J def's?
Why is B/2 independent of E def's?

Suggests (0,2) moduli space factors;
might be seen via study of U(1) actions.

- QSC for (0,2) theories that are not (2,2) deformations

- QSC for non-Kahler heterotic compactifications

- Heterotic GW invariants

Thank you for your time!