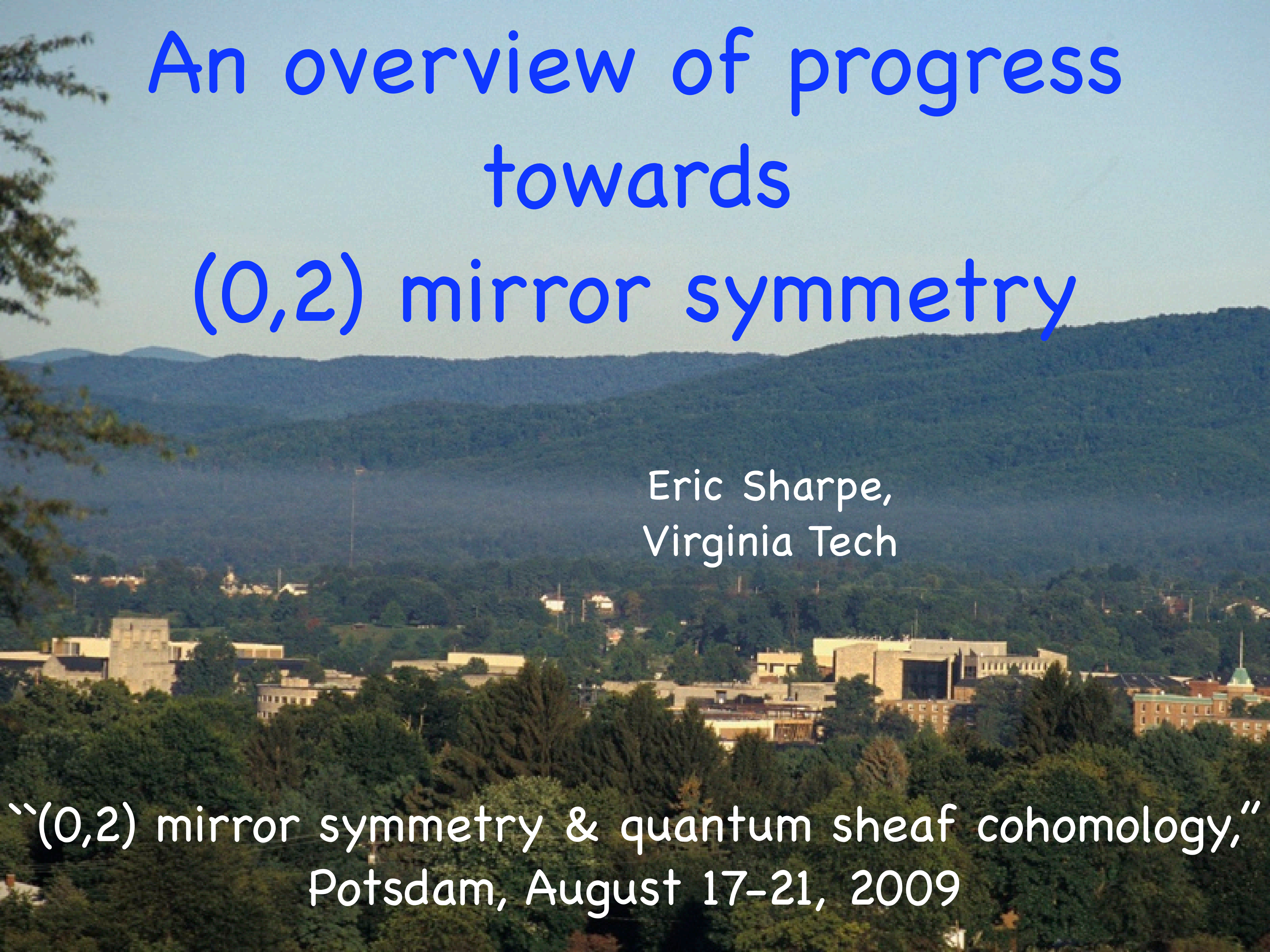

An overview of progress towards (0,2) mirror symmetry

Eric Sharpe,
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``(0,2) mirror symmetry & quantum sheaf cohomology,''
Potsdam, August 17-21, 2009

Mirror symmetry:

* CFT(X) = CFT(Y)

* Exchanges Hodge numbers

* Exchanges worldsheet instanton sums with classical computations

* Led to tremendous strides in enumerative geometry

The hope of this workshop is to generalize the results (and, hopefully, success!) of mirror symmetry.

(0,2) mirror symmetry is a generalization of ordinary mirror symmetry that is believed to occur in heterotic strings.

Some initial work was done in '94-'96;
but once string duality was discovered,
everyone's attention quickly shifted,
leaving unfinished business.

In recent years, several groups have been working out natural continuations of work done then.

The hope: enough basics have been done to start making serious progress.

Outline

-- review classic constructions of ordinary mirrors,
and in each case,
what is known about (0,2) analogues

-- outline quantum sheaf cohomology,
(0,2) A, B models

-- some modern ideas:
lift to LG models, fibered affine algebras, etc

Emphasis on listing open problems.

(0,2) mirror symmetry: the conjecture

Let (X_1,E_1) be a Calabi-Yau X_1 and stable holomorphic vector bundle E_1 of rk r, s.t. $ch_2(TX_1) = ch_2(E_1)$.

Claim there exists another such pair (X_2,E_2), where dim X_2 = dim X_1, rk E_2 = rk E_1, s.t.:

* $CFT(X_1,E_1) = CFT(X_2,E_2)$, hence
* total number of complex, Kahler, bundle moduli invariant
* $$h^i\left(X_1, \Lambda^j \mathcal{E}_1\right) = h^j\left(X_2, \left(\Lambda^i \mathcal{E}_2\right)^\vee\right)$$

(0,2) mirror symmetry: the conjecture

(0,2) mirror symmetry should reduce to
ordinary mirror symmetry
in the special case:

$E_1 = TX_1, \quad E_2 = TX_2$

(0,2) mirror symmetry: the conjecture

* Ordinary mirror symmetry exchanges
complex <-> Kahler;
this is why worldsheet instanton sums
are exchanged with classical computations.

* But in (0,2) mirrors, **no** physical reason
why complex, Kahler can't mix with each other
and with bundle moduli.
Always exchange quantum <-> quantum,
a priori never quantum <-> classical.

(0,2) mirror symmetry: the conjecture

Instead of exchanging (p,q) forms,
(0,2) mirror symmetry exchanges
sheaf cohomology:

$$H^j(X_1, \Lambda^i \mathcal{E}_1) \leftrightarrow H^j(X_2, (\Lambda^i \mathcal{E}_2)^\vee)$$

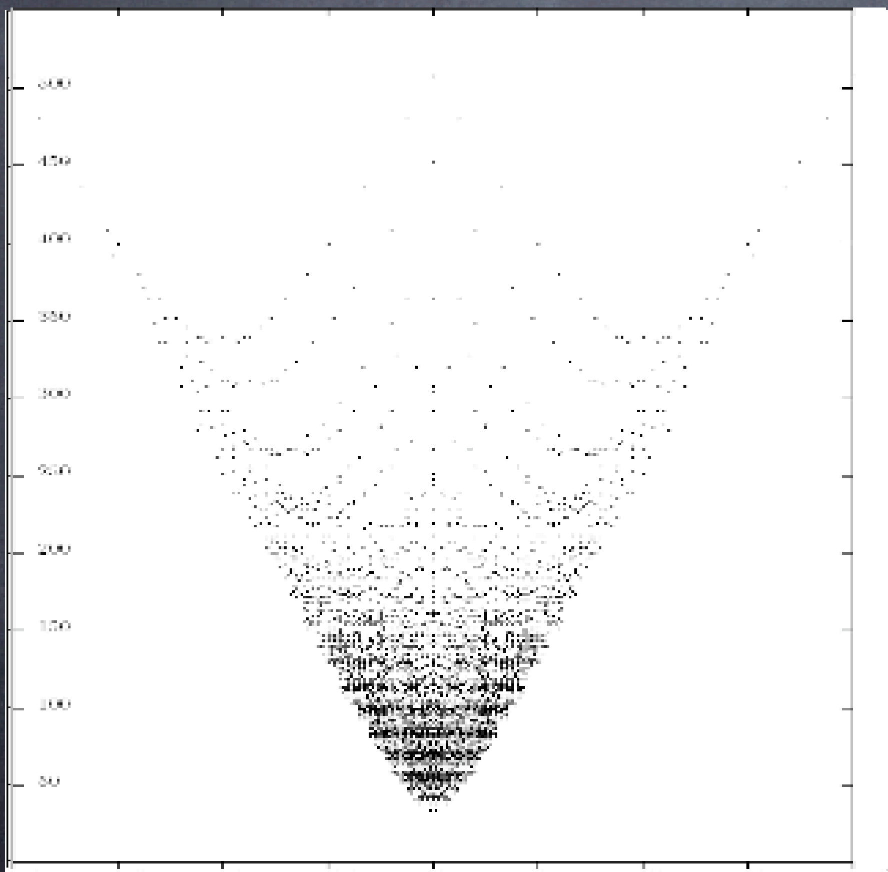
Note when $\mathcal{E}_i \cong TX_i$ this reduces to

$$H^{d-1,1}(X_1) \leftrightarrow H^{1,1}(X_2)$$

(for X_i Calabi-Yau)

Numerical tests of ordinary mirror symmetry

Shown are CY 3-folds:



(Klemm, Schimmrigk, NPB 411 ('94) 559-583)

Vertical axis: $h^{1,1} + h^{2,1}$

Horizontal axis: $2(h^{1,1} - h^{2,1})$

= 2 (# Kahler - # cpx defs)

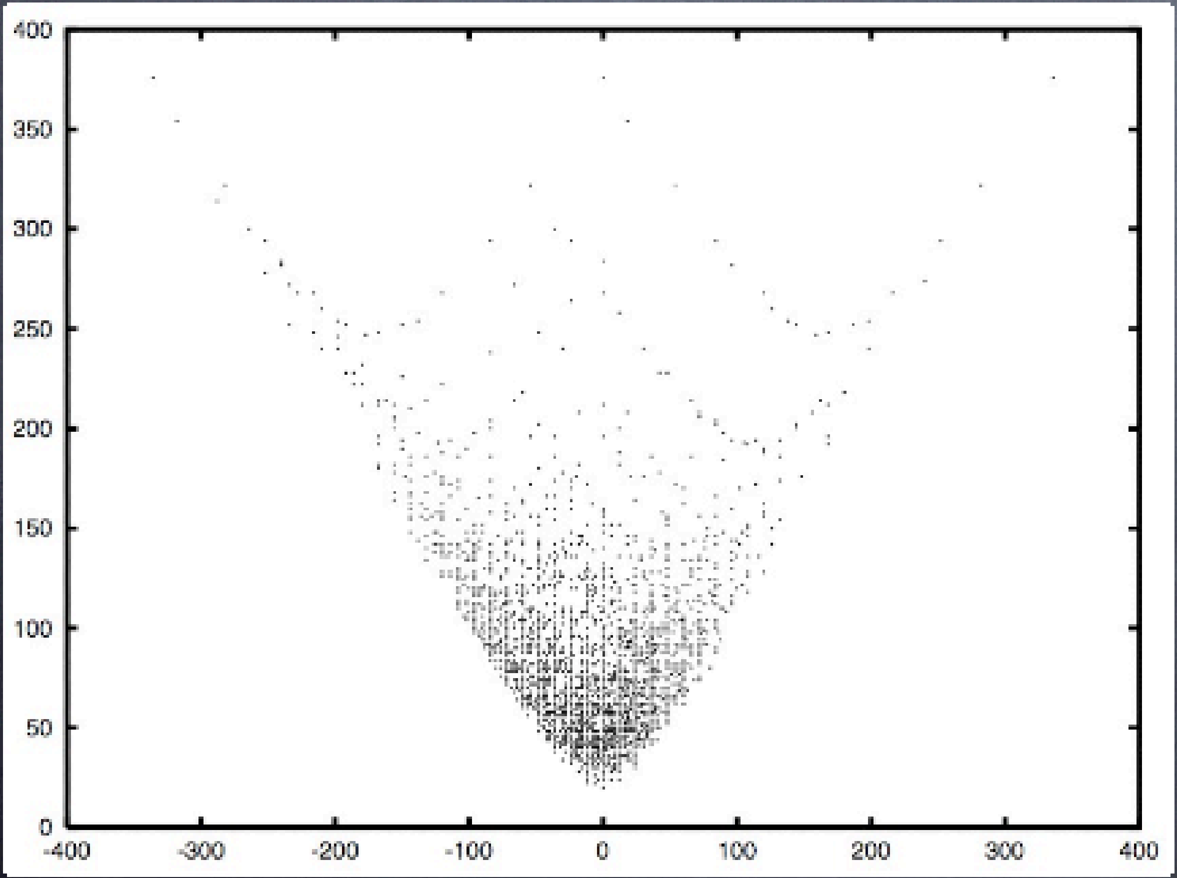
Mirror symm'

==> symm' across vert' axis

-- numerical evidence for mirror symmetry

Numerical tests of (0,2) mirror symmetry

Shown are CY 3-folds + bundles:



Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^\vee)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^\vee)$

where $\mathcal{E}$ is rk 4

-- numerical evidence for (0,2) mirrors

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

How to find mirrors?

One of the original methods:

"Greene-Plesser orbifold construction"

Idea: orbifold a hypersurface by automorphisms.

Example: quintic near Fermat point

$$Q_5 \subset \mathbf{P}^4 \quad \xleftrightarrow{\text{mirror}} \quad \widetilde{Q_5 / \mathbf{Z}_5^3}$$

(only useful for special cpx structures,
ie, near Fermat)

How to find mirrors?

(0,2) analogue of Greene-Plesser exists:

(Blumenhagen, Sethi, hepth/9611172)

Example: $$X = \mathbf{P}^5_{[1,1,1,1,2,2]}[4,4]$$

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_1^5 \mathcal{O}(1) \longrightarrow \mathcal{O}(5) \longrightarrow 0$$

is mirror to a $\mathbf{Z}_5$ orbifold

How to find mirrors?

Berglund - Hubsch transpositions

-- an attempt to get away from Fermat points

$$x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n} \leftrightarrow A \equiv \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_2 & 1 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & 1 \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

$$A^T \leftrightarrow y_n^{a_n} + \cdots + y_2^{a_2} y_3 + y_1^{a_1} y_2$$

Open problem: no known (0,2) analogue

How to find mirrors?

Batyrev's construction:

For a hypersurface in a toric variety,
mirror symmetry exchanges

polytope of ambient toric variety	$\longleftrightarrow$	dual polytope, for ambient t.v. of mirror

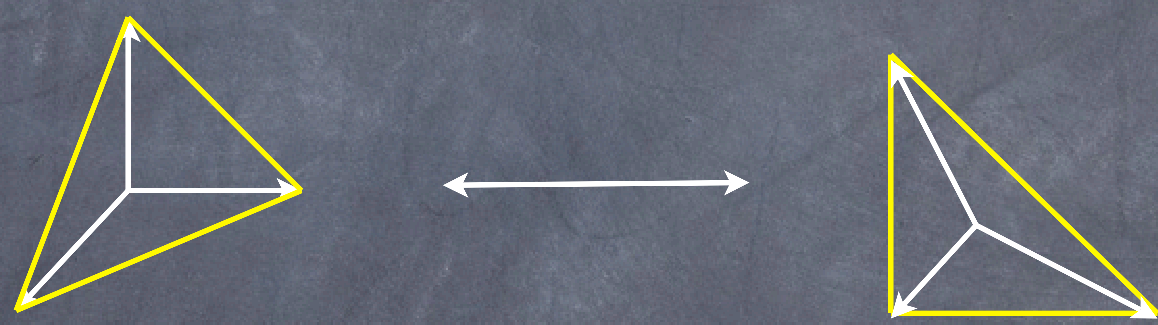
(generalization to CI exists,
but here I'll only describe hypersurfaces)

How to find mirrors?

Example of Batyrev's construction:

T^2 as deg 3 hypersurface in $\mathbf{P}^2$

$\mathbf{P}^2$:



$= \mathbf{P}^2/\mathbf{Z}_3$

$$P^0 = \{y \,|\, \langle x, y\rangle \geq -1 \,\forall x \in P\}$$

Result: deg 3 hypersurface in $\mathbf{P}^2$

mirror to

$\mathbf{Z}_3$ quotient of deg 3 hypersurface

(Greene, Plesser)

How to find mirrors?

Open problem:
No known (0,2) analogue of Batyrev's construction.

Speculation: Could there exist suitable polytopes for T-equivariant bundles on toric varieties?

Can describe such bundles by generalizing fans: associate a filtration of a fixed vector space to each toric divisor. (Klyachko)

Question: in any special cases, can one associate a polytope?

How to find mirrors?

In add'n, Batyrev's construction is only known in (2,2) cases for spaces, not stacks.

Evidence for existence of analogue for stacks:

* analogues of fans known for toric stacks; involve eg element of cyclic group assigned to edges

* mirrors to stacks often involve eg $\mathbf{Z}_N$-valued fields as products in polynomials

-- ingredients present, but never been assembled

How to find mirrors?

Monomial-divisor mirror map

(Aspinwall, Greene, Morrison, alg-geom/9309007)

-- a refinement of Batyrev's construction that maps specific cpx moduli to specific Kahler moduli

Open problem: no known (0,2) analogue

How to find mirrors?

Periods, Picard-Fuchs equations

Open problem: no known (0,2) analogue

How to find mirrors?

Gauged linear sigma models (GLSMs): (Witten, '93)

An extremely useful technology, still studied today, which made much progress possible.

(0,2) GLSMs **do** exist, and made many of the computations I'll describe possible.

(Distler, Kachru, '94)

How to find mirrors?

Hori-Vafa construction ((2,2) case):

Briefly, maps all cpx moduli to a single Kahler modulus point on the mirror.
(Sends A twist to B twist.)

Ex: $[\mathbf{C}^n // \mathbf{C}^\times]$

Build a LG model with

$$W = \exp(-Y_1) + \cdots + \exp(-Y_{n-1}) + q\exp(q_1 Y_1 + \cdots + q_{n-1} Y_{n-1})$$

How to find mirrors?

Hori-Vafa construction ((2,2) case)

-- (0,2) analogue **does** exist

(Adams, Basu, Sethi, hepth/0309226)

Current state of the art (I believe)
is that the construction doesn't quite uniquely determine the mirror,
instead one must do a bit of work at the end to nail down details.

How to find mirrors?

Example:

Take $X = \mathbf{P}^1 \times \mathbf{P}^1$

with $\mathcal{E}$ a deformation of the tangent bundle:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{\left[\begin{array}{cc} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{array}\right]} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

ABS predicted ``heterotic quant' cohom':"

$$\begin{aligned} \tilde{X}^2 &= q_2 \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= q_1 \end{aligned}$$

(a def' of the std q.c. ring of $\mathbf{P}^1 \times \mathbf{P}^1$)

ABS's predictions have since been verified and put on a more solid mathematical footing.

(ES, Katz, Sethi, Basu, Guffin, Melnikov, Adams, Distler)

The ``heterotic quantum cohomology'' rings are a deformation of classical product structures on the sheaf cohomology groups

$$H^{\cdot}(X, \Lambda^{\cdot} \mathcal{E}^{\vee})$$

``quantum sheaf cohomology''

(Combine minimal-area curves & gauge instantons.)

Quantum sheaf cohomology arises from correlation functions in a heterotic generalization of the A model TFT.

Std A twist:

$$\psi_-^i (\equiv \chi^i) \in \Gamma((\phi^* T^{0,1} X)^\vee) \qquad \psi_+^i (\equiv \psi_z^i) \in \Gamma(K \otimes \phi^* T^{1,0} X)$$
$$\psi_-^{\bar{\imath}} (\equiv \psi_{\bar{z}}^{\bar{\imath}}) \in \Gamma(K \otimes \phi^* T^{0,1} X) \qquad \psi_+^{\bar{\imath}} (\equiv \chi^{\bar{\imath}}) \in \Gamma((\phi^* T^{1,0} X)^\vee)$$

(0,2) A twist:

$$\lambda_-^a \in \Gamma((\phi^* \overline{\mathcal{E}})^\vee) \qquad \psi_+^i \in \Gamma(K \otimes \phi^* T^{1,0} X)$$
$$\lambda_-^{\bar{b}} \in \Gamma(K \otimes \phi^* \overline{\mathcal{E}}) \qquad \psi_+^{\bar{\imath}} \in \Gamma((\phi^* T^{1,0} X)^\vee)$$

Consistency conditions

(0,2) A model: $(X, \mathcal{E})$ s.t. $\Lambda^{\rm top} \mathcal{E}^{\vee} \cong K_X$
$${\rm ch}_2(\mathcal{E}) = {\rm ch}_2({\rm TX})$$

(On (2,2) locus, this is automatic.)

There is also a (0,2) B model:

(0,2) B model: $(X, \mathcal{E})$ s.t. $\Lambda^{\rm top} \mathcal{E} \cong K_X$
$${\rm ch}_2(\mathcal{E}) = {\rm ch}_2({\rm TX})$$

(On (2,2) locus, reduces to standard $K_X^{\otimes 2} \cong \mathcal{O}_X$)

Symmetry: (0,2) A $(X, \mathcal{E})$ = (0,2) B $(X, \mathcal{E}^{\vee})$
(modulo regularization issues)

Back to A twist.
Correlation functions in `standard' cases:

Std A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1,q_1}(\mathcal{M}_d) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M}_d)$$

$$= \sum_d \int_{\mathcal{M}_d} (\mathrm{top-form})$$

(0,2) A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1}\left(\mathcal{M}_d, \Lambda^{q_1} \mathcal{F}^\vee\right) \wedge \cdots \wedge H^{p_m}\left(\mathcal{M}_d, \Lambda^{q_m} \mathcal{F}^\vee\right)$$

where $\mathcal{F} \equiv R^0 \pi_* (\alpha^* \mathcal{E})$

Use $\left. \begin{array}{c} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$

If there are `excess' zero modes, must work a little harder:

Std A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1,q_1}(\mathcal{M}_d) \wedge \cdots \wedge H^{p_m,q_m}(\mathcal{M}_d) \wedge \mathrm{Eul}(\mathrm{Obs})$$

$$= \sum_d \int_{\mathcal{M}_d} (\mathrm{top} - \mathrm{form})$$

(0,2) A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1}(\Lambda^{q_1} \mathcal{F}^\vee) \wedge \cdots \wedge H^{p_m}(\Lambda^{q_m} \mathcal{F}^\vee) \wedge H^n(\Lambda^n \mathcal{F}^\vee \otimes \Lambda^n \mathcal{F}_1 \otimes \Lambda^n (\mathrm{Obs})^\vee)$$

Use $\left. \begin{array}{c} \Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \Lambda^{\mathrm{top}} \mathcal{F}^\vee \otimes \Lambda^{\mathrm{top}} \mathcal{F}_1 \otimes \Lambda^{\mathrm{top}} (\mathrm{Obs})^\vee \cong K_{\mathcal{M}}$

Reduces to (2,2) by virtue of Atiyah classes.

Quantum sheaf cohomology

In computing ordinary quantum cohomology rings, tech issues such as compactifying moduli spaces of holomorphic maps into a cpx manifold arise.

In the heterotic case, there are also sheaves $\mathcal{F}$ over those moduli spaces, which have to be extended over the compactification, in a way consistent with e.g.

$$\Lambda^{\rm top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

But, this can be done....

Quantum sheaf cohomology

Need to not only compactify, but also extend induced sheaves, so as to preserve properties eg

$$\Lambda^{\mathrm{top}} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

* If the moduli space admits `universal instanton,' automatic.

* LSM moduli spaces do not.

But, abelian GLSMs naturally provide suitable sheaves over the moduli space regardless.

(Expand Fermi fields in zero modes.)

Quantum sheaf cohomology

Open problems:

-- What is analogue for nonabelian GLSM's?
For Grassmannians, flag manifolds,
w/ (0,2) bundle **or** homogeneous bundle?

-- What are analogues for Klyachko's T-equivariant bundles on toric varieties?

Quantum sheaf cohomology

ABS Example:

Take $X = \mathbf{P}^1 \times \mathbf{P}^1$

with $\mathcal{E}$ a deformation of the tangent bundle:

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \stackrel{\left[\begin{array}{cc} x_1 & \epsilon_1 x_1 \\ x_2 & \epsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{array} \right]}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

Can use methods outlined to verify ABS:

$$\begin{aligned} \tilde{X}^2 &= q_2 \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= q_1 \end{aligned}$$

Quantum sheaf cohomology

Strictly speaking, all I've outlined is a computation of nonpert' corrections to certain special correlators.
-- do the OPE's close into a ring?

Ordinarily argue using (2,2) susy, but only (0,2) here.

It has been shown that CFT + (0,2) is sufficient for the OPE's to close properly, so one does get a ring, at least for def's of tangent bundle, rk < 8.

(Adams, Distler, Ernebjerg, hepth/0506263)

Quantum sheaf cohomology

More recent developments
-- more complicated examples, etc
-- in recent work of
Guffin, McOrist, Melnikov, Sethi

(will be reported on in their talks)

Quantum sheaf cohomology

Alternative applications:

There exists a rewriting of Witten's twistor string theory in terms of heterotic strings, which uses precisely this technology.

$$X = \mathbf{P}^3, \quad \mathcal{E} = \mathcal{O}(1)^{\oplus 4}$$

(Mason, Skinner, 0708.2276)

Open problems:

* Need Pfaffians for higher-genus computations (all existing computations are genus zero)

* Then, couple to worldsheet gravity.

This would enable us to truly generalize Gromov-Witten theory.

Stability

To get a CFT, the heterotic bundle + connection must satisfy

DUY: $$g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0$$

equiv'ly, Mumford-Takemoto stability

(at least, close to large radius)

-- explicit metric dependence

-- Kahler cone breaks up into subcones

-- D-terms in low energy gauge theory

Open problems in stability:

-- What are the quantum corrections?
Is there an analogue of Douglas's pi-stability ansatz?

(Partial results: Anderson, Gray, Lukas, Ovrut, 0905.1748, 0903.5088)

-- How does quantum sheaf cohomology change as cross subcone walls?

(``heterotic flop")

Strominger-Yau-Zaslow

There is, at least very formally, an obvious extension of SYZ to heterotic cases:
describe a bundle over torus fibration in terms of flat connection on torus fibers,
then, apply heterotic T-duality fiberwise.

Open problem:
Mark Gross and others have done a lot of work on understanding SYZ in the (2,2) case;
can any of it be extended to (0,2) ?

Another approach to (ordinary) mirrors:

Lift spaces to (UV) LG models,
and then construct mirror symmetry as a duality between LG models.

(P Clarke, 0803.0447)

I'll outline the idea over the next several slides....

Example of lifting to LG:

LG model on X = Tot($\mathcal{E}^\vee \stackrel{\pi}{\longrightarrow} B$)

with W = p π^*s

↓ renormalization group flow

string on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

Computational advantages:

For example, consider curve-counting in a deg 5 (quintic) hypersurface in $\mathbf{P}^4$
-- need moduli space of curves in quintic, rather complicated

Can replace with LG model on

$$\mathrm{Tot}\left(\mathcal{O}(-5) \to \mathbf{P}^4\right)$$

and here, curve-counting involves moduli spaces of curves on $\mathbf{P}^4$, much easier

(Kontsevich: early '90s; physical LG realization: ES, Guffin, '08)

Application to mirror symmetry:

Instead of directly dualizing spaces,
replace spaces with corresponding LG models,
and dualize the LG models.

(P Clarke, '08)

* Resulting picture is often easier to understand
* Technical advantage: also encapsulates cases in which mirror isn't an ordinary space (but still admits a LG description)

There also exist heterotic LG models:

* a space X
* a holomorphic vector bundle $\mathcal{E} \rightarrow X$
  (satisfying same constraints as before)
* some potential-like data:

$$E^a \in \Gamma(\mathcal{E}), \quad F_a \in \Gamma(\mathcal{E}^\vee)$$

such that $\sum_a E^a F_a = 0$

(Recover ordinary LG when $\mathcal{E} = TX$,

$E^a \equiv 0$ and $F_i = \partial_i W$)

Heterotic LG models are related to heterotic strings via renormalization group flow.

Example:

A heterotic LG model on $X = \mathrm{Tot}\left(\mathcal{F}_1 \xrightarrow{\pi} B\right)$

with $\mathcal{E}' = \pi^* \mathcal{F}_2$ & $F_a \equiv 0, \quad E^a \neq 0$

↓ Renormalization group

A heterotic string on B

with $\mathcal{E} = \mathrm{coker}\left(\mathcal{F}_1 \longrightarrow \mathcal{F}_2\right)$

Open problem:

Can (0,2) mirrors be constructed as a duality between LG models on different spaces, generalizing P Clarke's construction?

Something else to ponder....

Heterotic flux compactifications

Following Strominger's ancient paper,
one would like to consider heterotic compactifications
on complex, **non**-Kahler spaces with
trivial canonical bundle.

Many papers have been written by e.g. Becker^2.

Basic issue: cannot go to large-radius limit,
these can only exist for finite radius,
where no control over quantum corrections,
and math may or may not be valid.

Heterotic flux compactifications

Partial progress by A Adams et al
-- build using analogues of GLSM's.

ie, have a UV theory can control,
and then RG fixes the details appropriately.

Open problems:

* How do (0,2) mirrors work here?
* Could this construction help in building mirrors?

Could thinking about fibered WZW models help?

Let P be a principal G bundle over X,
with connection A.

Replace the left-movers of ordinary heterotic with WZW model with left-multiplication gauged with A.

NLSM on X

$$\frac{1}{\alpha'}\int_\Sigma \left(g_{i\bar{\jmath}} \partial_\alpha \phi^i \partial^\alpha \phi^{\bar{\jmath}} + \cdots \right)$$

$$- \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}\ \left(g^{-1}\partial g g^{-1}\overline{\partial} g \right) - \frac{ik}{12\pi}\int_B d^3 y \epsilon^{ijk}\mathrm{Tr}\ \left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g \right)$$

WZW

$$- \frac{k}{2\pi}\mathrm{Tr}\ \left((\partial\phi^\mu) A_\mu \overline{\partial} g g^{-1} + \frac{1}{2}(\partial\phi^\mu \overline{\partial}\phi^\nu) A_\mu A_\nu \right)$$

Gauge left-multiplication

Could thinking about fibered WZW models help?

Result is a fibered current algebra.

If at level k, then anomaly cancellation becomes

$$k\,\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX)$$

Such constructions needed to realize many $E_8 \times E_8$ gauge fields.

Open problems:

* How do (0,2) mirrors work here?
* Could this construction help in building mirrors?

UNDER CONSTRUCTION

Potential new heterotic CFT's: heterotic strings on gerbes

Prototype: $\mathcal{O}(1) \longrightarrow \mathbf{P}_{[k,k,\cdots,k]}$ "$\mathcal{O}(1/k)$"

-- understanding of some of the 2d (0,4) theories appearing in geometric Langlands program

-- genuinely new string compactifications

String duality

Open problem:

What are string duals of (0,2) mirrors?

Ex: Heterotic - type II exchanges $\alpha' \leftrightarrow \phi$
so is this some symmetry of D-branes?

Summary

-- overview of (0,2) mirrors

-- numerical evidence

-- constructions: Greene-Plesser, Berglund-Hubsch,
Batyrev-Borisov, Hori-Vafa

-- quantum sheaf cohomology

-- the computations

-- (0,2) A, B models

-- stability

-- lifting to LG

-- SYZ, heterotic flux compactifications,
fibered WZW's, strings on gerbes