Recent developments in 2d (0,2) theories

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Geometry in Gauge Theories and String Theory

Over the last decade, there's been a *tremendous* amount of progress in perturbative string compactifications.

A few of my favorite examples:

- nonpert' realizations of geometry (Pfaffians, double covers)
  (Hori-Tong '06, Caldararu et al '07,…)
- perturbative GLSM's for Pfaffians (Hori '11, Jockers et al '12,…)
- non-birational GLSM phases - physical realization of homological projective duality (Hori-Tong '06, Caldararu et al '07, Ballard et al '12; Kuznetsov '05-'06,…)
- examples of closed strings on noncommutative res'ns
  (Caldararu et al '07, Addington et al '12, ES '13)
- localization techniques: new GW & elliptic genus computations, role of Gamma classes, …
  (Benini-Cremonesi '12, Doroud et al '12; Jockers et al '12, Halverson et al '13, Hori-Romo '13, Benini et al '13, ….)
- heterotic strings: nonpert' corrections, 2d dualities/trialities, non-Kahler moduli (many)

I'll focus today on just one….

Today I'll restrict to

- heterotic strings: nonpert' corrections, 2d dualities/trialities, non-Kahler moduli

My goal today: an overview of progress towards computing worldsheet instanton effects, one of the outstanding problems in perturbative heterotic string compactifications.

Briefly, we need to generalize instanton corrections and mirror symmetry to heterotic theories, and some progress has been made.

Review gen'l aspects next….

Some background.

In 10d, a heterotic string describes metric & gauge field.

To compactify, must specify not only a space X,
but also a bundle $\mathcal{E}$ on that space,
satisfying consistency conditions

$$[\mathrm{tr}\, F \wedge F] \;=\; [\mathrm{tr}\, R \wedge R]$$

Described on worldsheet by 2d (0,2) susy theory.

Simplest case: $\mathcal{E} = TX$, corresponding to (2,2) susy.
"embed the spin connection in gauge connection"

Simplest case: compactification on a Calabi-Yau with gauge bundle = tangent bundle
(`embedding the spin connection' = (2,2) locus)

In this case, we know basics:

- massless states (inc. moduli)

— counted by cohomology of the CY; `chiral ring'

- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

Nonperturbative corrections = GW inv'ts

$\overline{\mathbf{27}}^3$ = A model TFT computation

$\mathbf{27}^3$ = B model TFT computation

More gen'l case: compactification on a Calabi-Yau with gauge bundle $\neq$ tangent bundle

(Worldsheet has (0,2) susy.)

- massless states (inc. moduli)

— counted by bundle-valued forms on the CY

- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

Nonperturbative corrections $\neq$ GW inv'ts

$\overline{\mathbf{27}}^3$ = A/2 model computation

$\mathbf{27}^3$ = B/2 model computation

- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

Nonperturbative corrections $\neq$ GW inv'ts

$\overline{\mathbf{27}}^3$ = A/2 model computation

$\mathbf{27}^3$ = B/2 model computation

But we know far less for (0,2) than for (2,2)!

Understanding these nonperturbative corrections is a central issue in perturbative heterotic strings on CYs.

quantum sheaf cohomology

(0,2) mirror symmetry

generalizing ordinary quantum cohomology & mirror symmetry.

My goal today is to survey WIP on applying susy localization to (twisted) (0,2) theories to derive nonperturbative contributions to Yukawa couplings.

- Survey of couplings and open problems
- A/2, B/2 pseudo-topological field theories
- Susy localization in A/2 model for def's of (2,2) theories
- Examples: $\mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{F}_n, \quad G(k,n)$

  — new expressions for old results: JK residues

  — new results: nonabelian GLSM's
- Analogous computations in dual B/2 theories

First, before computing Yukawa couplings,
how to count massless states?

Let X be Calabi-Yau 3-fold, for simplicity.

Let $\mathcal{E}$ denote the gauge bundle.

(Distler-Greene '88)

Rank 3 bundle: low-energy $E_6 \times E_8$

$$\overline{\mathbf{27}} \sim H^1(X, \mathcal{E}^*) \qquad \mathbf{27} \sim H^1(X, \mathcal{E})$$

(2,2): $\sim H^1(X, T^*X)$ Kähler $\qquad \sim H^1(X, TX)$ complex

Rank 4 bundle: low-energy $\mathrm{Spin}(10) \times E_8$

$$\mathbf{16} \sim H^1(X, \mathcal{E}) \qquad \mathbf{10} \sim H^1(X, \wedge^2 \mathcal{E})$$

Rank 5 bundle: low-energy $SU(5) \times E_8$

$$\mathbf{10} \sim H^1(X, \mathcal{E}) \qquad \overline{\mathbf{5}} \sim H^1(X, \wedge^2 \mathcal{E})$$

What are their couplings?

What are their couplings?

Suppose bundle is rank 3, for simplicity,
so that we have low-energy $E_6 \times E_8$.

$$\overline{\mathbf{27}}^3 = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \mathcal{O}(q) \quad \text{where} \quad \omega_i \in H^1(X, \mathcal{E}^*)$$

No perturbative loop corrections, but there are nonperturbative corrections.
(Dine-Seiberg-Wen-Witten '86)

Example: (2,2) quintic

(Candelas, de la Ossa, Green, Parkes, '91)

$$\overline{\mathbf{27}}^3 = \underbrace{5}_{\text{intersection number}} + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1-q^k} = 5 + 2875 q + 4876875\, q^2 + \cdots$$

intersection number
(Strominger '85)

nonperturbative contributions

n_k = Gromov-Witten invariants

How to compute nonperturbative corrections in (0,2) cases?

How to compute nonperturbative corrections in (0,2) cases?

Historically, on the (2,2) locus,
used mirror symmetry.

For (0,2), would need a generalization called
(0,2) mirror symmetry.

Some results do exist — state of the art is a version of Batyrev's mirror map due to Melnikov-Plesser '10 — but we have not yet worked out analogue of flat coordinates or how to compute nonperturbative corrections using (0,2) mirrors alone.

We'll do this directly instead….

How to compute nonperturbative corrections in (0,2) cases?

It's convenient to work in an analogue of a TFT.

On (2,2) locus,

$$\overline{\mathbf{27}}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\rm phys} \;=\; \langle V^3 \rangle_{\rm A\,TFT}$$

$$\mathbf{27}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\rm phys} \;=\; \langle V^3 \rangle_{\rm B\,TFT}$$

& the TFT expressions are convenient for computations.

There are analogues for more general (0,2) theories;
these are A/2, B/2 pseudo-TFT's, which also have the property

$$\overline{\mathbf{27}}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\rm phys} \;=\; \langle V^3 \rangle_{\rm A/2\,TFT}$$

$$\mathbf{27}^3 \;=\; \langle V_f^{\mathbf{16}} V_b^{\mathbf{10}} V_f^{\mathbf{16}} \rangle_{\rm phys} \;=\; \langle V^3 \rangle_{\rm B/2\,TFT}$$

The A/2, B/2 pseudo-TFT's

These (0,2) NLSM's have two anomalous global U(1)'s:

- a right-moving $U(1)_R$
- a canonical left-moving U(1), rotating the phase of all left fermions, which becomes $U(1)_L$ on (2,2) locus

If $\det \mathcal{E}^{\pm 1} \cong K_X$, then a nonanomalous U(1) exists along which we can twist right & left moving fermions.

There are two distinct possibilities, which on (2,2) locus become the A, B model TFT's, and are called the A/2, B/2 models.

A little more explicitly:

(0,2) NLSM has Lagrangian density

$$\mathcal{L} \;=\; g_{i\bar{\jmath}}\partial\phi^{\bar{\jmath}}\overline{\partial}\phi^{i} \;+\; i g_{i\bar{\jmath}}\psi_{+}^{\bar{\jmath}} D_{-}\psi_{+}^{i} \;+\; i h_{a\bar{b}}\lambda_{-}^{\bar{b}} D_{+}\lambda_{-}^{a} \;+\; F_{i\bar{\jmath}a\bar{b}}\psi_{+}^{i}\psi_{+}^{\bar{\jmath}}\lambda_{-}^{a}\lambda_{-}^{\bar{b}}$$

$$\psi_{+} \;\sim\; TX \qquad\qquad \lambda_{-} \;\sim\; \mathcal{E}$$

subject to Green-Schwarz condition: $\mathrm{ch}_2(TX) = \mathrm{ch}_2(\mathcal{E})$

A/2 twist: take $\psi_{+}^{i}, \lambda_{-}^{\overline{a}}$ to be scalars

B/2 twist: take $\psi_{+}^{\bar{\imath}}, \lambda_{-}^{\overline{a}}$ to be scalars

so we get a scalar half of susy — but this BRST operator is purely right-moving, so this not a standard TFT.

In order for this twist to be anomaly-free, there are constraints..

A/2 model: Exists when $(\det \mathcal{E})^{-1} \cong K_X$

(on (2,2) locus, always possible; reduces to A model)

States: $H^{\bullet}\left(X, \wedge^{\bullet}\mathcal{E}^*\right)$

B/2 model: Exists when $\det \mathcal{E} \cong K_X$

(on (2,2) locus, requires $K_X^{\otimes 2} \cong \mathcal{O}_X$; reduces to B model)

States: $H^{\bullet}\left(X, \wedge^{\bullet}\mathcal{E}\right)$

Exchanging $\mathcal{E} \leftrightarrow \mathcal{E}^*$ swaps the A/2, B/2 models.

(Physically, just a complex conjugation of left movers.)

What do the A, A/2 model correlation functions look like?

Classical contributions, schematically:

A model: Classical contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \int_X \omega_1 \wedge \cdots \wedge \omega_n \;=\; \int_X (\mathrm{top} - \mathrm{form})$$

$$\omega_i \in H^{p_i, q_i}(X)$$

A/2 model: Classical contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \int_X \omega_1 \wedge \cdots \wedge \omega_n \qquad \omega_i \in H^{q_i}\left(X, \wedge^{p_i} \mathcal{E}^*\right)$$

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\mathrm{top}}(X, \wedge^{\mathrm{top}} \mathcal{E}^*) = H^{\mathrm{top}}(X, K_X)$

using the anomaly constraint $\det \mathcal{E}^* \cong K_X$

Again, a top form, so get a number.

What do the A, A/2 model correlation functions look like?

Instanton sectors, schematically:

A model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \;=\; \int_{\mathcal{M}} (\mathrm{top} - \mathrm{form})$$

$$\omega_i \in H^{p_i, q_i}(\mathcal{M})$$

where $\mathcal{M}$ is moduli space of worldsheet instantons.

A/2 model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \qquad \omega_i \in H^{q_i}(\mathcal{M}, \wedge^{p_i} \mathcal{F}^*)$$

where $\mathcal{F}$ is sheaf on $\mathcal{M}$ induced by $\mathcal{E}$.

Now, $\quad \omega_1 \wedge \cdots \wedge \omega_n \in H^{\mathrm{top}}(\mathcal{M}, \wedge^{\mathrm{top}} \mathcal{F}^*)$

so need to explain how to get top-form etc….

What do the A, A/2 model correlation functions look like?

To actually define A model correlation functions,
need to compactify $\mathcal{M}$.

To actually define A/2 model correlation functions,
need to not only compactify $\mathcal{M}$,
but also extend $\mathcal{F}$ over compactification divisor,
consistent with symmetries.

Then, formally, get a top-form so long as no anomalies:

$$\left. \begin{array}{c} \wedge^{\rm top} \mathcal{E}^* \cong K_X \\ {\rm ch}_2(\mathcal{E} = {\rm ch}_2(TX) \end{array} \right\} \stackrel{\rm GRR}{\Longrightarrow} \wedge^{\rm top} \mathcal{F}^* \cong K_{\mathcal{M}}$$

All of this has been done.

(Katz-ES hepth/0406226, …..,
Donagi-Guffin-Katz-ES 1110.3751, .3752)

Today I'll focus on susy localization computations.

Susy localization

I'll first discuss A/2 theories obtained by deforming off the (2,2) locus, generalizing A model susy localization described in

Benini-Zaffaroni 1504.03698 Closset-Cremonesi-Park 1504.06308

Corresponding (0,2) GLSM's will have a Coulomb branch, along which we shall work.

Schematically, correlation functions take general form

$$\langle f(\sigma) \rangle \; = \; \sum_{\mathfrak{m} \in \mathbb{Z}} \mathrm{JK} - \mathrm{Res}_{\sigma = 0} \left\{ Z^{1-\mathrm{loop}} q^{\mathfrak{m}} f(\sigma) \right\}$$

for $$Z^{1-\mathrm{loop}} \; = \; \frac{\det \mathcal{O}_{\mathrm{fermi}}}{\det \mathcal{O}_{\mathrm{bose}}}$$

Susy localization

$$Z^{1-\mathrm{loop}} \;=\; \frac{\det \mathcal{O}_{\mathrm{fermi}}}{\det \mathcal{O}_{\mathrm{bose}}}$$

For deformations off the (2,2) locus, in a GLSM,
ψ_+, ψ_- have same gauge charges.

Fermi interactions: $\overline{\psi}_-^i \psi_+^j E_i^j \;+\; \overline{\psi}_+^j \psi_-^i (E_i^j)^*$

$$\mathcal{O}_{\mathrm{fermi}} \;=\; \left[\begin{array}{ccccc} E_1^1 & D_+ & E_1^2 & 0 & \cdots \\ D_- & (E_1^1)^* & 0 & (E_2^1)^* & \cdots \\ E_2^1 & 0 & E_2^2 & D_+ & \cdots \\ 0 & (E_1^2)^* & D_- & (E_2^2)^* & \cdots \\ \vdots & & & & \ddots \end{array} \right]$$

$$\det \mathcal{O}_{\mathrm{fermi}} \;=\; (S(\det E))^{|b+1|} \prod_{n \geq 1} \left[\sum_{k=0}^{N} t_n^{2k} \left(\sum_{i_1 < i_2 < \cdots i_k, j_1 < j_2 < \cdots j_k} \left| \tilde{E}_{i_1 \cdots i_k j_1 \cdots j_k} \right|^2 \right) \right]^{2n+|b+1|}$$

where $\quad t_n \;=\; n(n + |b+1|) \qquad\qquad b \;=\; Q(\mathfrak{m})$

Susy localization

$$Z^{1-\text{loop}} = \frac{\det \mathcal{O}_{\text{fermi}}}{\det \mathcal{O}_{\text{bose}}}$$

Bosonic potential:

$$|E^i(\phi)|^2 = \sum_i \left(\sum_j |E^i_j|^2 |\phi^j|^2 \right) + \sum_{i \neq j} \left(\sum_k (E^k_i)^* E^k_j \right) \overline{\phi}^{\bar{\imath}} \phi^j$$

$$\mathcal{O}_{\text{bose}} = \begin{bmatrix} -D^2 + |E^1_1|^2 + \cdots + |E^N_1|^2 & (E^1_1)^* E^1_2 + \cdots + (E^N_1)^* E^N_2 & \cdots \\ E^1_1 (E^1_2)^* + \cdots + E^N_1 (E^N_2)^* & -D^2 + |E^1_2|^2 + \cdots + |E^N_2|^2 & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

$$\det \mathcal{O}_{\text{bose}} = \prod_{n \geq 0} \left[\sum_{k=0}^{N} t_n^{2k} \left(\sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{E}_{i_1 \cdots i_k j_1 \cdots j_k} \right|^2 \right) \right]^{2n + |b| + 1}$$

where $\qquad t_n = \frac{1}{2} \left(2n(n+1) + (2n+1)|b| - b \right)$

Susy localization

Putting this together, can show

$$Z^{1-\text{loop}} = \frac{\det \mathcal{O}_{\text{fermi}}}{\det \mathcal{O}_{\text{bose}}} = \left(\frac{1}{\det E}\right)^{Q(\mathfrak{m})+1}$$

so *schematically* correlation functions take form

$$\begin{aligned}\langle f(\sigma)\rangle &= \sum_{\mathfrak{m}\in\mathbb{Z}} \mathrm{JK}-\mathrm{Res}_{\sigma=0}\left\{Z^{1-\text{loop}} q^{\mathfrak{m}} f(\sigma)\right\} \\ &= \sum_{\mathfrak{m}\in\mathbb{Z}} \mathrm{JK}-\mathrm{Res}_{\sigma=0}\left\{\left(\frac{1}{\det E}\right)^{Q(\mathfrak{m})+1} q^{\mathfrak{m}} f(\sigma)\right\}\end{aligned}$$

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Build a (0,2) theory that deforms (2,2) model.

Math: $$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$x, \tilde{x}$ vectors of homogeneous coordinates,

A, B, C, D 2×2 matrices describing deformation

(2,2) locus: $A = D = I_{2\times 2}, \quad B = C = 0$

Physics….

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Build a (0,2) theory that deforms (2,2) model.

Physics:

$x^i, \tilde{x}^i$ chiral superfields charge (1,0), (0,1)

$\Lambda^i, \tilde{\Lambda}^i$ Fermi superfields charge (1,0), (0,1) s.t.

$$\overline{D}_+ \Lambda^i \;=\; A^i_j \sigma x^j + B^i_j \tilde{\sigma} x^j \qquad \overline{D}_+ \tilde{\Lambda}^i \;=\; C^i_j \sigma \tilde{x}^j + D^i_j \tilde{\sigma} \tilde{x}^j$$

σ neutral (adj-valued) chiral superfield

no superpotential

On (2,2) locus, x^i, Λ^i combine into (2,2) chiral superfields,
$\tilde{x}^i, \tilde{\Lambda}^i$ combine into (2,2) chiral superfields, and
σ part of (2,2) vector multiplet.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Localization computation: (genus zero)

$$\langle f(\sigma, \tilde{\sigma}) \rangle \; = \; \sum_{\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{Z}} \mathrm{JK} - \mathrm{Res}_{\sigma = \tilde{\sigma} = 0} \left\{ \left(\frac{1}{\det E} \right)^{\mathfrak{m}_1 + 1} \left(\frac{1}{\det \tilde{E}} \right)^{\mathfrak{m}_2 + 1} q^{\mathfrak{m}_1} \tilde{q}^{\mathfrak{m}_2} f(\sigma, \tilde{\sigma}) \right\}$$

for $\quad E = A\sigma + B\tilde{\sigma}, \quad \tilde{E} = C\sigma + D\tilde{\sigma}$

Note: Looks like a TFT result — no propagators, no worldsheet position dependence — but this is not quite TFT.

“Non-topological TFT”

How can that be?

“Non-topological TFT”

The basic reason we’re getting a TFT-like structure, albeit not in an actual TFT, is that the OPE’s close on dim zero A/2 op’s.

(Adams-Distler-Ernebjerg ’05) argued that e.g. in an open patch on moduli space containing (2,2) locus, the OPE’s of the A/2 model operators close into other A/2 model operators.

For conformal cases, combination of

- right-moving N=2 algebra to bound dimensions
- worldsheet conformal invariance to relate left, right dim’s

to argue closure on patches.

Since operators have dim’ zero, & OPE’s close, no worldsheet dependence in correlation functions.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Let's take another look at the result:

$$\langle f(\sigma,\tilde{\sigma})\rangle \;=\; \sum_{\mathfrak{m}_1,\mathfrak{m}_2\in\mathbb{Z}} \mathrm{JK}-\mathrm{Res}_{\sigma=\tilde{\sigma}=0}\left\{\left(\frac{1}{\det E}\right)^{\mathfrak{m}_1+1}\left(\frac{1}{\det \tilde{E}}\right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1}\tilde{q}^{\mathfrak{m}_2} f(\sigma,\tilde{\sigma})\right\}$$

for $\quad E = A\sigma + B\tilde{\sigma}, \quad \tilde{E} = C\sigma + D\tilde{\sigma}$

Inserting a factor of, say, $\det E$ in the correlation f'n is equivalent to shifting q.

Quantum sheaf cohomology ring rel'ns:

$$\det E \;=\; q, \quad \det \tilde{E} \;=\; \tilde{q}$$

This result already known (for all toric varieties w/ def's):

Physics: McOrist-Melnikov 0810.0012 Math: Donagi-Guffin-Katz-ES 1110.3751, .3752

but the derivation is new.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Compare

Quantum sheaf cohomology (q.s.c.) ring rel'ns:

$$\det(A\sigma + B\tilde{\sigma}) \; = \; q, \quad \det(C\sigma + D\tilde{\sigma}) \; = \; \tilde{q}$$

Ordinary quantum cohomology ring rel'ns:

$$\sigma^2 \; = \; q, \quad \tilde{\sigma}^2 \; = \; \tilde{q}$$

On the (2,2) locus, where $A = D = I_{2\times 2}, \quad B = C = 0$

quantum sheaf cohomology reduces to
ordinary quantum cohomology.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

2-pt correlation functions:

$$\langle \sigma\sigma \rangle = -\alpha^{-1}\Gamma_1 \qquad \langle \sigma\tilde{\sigma} \rangle = \alpha^{-1}\Delta \qquad \langle \tilde{\sigma}\tilde{\sigma} \rangle = -\alpha^{-1}\Gamma_2$$

where

$$\Gamma_1 = \gamma_{AB} \det D - \gamma_{CD} \det B \qquad \Gamma_2 = \gamma_{CD} \det A - \gamma_{AB} \det C$$

$$\gamma_{AB} = \det(A+B) - \det A - \det B$$

$$\gamma_{CD} = \det(C+D) - \det C - \det D$$

$$\Delta = \det A \det D - \det B \det C$$

$$\alpha = \Delta^2 - \Gamma_1\Gamma_2$$

$\{\alpha = 0\}$ = locus where bundle degenerates

JK residue results match Cech cohomology computation.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

2-pt correlation functions:

$$\langle \sigma \sigma \rangle \; = \; -\alpha^{-1} \Gamma_1 \qquad \langle \sigma \tilde{\sigma} \rangle \; = \; \alpha^{-1} \Delta \qquad \langle \tilde{\sigma} \tilde{\sigma} \rangle \; = \; -\alpha^{-1} \Gamma_2$$

Can show these 2-pt functions obey

$$\langle \det(A\sigma + B\tilde{\sigma}) \rangle \; = \; 0 \qquad \langle \det(C\sigma + D\tilde{\sigma}) \rangle \; = \; 0$$

matching classical limit of q.s.c. relations.

Can also compute higher-pt functions.
They also match Cech computations, and obey suitable OPE's;
for brevity, let's move on.

Example: Hirzebruch surfaces $\mathbb{F}_n$

Build a (0,2) theory that deforms (2,2) model.

Math:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ \gamma_1 w + s f_n(x_1,x_2) & \beta_1 w + s g_n(x_1,x_2) \\ \gamma_2 s & \beta_2 s \end{array} \right]$$

Physics:

$$\overline{D}_+ \Lambda^i = A^i_j \sigma x^j + B^i_j \tilde{\sigma} x^j$$

$$\overline{D}_+ \Lambda_w = \sigma(\gamma_1 w + s f_n) + \tilde{\sigma}(\beta_1 w + s g_n)$$

$$\overline{D}_+ \Lambda_s = \sigma \gamma_2 s + \tilde{\sigma} \beta_2 s$$

— depends upon deg n polynomials f_n, g_n ;
however, they don't contribute to correlation functions:

Math: 1110.3751, .3752

Also follows from susy localization

Example: Hirzebruch surfaces $\mathbb{F}_n$

Localization computation:

$$\langle f(\sigma,\tilde{\sigma})\rangle \;=\; \sum_{\mathfrak{m}_1,\mathfrak{m}_2\in\mathbb{Z}} \mathrm{JK}-\mathrm{Res}_{\sigma=\tilde{\sigma}=0}\left\{\left(\frac{1}{\det E}\right)^{\mathfrak{m}_1+1}\left(\frac{1}{Q_w}\right)^{n\mathfrak{m}_1+\mathfrak{m}_2+1}\left(\frac{1}{Q_s}\right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1}\tilde{q}^{\mathfrak{m}_2} f(\sigma,\tilde{\sigma})\right\}$$

$$\begin{aligned} E &= A\sigma + B\tilde{\sigma} \\ Q_w &= \gamma_1\sigma + \beta_1\tilde{\sigma} \\ Q_s &= \gamma_2\sigma + \beta_2\tilde{\sigma} \end{aligned}$$

Can read off quantum sheaf cohomology ring rel'ns:

$$(\det E)Q_w^n \;=\; q \qquad\qquad Q_sQ_w \;=\; \tilde{q}$$

— matches previous results of McOrist-Melnikov; Donagi-Guffin-Katz-ES

— reduce to ordinary quantum cohomology on (2,2) locus

Example: Hirzebruch surfaces $\mathbb{F}_n$

2-pt correlation functions:

$$
\begin{aligned}
\langle \sigma\sigma \rangle &= \alpha^{-1} \left[\Delta - \beta_1\beta_2 \det(A+B) + (\gamma_1+\beta_1)(\gamma_2+\beta_2) \det B \right] \\
\langle \sigma\tilde{\sigma} \rangle &= \alpha^{-1} \Delta \\
\langle \tilde{\sigma}\tilde{\sigma} \rangle &= \alpha^{-1} \left[\Delta + \gamma_1\gamma_2 \det(A+B) - (\gamma_1+\beta_1)(\gamma_2+\beta_2) \det A \right]
\end{aligned}
$$

where

$$
\Delta = \beta_1\beta_2 \det A - \gamma_1\gamma_2 \det B
$$

$$
\alpha = \Phi_1\Phi_2
$$

for

$$
\begin{aligned}
\Phi_i &= \beta_i^2 \det A - \beta_i\gamma_i\gamma_{AB} + \gamma_i^2 \det B \\
\gamma_{AB} &= \det(A+B) - \det A - \det B
\end{aligned}
$$

JK residue results match Cech cohomology computation.

For a general toric variety + deformation of tangent bundle,

$$\langle f(\sigma) \rangle \; = \; \sum_{\mathfrak{m}_1, \cdots \in \mathbb{Z}} \mathrm{JK} - \mathrm{Res}_{\sigma=0} \left[\prod_{a,\alpha} \left(\frac{1}{\det M_{(\alpha)}} \right)^{Q^a_\alpha(\mathfrak{m}_a)+1} q_a^{\mathfrak{m}_a} \right] f(\sigma)$$

We've sketched an argument that this matches result of McOrist-Melnikov '08:

$$\langle f(\sigma) \rangle \; = \; \sum_{\sigma | J=0} f(\sigma) \left((\det J_{a,b}) \prod_\alpha \det M_{(\alpha)} \right)^{-1}$$

where $$J_a \; = \; \ln \left(q_a^{-1} \prod_\alpha M_{(\alpha)}^{Q^a_\alpha} \right)$$

Now, getting new expressions for old results is nice,
but, what's even better is that we can also get new results…..

Nonabelian cases

So far, we have discussed the results of applying susy localization to A/2 theories describing toric varieties.

Next: Grassmannians

Understanding A/2 twists of Grassmannians has been an open problem for many years, as older GLSM techniques don't easily apply.

We'll see that susy localization allows us to quickly derive results not previously obtainable.

Basic example, (2,2): G(k,n) = Grassmannian of k planes in $\mathbb{C}^n$

Physics: U(k) gauge theory

n chiral multiplets in fundamental rep'

(0,2) deformation:

U(k) gauge theory

n chiral multiplets ϕ^i in fundamental rep'

n Fermi multiplets Λ^i in fundamental rep'

$$\overline{D}_+ \Lambda^i_a \;=\; \sigma^b_a \phi^i_b \;+\; B^i_j (\mathrm{Tr}\,\sigma) \phi^j_a$$

The B's define deformation off (2,2) locus.

Can show, total num' of deformations = $n^2 - 1$

(for $1 < k < n-1$)

— overall trace of B defines trivial deformation; rest interesting

General formula for A/2 correlation functions:

$$\langle f(\sigma_1, \cdots, \sigma_k) \rangle \; = $$

$$\frac{1}{k!} \sum_{\mathfrak{m}_1, \cdots, \mathfrak{m}_k \in \mathbb{Z}} \mathrm{JK} - \mathrm{Res}_{\sigma_i = 0} \left\{ q^{\sum \mathfrak{m}_i} \left(\prod_{\alpha \neq \beta} (\sigma_\alpha - \sigma_\beta) \right) \prod_{\alpha=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_\alpha)} \right)^{\mathfrak{m}_\alpha + 1} f(\sigma) \right\}$$

where $\tilde{E}^i_j(\sigma) \; = \; \sigma \delta^i_j \; + \; B^i_j \left(\sum_\alpha \sigma_\alpha \right)$

Q.s.c. relations: $\det \tilde{E}(\sigma_\alpha) \; = \; q$ for all α

We'll see more meaningful expressions shortly….

We'll focus on cases in which B is diagonal, for simplicity.

Example: Deformation of TG(2,4)

Classical correlation functions ($\mathfrak{m}_1 = \mathfrak{m}_2 = 0$)

$$\langle \sigma_1^4 \rangle \;=\; \Delta^{-1}\left(I_3 + 2I_3^2 + 4I_3I_2 - 2I_1 + 2I_2^2 + 2I_3I_1 - 4\det B + 2I_2I_1 - 2I_3\det B\right)$$

$$\langle \sigma_1^3\sigma_2 \rangle \;=\; \Delta^{-1}\left(-1 - 3I_3 - 2I_3^2 - 3I_2 - 4I_3I_2 - 2I_2^2 - I_1 - 2I_3I_1 + 4\det B - 2I_2I_1 + 2I_3\det B\right)$$

$$\langle \sigma_1^2\sigma_2^2 \rangle \;=\; \Delta^{-1}\left(2 + 4I_3 + 2I_3^2 + 4I_2 + 4I_3I_2 + 2I_1 - 4\det B + 2I_2^2 + 2I_3I_1 + 2I_2I_1 - 2I_3\det B\right)$$

$$\langle \sigma_1\sigma_2^3 \rangle \;=\; \langle \sigma_1^3\sigma_2 \rangle \qquad\qquad \langle \sigma_2^4 \rangle \;=\; \langle \sigma_1^4 \rangle$$

$$\Delta \;=\; 2\prod_{i<j}(1 + B_{ii} + B_{jj})$$

is the locus on which bundle degenerates.

$$I_1 \;=\; \sum_{i<j<k} B_{ii}B_{jj}B_{kk} \qquad I_2 \;=\; \sum_{i<j} B_{ii}B_{jj} \qquad I_3 \;=\; \sum_i B_{ii} \;=\; \mathrm{tr}\, B$$

are coefficients in the characteristic polynomial of B.

How can we interpret those correlation functions usefully?
How can we compare to ordinary cohomology, on (2,2) locus?

We can naturally group according to Young diagrams.
Using Schur polynomials,

$$\sigma_{\ydiagram{1}} = \sigma_1 + \sigma_2$$

$$\sigma_{\ydiagram{2}} = \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2$$

$$\sigma_{\ydiagram{1,1}} = \sigma_1\sigma_2$$

$$\sigma_{\ydiagram{2,1}} = \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2$$

$$\sigma_{\ydiagram{2,2}} = \sigma_1^2\sigma_2^2$$

Cohomology of G(k,n) is naturally in 1-1 correspondence with Young diagrams inside k x (n-k) box.

G(2,4): $\ydiagram{2,2}$

so on (2,2) locus, for example,

$$\sigma_{\ydiagram{3}} = 0$$

$$\sigma_{\ydiagram{3,1}} = 0$$

Classical correlation functions on (2,2) locus:

$$\langle \sigma_{\ydiagram{2,2}} \rangle = +1 \qquad \langle \sigma_{\ydiagram{4}} \rangle = 0 = \langle \sigma_{\ydiagram{3,1}} \rangle$$

which imply OPE $\sigma_{\ydiagram{3}} \cdot \sigma_{\ydiagram{1}} = 0$

hence $\sigma_{\ydiagram{3}} = 0$ Agree!

(2,2) locus: $\sigma_{\ydiagram{3}} \;=\; \sigma_{\ydiagram{4}} \;=\; \sigma_{\ydiagram{3,1}} \;=\; 0$ classically

$\sigma_{\ydiagram{4}} - \sigma_{\ydiagram{3,1}} \;=\; 2q$ nonpert'ly

(0,2):

(a) $$(1 + I_1 + I_2 + I_3)\sigma_{\ydiagram{3}} \;+\; (I_3 + 2I_2 + 2I_1)\sigma_{\ydiagram{2,1}} \;=\; 0$$

(b) $$(1 + I_1 + I_2 + I_3)\sigma_{\ydiagram{4}} \;+\; (1 + 3I_1 + 3I_2 + 2I_3)\sigma_{\ydiagram{3,1}} \;+\; (I_3 + 2I_2 + 2I_1)\sigma_{\ydiagram{2,2}} \;=\; 0$$

(c) $$(1 + I_3 + I_2 + I_1 + 2\det B)\sigma_{\ydiagram{4}} \;+\; (-1 + I_2 + 3I_1 + 6\det B)\sigma_{\ydiagram{3,1}}$$
$$+ (-I_3 + 2I_1 + 4\det B)\sigma_{\ydiagram{2,2}} \;=\; 2q$$

(derived from $\det E = q$)

More generally, for any deformation of given form of TG(k,n), we've recently argued that

classical sheaf cohomology ring =

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(k(n-k))}] / \left(D_{k+1}, \cdots, D_{k(n-k)}, R_{n-k+1}, \cdots, R_{k(n-k)} \right)$$

$$D_m = \det(\sigma_{1+j-i})_{1 \leq i,j \leq m}$$

$$R_r = \sum_{i=0}^{\min(r,n)} I_{n-i} \sigma_{(r-i)} \sigma_{(1)}^i$$

for $\sigma_{(1)} = \sigma_{\Box}$ and so forth, and

(I_i) the coefficients in the characteristic polynomial of B.

More generally, for any deformation of given form of TG(k,n), we've argued that

classical sheaf cohomology ring =

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(k(n-k))}] / \left(D_{k+1}, \cdots, D_{k(n-k)}, R_{n-k+1}, \cdots, R_{k(n-k)} \right)$$

On (2,2) locus, $R_r = \sigma_{(r)}$

and the above becomes a standard presentation of the ordinary cohomology of G(k,n).

So: matches (2,2) locus.

Gen'l expression for quantum sheaf cohomology ring:

Analogues for the B/2 model

So far I've only discussed susy localization in the A/2 model, for deformations of (2,2) theories.

We can also apply the same ideas to B/2 twists of `dual' theories.

Which theories?

Recall mentioned earlier that

$$A/2(X, \mathcal{E}) \;=\; B/2(X, \mathcal{E}^*)$$

so we're going to be able to apply B/2 to spaces with deformations of cotangent bundles — no (2,2) locus.

Analogues for the B/2 model

Quick aside: how is this related to (0,2) mirror symmetry?

Suppose (0,2) NLSM's on $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ define same SCFT.

$$
\begin{tikzcd}
A/2(X, \mathcal{E}) \arrow[r, equal, "\text{(0,2) mirror}"] \arrow[d, equal] & B/2(Y, \mathcal{F}) \arrow[d, equal] \\
B/2(X, \mathcal{E}^*) \arrow[r, equal, "\text{(0,2) mirror}"] & A/2(Y, \mathcal{F}^*)
\end{tikzcd}
$$

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Math:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2 \stackrel{*}{\longrightarrow} \mathcal{O}^2 \longrightarrow 0$$

$$* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$$

$x, \tilde{x}$ vectors of homogeneous coordinates,

A, B, C, D 2×2 matrices describing deformation

No (2,2) locus; but cotangent bundle at

$$A = D = I_{2\times 2}, \quad B = C = 0$$

Physics….

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Physics:

$x^i, \tilde{x}^i$ chiral superfields charge (1,0), (0,1)

$p, \tilde{p}$ neutral chiral superfields

$\Lambda_i, \tilde{\Lambda}_i$ Fermi superfields charge (-1,0), (0,-1)

plus (0,2) superpotential

$$W \;=\; \Lambda_i F^i_j x^j \;+\; \tilde{\Lambda}_i \tilde{F}^i_j \tilde{x}^j$$

where

$$F^i_j \;=\; A^i_j p \;+\; B^i_j \tilde{p}$$

$$\tilde{F}^i_j \;=\; C^i_j p \;+\; D^i_j \tilde{p}$$

(Compare A/2 version: there, no superpotential, and charges matched.)

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Unlike the A/2 case, here there is no σ field — no adjoint-valued scalar that is part of vector multiplet on (2,2) locus.

So, no Coulomb branch along which to compute.

Instead, have p field, which plays a `dual' role.

In effect, the Coulomb branch replaced by (part of) Higgs branch.

Analogues for the B/2 model

So, for example, we should be able to compute B/2 correlation functions for deformations of cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

Result:

Correlation functions are given by:

$$\langle f(p,\tilde{p}) \rangle \;=\; \sum_{\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{Z}} \mathrm{JK}-\mathrm{Res}_{p=\tilde{p}=0} \left\{ \left(\frac{1}{\det F} \right)^{\mathfrak{m}_1+1} \left(\frac{1}{\det \tilde{F}} \right)^{\mathfrak{m}_2+1} q^{\mathfrak{m}_1} \tilde{q}^{\mathfrak{m}_2} f(p,\tilde{p}) \right\}$$

where $\quad F \;=\; Ap + B\tilde{p} \qquad \tilde{F} \;=\; Cp + D\tilde{p}$

— equivalent to results in dual A/2 model, as expected

Other cotangent bundle deformations similar.

Summary

- Open problems in heterotic compactifications
- A/2, B/2 pseudo-topological field theories
- Susy localization in A/2 model for def's of (2,2) theories
- Examples: $\mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{F}_n, \quad G(k,n)$

    — new expressions for old results: JK residues

    — new results: nonabelian GLSM's

- Analogous computations in dual B/2 theories