An introduction to decomposition

String and M-theory: the new geometry of the 21st century II
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An overview of hep-th/0502027, 0502044, 0502053, 0606034,
0709.3855, 1012.5999, 1307.2269, 1404.3986, ... (many ...),
2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423

My talk today is an overview of **decomposition**,
a new notion in quantum field theory (QFT).

Briefly, decomposition is the observation that some QFTs
(possessing a higher-form symmetry)
are secretly equivalent to
sums of other QFTs, known as ‘universes.’

When this happens, we say the QFT `decomposes.’
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al ’06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

Math analogue:

If a space X has m connected components, then $\dim H^0(X) = m$

— multiple degree-zero elements of cohomology

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al ’06)

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i Z_i = \sum_i \sum \exp(-\beta H_i)$$

(on a connected spacetime)

Now, given a nontrivial structure, expect a symmetry....

3) In (n+1) spacetime dimensions, has a (possibly noninvertible) n-form symmetry.

I’ll explain what that is in a few minutes....

Decomposition & higher-form symmetries go hand-in-hand.

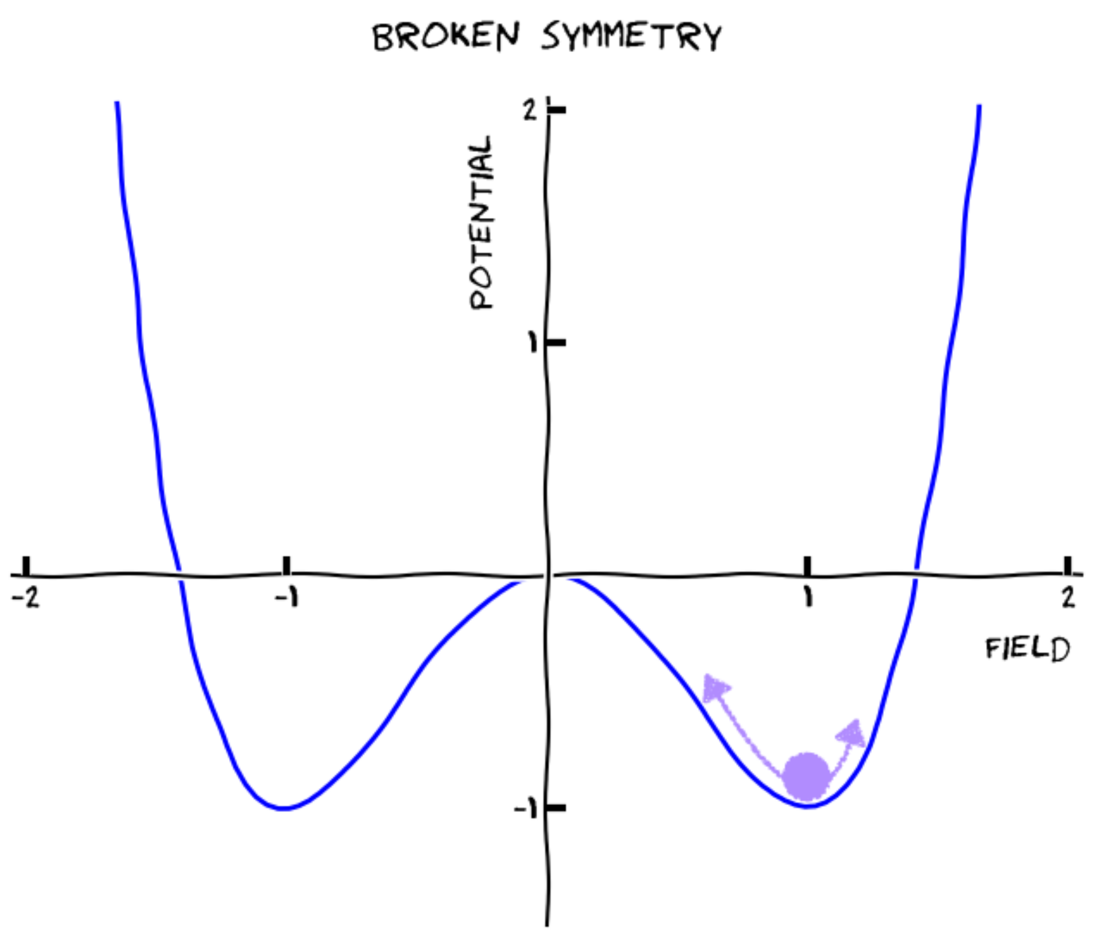
Decomposition ≠ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

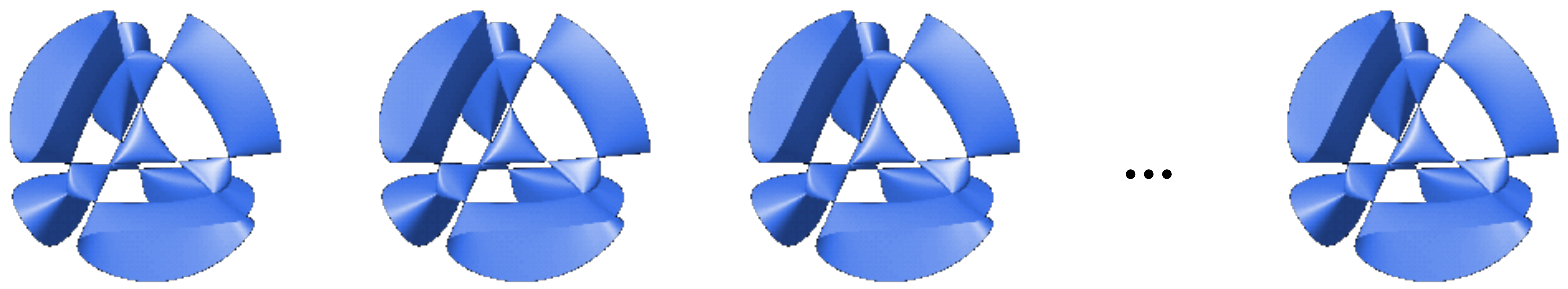


Decomposition:

Universes:

- separated by nondynamical domain walls
- disjoint at all energy scales
- multiple different QFTs present

Prototype:

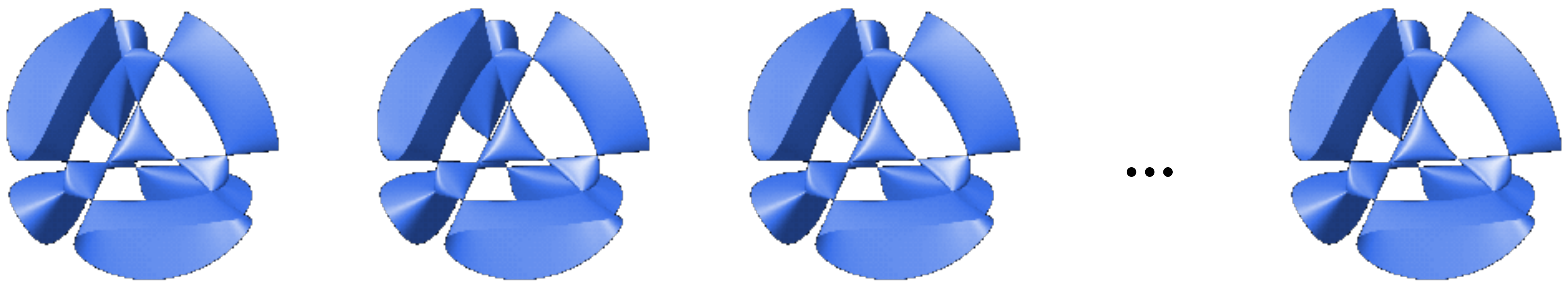


(see e.g. Tanizaki-Unsal 1912.01033)

Decomposition $\neq$ spontaneous symmetry breaking

Note that they both have an order parameter, so be careful when distinguishing.

Ex: sigma model on disjoint union of n spaces ('universes')



Have topological projectors Π_i $\quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1$

Have order parameter X

$$X = \sum_{i=0}^{n-1} \xi^i \Pi_i, \quad \xi = \exp(2\pi i/n)$$

Vev in ith universe: $\quad \langle \Pi_i X \rangle = \langle \xi^i \Pi_i \rangle = \xi^i$

So, could be described as spontaneously broken phase
— but that clearly does **not** capture the physics.

I mentioned higher-form symmetries. What's a one-form symmetry?....

What is a one-form symmetry?

For this talk, *intuitively*, this will be a \`group' that exchanges nonperturbative sectors.

Example: G gauge theory in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume K abelian,
but decompositions exist more generally.)

Then, at least for K central, nonperturbative sectors are invariant under

$$(G - \text{bundle}) \mapsto (G - \text{bundle}) \otimes (K - \text{bundle})$$

$$A \mapsto A + A'$$

At least when K central, this is the action of the \`group' of K-bundles.

That group is denoted BK or $K^{(1)}$

(Technically, is a 2-group, only weakly associative.)

One-form symmetries can also be seen in algebra of topological local operators,
as we'll see later.

What sort of QFTs admit a decomposition ? ....

The QFTs I'm interested in, which have a decomposition,
are (1+1)-dimensional theories with "global 1-form symmetries,"
and can be described in several ways, such as

(Pantev, ES '05;
Hellerman et al '06)

- Gauge theory w/ trivially-acting subgroup
- Theory w/ restriction on instantons
- Sigma models on gerbes

  = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$

We'll see in this talk how decomposition (into 'universes') relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Outline of the rest of the talk:

- Outline of decomposition in 2d & 1-form symmetries

Next:
- Examples: G gauge theories with trivially-acting $K \subset G$
- Overview of literature
- Examples: orbifolds
- Examples: unitary TQFTs
- Original motivation: string compactifications on stacks & gerbes
- Outline of application to anomaly resolution

Decomposition in (1+1)-d gauge theories

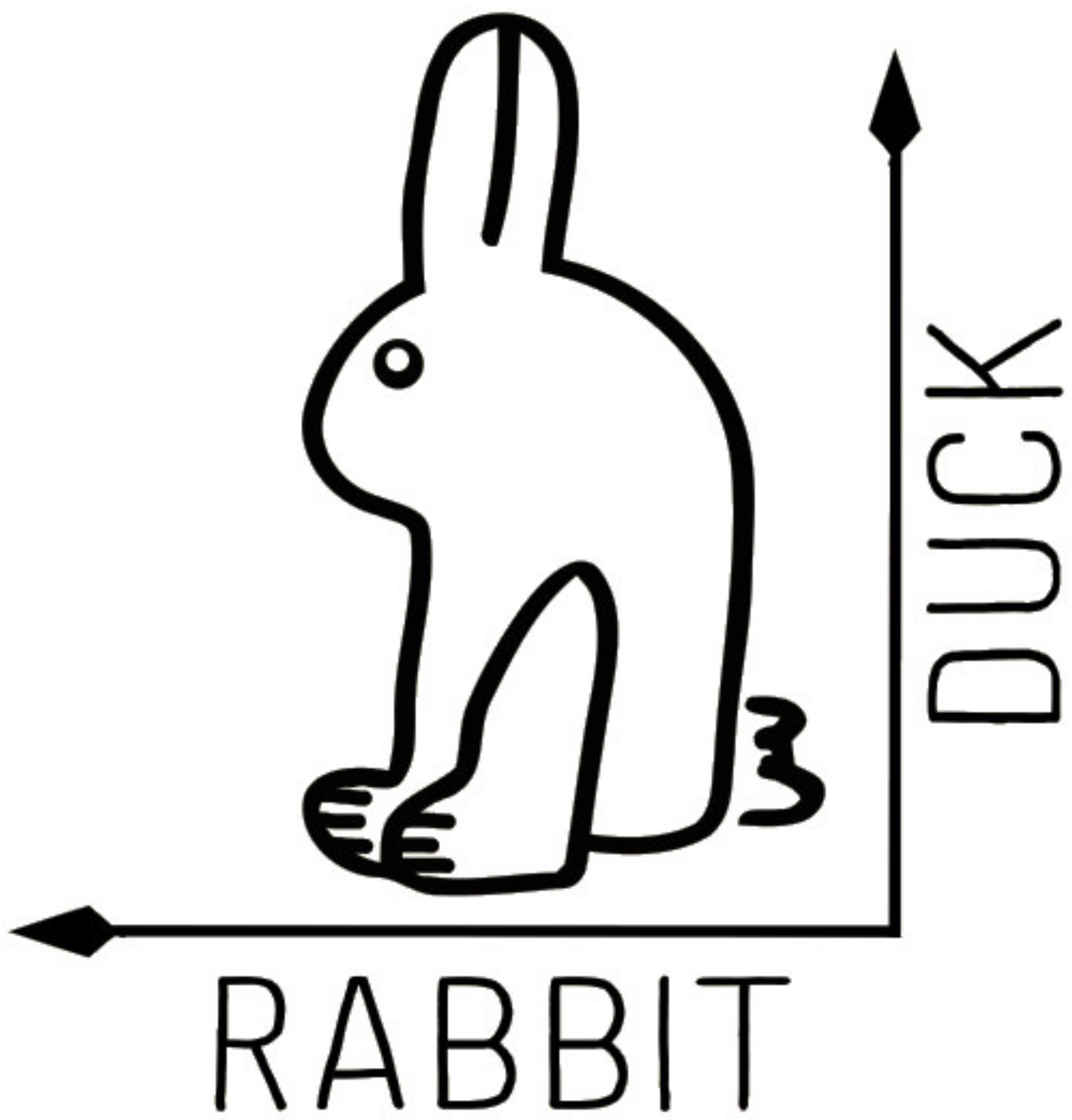
(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

So far, this sounds like just one QFT.



However, I'll outline how, from another perspective,
QFTs of this form are also each
a disjoint union of other QFTs;
they "decompose."

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center.

Has BK 1-form symmetry.

Claim this theory decomposes.
Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn's theorem.

Universes $\longleftrightarrow$ Irreducible representations of K

Partition functions & relation of decomp' to restrictions on instantons....

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Statement of decomposition:

$$\text{QFT}(G-\text{gauge theory}) = \coprod_{\text{irreps } K = \hat{K}} \text{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

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Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset$ $SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Statement of decomposition:

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Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

("multiverse interference" cancels out some sectors)

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

projection operator

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union

Decomposition in (1+1)-d gauge theories

(Hellerman et al '06)

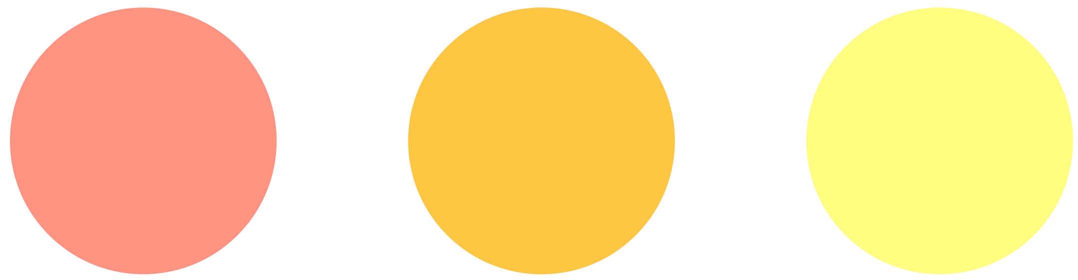
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} \quad = \quad \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union of several QFTs / universes = `One' QFT with a restriction on nonperturbative sectors = `multiverse interference'

Schematically,
two theories combine to form a distinct third:

universe ($SO(3)_+$)



universe ($SO(3)_-$)

multiverse interference effect ($SU(2)$)

Example:

Pure nonsusy 2d SU(2) Yang-Mills

Decomposition: $SU(2) = SO(3)_+ + SO(3)_-$ due to global $B\mathbb{Z}_2$ center symmetry

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$

In fact, this is easy to generalize...

Example:

Pure nonsusy 2d G Yang-Mills

More generally, if G has center K,

a pure 2d nonsusy $G-$gauge theory has BK symmetry,

and decomposes as

$$G \;=\; \coprod_{\theta \in \hat{K}} (G/K)_\theta$$

where the θ are discrete theta angles,

coupling to analogues of Stiefel-Whitney classes.

Hilbert spaces...

Example:

Pure nonsusy 2d G Yang-Mills

Hilbert spaces:

The Hilbert space of a pure G YM theory is $\mathscr{H}(G) =$ L² class f'ns on G

These decompose under action of center: $f(gz) = \theta(z) f(g)$

$\mathscr{H}\left((G/K)_\theta\right) =$ L² class f'ns on G such that $f(gz) = \theta(z) f(g)$

As a result, $$\mathscr{H}(G) = \sum_{\theta \in \hat{K}} \mathscr{H}\left((G/K)_\theta\right)$$

which is consistent with decomposition: $$G = \coprod_{\theta \in \hat{K}} (G/K)_\theta$$

Example:

Pure nonsusy 2d G Yang-Mills

So far I've described one version of decomposition for pure nonsusy 2d Yang-Mills, which uses center one-form symmetries.

There exists a more extreme decomposition,
into invertible field theories indexed by irreps of G
(Nguyen-Tachikawa-Unsal '21).

I'll briefly touch on that when I discuss unitary 2d TQFTs.

Supersymmetric gauge theories work the same way.

Briefly:

- We can apply susy localization and see decomposition explicitly in partition f'ns. (ES '14)

- We can also see the result in mirrors, both abelian (Hori-Vafa '00) and nonabelian (Gu-ES '18)

- Using knowledge of discrete theta angles & susy breaking from eg mirrors, one can also compute elliptic genera using decomposition. (Eager-ES '20)

Decomposition in (1+1)-d gauge theories

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (Honda et al '21)
- Plus version for (3+1)d theories w/ 3-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

Applications include:

- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ...)
- Elliptic genera (Eager et al '20)
- Anomalies (Robbins et al '21)

At the end, we'll look at application to anomalies....

Outline of the rest of the talk:

- Outline of decomposition in 2d & 1-form symmetries
- Examples: G gauge theories with trivially-acting $K \subset G$
- Overview of literature

Next:
- Examples: orbifolds
- Examples: unitary TQFTs
- Original motivation: string compactifications on stacks & gerbes
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Decomposition in (1+1)-d gauge theories

Let’s apply to orbifolds.

An orbifold $[X/G]$ is a G-gauge theory for a finite group G,
specifically, a G-gauged sigma model into a (target) space X.

What does that mean?

An ordinary sigma model is a path integral over maps into target space X:

$$\Sigma \longrightarrow X$$

When we gauge G, we identify field configurations related by G.

In the orbifold $[X/G]$, we allow for branch cuts defined by elements of G.

Example: $T^2 \longrightarrow X$

g  $\longrightarrow X$

h

Decomposition in orbifolds in (1+1) dimensions

Let's begin by discussing ordinary orbifolds w/o extra phases.
(There exist more complicated versions, which are needed for e.g. anom' res'n, but let's start here, and build up.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central) (K, Γ, G finite)

Decomposition implies

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

G acts on $\hat{K}$: $\rho(k) \mapsto \rho(hkh^{-1})$ for $h \in \Gamma$ a lift of $g \in G$

$\hat{\omega}$ = phases called "discrete torsion" — we'll see more later.

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

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$$\mathrm{QFT}\left([X/\Gamma]\right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

If K is in the center of Γ, then the G action on $\hat{K}$ is trivial, and decomposition specializes to

$$\mathrm{QFT}\left([X/\Gamma]\right) \; = \; \mathrm{QFT}\left(\coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

— a disjoint union, as many elements as $\hat{K}$

More gen'ly, get both copies and covers of $[X/G]$, as we shall see.

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

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(Helllerman et al '06)

$\hat{K}$ = set of iso classes of irreps of K

Universes (summands of decomposition)
correspond to orbits of G action on $\hat{K}$.

We'll see explicit formulas for projectors and $\hat{\omega}$ later....

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

(Hellerman et al '06)

Boundaries also decompose.

The boundary of that theory in (1+1) dims can have e.g. fermions on which Γ acts.

Although $K \subset \Gamma$ acts trivially on the bulk d.o.f.,
it can act *non*trivially on boundary d.o.f.

To compute which universe a given boundary lies in,
restrict the Γ action to K,
at which point it becomes a representation of K.
Then, compare orbits — universes correspond to orbits in $\hat{K}$.

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (K need not be central)

(Hellerman et al '06)

Boundaries also decompose.

A quick note — boundaries can be understood in terms of K theory, and this boundary decomposition reflects math of K theory.

Technically, $[X/\Gamma]$ is an example of a "gerbe" on $[X/G]$, essentially, a fiber bundle in which the fibers are \`groups' BK of 1-form symmetries.

($BK = [\text{point}/K]$, which is why have triv'ly acting K.)

Fun math fact: K theory of a gerbe = disjoint union of K theory of underlying spaces/orbifolds, in the same fashion as we have described.

(Decomposition gives a physical explanation for this facet of K theory.)

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

(consequence of a general formula)

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \; = \; \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \; = \; 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} \; = \; \frac{1}{2}(1 \pm \hat{z})$$

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_{+} + \Pi_{-} = 1$$

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \underset{h}{\boxed{}} \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

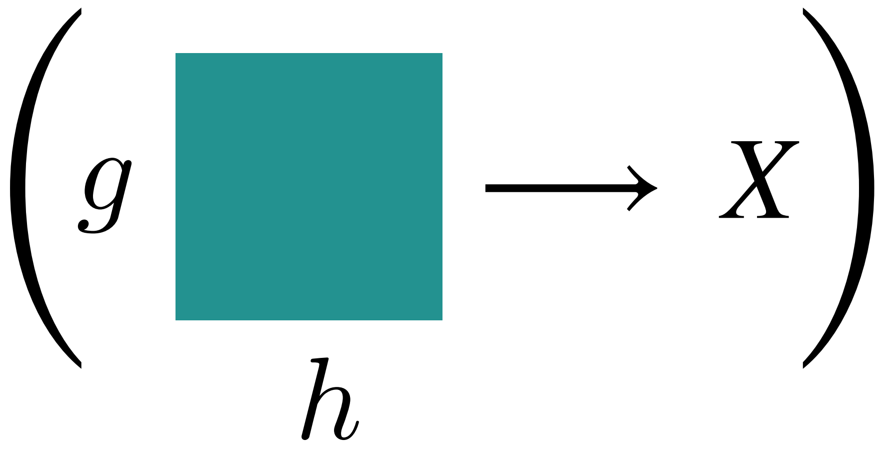
$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

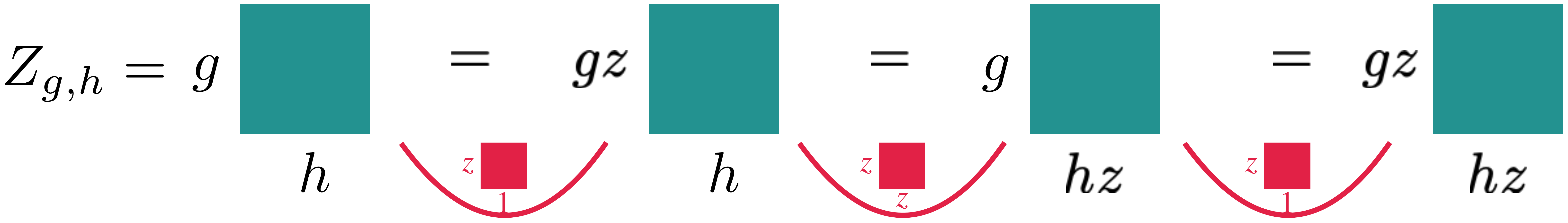
$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{gh \in D_4, gh = hg} Z_{g,h}$$

where $Z_{g,h} =$



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{gh \in D_4, gh=hg} Z_{g,h}$$ where $Z_{g,h} = \left(g \, \underset{h}{\blacksquare} \longrightarrow X \right)$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a} \, \underset{\overline{b}}{\blacksquare}$ $\quad \overline{a} \, \underset{\overline{ab}}{\blacksquare}$ $\quad \overline{b} \, \underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $\quad Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h)\, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

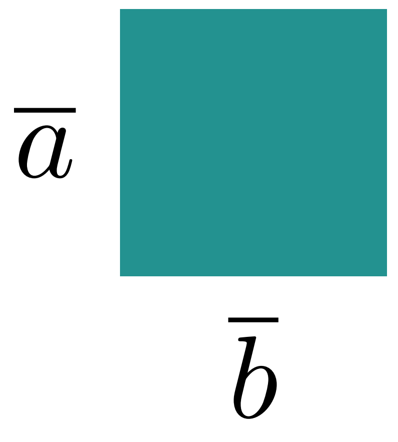
(T Pantev, ES '05)

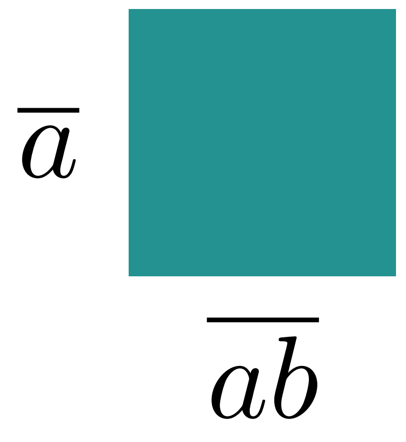
$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

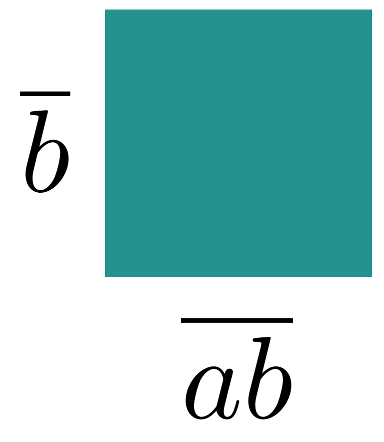
$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, discrete torsion $\in \; H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \; = \; \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors







which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

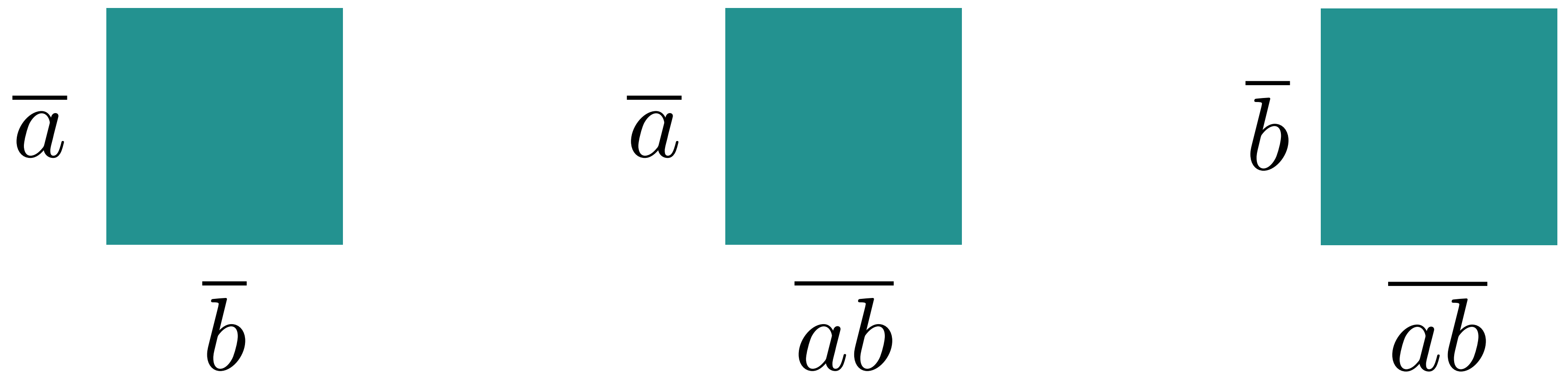
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$=\; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

Discrete torsion is $H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

I'll come back to dilaton shifts later in discussing example of
orbifold of a point (= (1+1)-d Dijkgraaf-Witten theory).

Example, cont'd

Quick aside on symmetries:

I mentioned that the one-form symmetry $B\mathbb{Z}_2$ shows up in permutations of nonperturbative sectors.

It also shows up in the dimension-zero twist field: $\hat{z}^2 = 1$

We'll come back to this later.

Example, cont'd

Massless spectra for $X = T^6$

(T Pantev, ES '05)

Massless spectrum of D_4 orbifold

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
\;=\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\;+\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components /

cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb'

w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb'

w/ d.t.

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) \;=\; \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H} =$ eight-element gp of unit quaternions, where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts (Hellerman et al '06)

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

Write $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$

Dimension-zero twist fields: $1, \quad \sigma_{-1}, \quad \sigma_{[i]}$

obeying $\sigma_{-1}^2 = 1, \ \sigma_{-1}\sigma_{[i]} = \sigma_{[i]}, \ \sigma_{[i]}^2 = (1/2)\left(1 + \sigma_{-1}\right)$

Projectors:

$$\Pi_{\pm} = \frac{1}{4}\left(1 + \sigma_{-1} \pm 2\sigma_{[i]}\right), \ \Pi_2 = \frac{1}{2}\left(1 - \sigma_{-1}\right)$$

(project onto $[X/\mathbb{Z}_2]$) (projects onto X)

which are easily checked to be idempotents. Partition functions....

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

Write $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$

Partition function on T^2: Denote generator of $\mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ by ξ

$$Z_{T^2}([X/\mathbb{H}]) = \frac{1}{|\mathbb{H}|} \sum_{gh=hg} Z_{g,h} = \frac{1}{|\mathbb{H}|}\left((16)\ {}^{1}\underset{1}{\square} + (8)\ {}^{1}\underset{\xi}{\square} + (8)\ {}^{\xi}\underset{\xi}{\square} \right)$$

$$= 2 Z_{T^2}\left([X/\mathbb{Z}_2]\right) + Z_{T^2}(X)$$

Works!

Higher genus partition functions also work (w/ dilaton shifts), see hep-th/0606034 sect 5.4.

Another example: Triv'ly acting subgroup not in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,
where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially. (Hellerman et al '06)

Decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

One-form symmetries:

Recall this theory has dimension-zero twist fields: $1, \quad \sigma_{-1}, \quad \sigma_{[i]}$

obeying $\quad \sigma_{-1}^2 = 1, \;\; \sigma_{-1}\sigma_{[i]} = \sigma_{[i]}, \;\; \sigma_{[i]}^2 = (1/2)\left(1 + \sigma_{-1}\right)$

This describes a noninvertible one-form symmetry,
which includes a $B\mathbb{Z}_2$ as a subset: $\sigma_{-1}^2 = 1$.

Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

where $\hat{K}$ = irreps of K
$\hat{\omega}$ = discrete torsion on universes

Example: $[X/D_4]$

Here, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially on $\hat{K} = \mathbb{Z}_2$ so RHS = 2 copies of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$

Example: $[X/\mathbb{H}]$

Here, $G = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

$$\text{so RHS } = X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2]$$

Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\hat{\omega}}\right)$$

where $\hat{K}$ = irreps of K

$\hat{\omega}$ = discrete torsion on universes

Projectors:

For $R = \oplus_i R_i$, $R_i \in \hat{K}$ related by the action of G, we have

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

where $\hat{K}$ = irreps of K
$\hat{\omega}$ = discrete torsion on universes

What about the $\hat{\omega}$? Where did that come from?

Its discussion is more technical & abstract, so I wanted to delay until after examples, but now that we've seen some examples, I'll outline it....

Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Origin of $\hat{\omega}$: (apologies for the math, I'll try to be quick)

Let $\{\rho_a\}$ be a collection of irreps in $\hat{K}$ chosen s.t. $[\rho_a]$ represent orbits of G on $\hat{K}$.

For each a, let $H_a \subset G$ be the stabilizer of $[\rho_a] \subset \hat{K}$ (ie, the subgp that leaves it inv't).

$$\Delta_a = \pi^* H_a \subset \Gamma, \quad s_a : H_a \to \Delta_a \text{ a section}$$

Write $\rho_a : K \to \mathrm{End}(V_a)$, then there are intertwiners $f_a : H_a \to \mathrm{End}(V_a)$,

$$\begin{tikzcd} V_a \arrow[r, "\rho_a(k)"] \arrow[d, "f_a(h)"'] & V_a \arrow[d, "f_a(h)"] \\ V_a \arrow[r, "\rho_a(s_a(h)^{-1} k s_a(h))"] & V_a \end{tikzcd}$$

Define a projective rep' $\tilde{\rho}_a$ of Δ_a by $\tilde{\rho}_a(\,\iota(k)\, s_a(h)\,) = \rho_a(k) f_a(h)^{-1}$

Can show $\tilde{\rho}_a(g_1)\,\tilde{\rho}_a(g_2) = \hat{\omega}(g_1, g_2)\,\tilde{\rho}_a(g_1 g_2)$ where $\hat{\omega}$ is a cocycle & pullback from H_a

Decomposition in orbifolds in (1+1)-dims without discrete torsion

(Hellerman et al '06)

Consider $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$.

Origin of $\hat{\omega}$: (apologies for the math, I'll try to be quick)

Let $\{\rho_a\}$ be a collection of irreps in $\hat{K}$ chosen s.t. $[\rho_a]$ represent orbits of G on $\hat{K}$.

For each a, let $H_a \subset G$ be the stabilizer of $[\rho_a] \subset \hat{K}$ (ie, the subgp that leaves it inv't).

$$\Delta_a = \pi^* H_a \subset \Gamma, \quad s_a : H_a \to \Delta_a \text{ a section}$$

Explicitly,

$$\hat{\omega}(h_1, h_2)\, I = f_a(h_1)^{-1} f_a(h_2)^{-1} f_a(h_1 h_2)\, \rho_a \left(s_a(h_1) s_a(h_2) s_a(h_1 h_2)^{-1} \right)^{-1} \quad \text{where } h_1, h_2 \in H_a$$

If K is in the center of Γ, then G acts triv'ly on $\hat{K}$, all irreps are 1d,

and then $\hat{\omega}$ is the (inverse of the) image of the characteristic class of the K-gerbe:

$$H^2([X/G], K) \xrightarrow{\rho_a} H^2([X/G], U(1))$$

You won't need this level of detail for this talk, but I wanted to stress that it exists.

Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) = \mathrm{Hom}(G, \hat{K})$$

Cases:

1) If $\iota^*\omega \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{G}\right]_{\hat{\omega}}\right)$$

2) If $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, \beta(\omega)}}{\mathrm{Ker}\, \beta(\omega)}\right]_{\hat{\omega}}\right)$$

3) If $\iota^*\omega = 0$ and $\beta(\omega) = 0$, then $\omega = \pi^*\overline{\omega}$ for $\overline{\omega} \in H^2(G, U(1))$ and

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\overline{\omega}+\hat{\omega}}\right)$$

Projectors....

Decomposition in orbifolds in (1+1)-dims with discrete torsion

(Robbins et al '21)

Consider $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$, and define $G = \Gamma/K$.

$$1 \longrightarrow K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1 \quad \text{(assume central)}$$

$$H^2(G, U(1)) \xrightarrow{\pi^*} \left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \xrightarrow{\beta} H^1(G, H^1(K, U(1)))$$

Projectors:

For $R = \bigoplus_i R_i$, $R_i \in \hat{K}$ related by the action of G, we have

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \frac{\chi_{R_i}(k^{-1})}{\omega(k, k^{-1})} \tau_k$$

Decomposition in orbifolds in (1+1)-dims with discrete torsion

An important special case: $[\mathrm{point}/G]_\omega$

Decomposition implies $\mathrm{QFT}\left([\mathrm{point}/G]_\omega\right) = \coprod \mathrm{QFT}(\mathrm{point})$

(as many copies as ω-proj' irreps of G)

up to overall dilaton shifts.

In math, this is a gen'l property of the center of the (twisted) group algebra $\mathbb{C}[G]_\omega$:
it has a basis corresponding to twist fields,
and another basis of projectors.

QFT(point) is an example of an \`invertible' field theory.

This is also two-dimensional Dijkgraaf-Witten theory, a 2d unitary TQFT....

Decomposition in orbifolds in (1+1)-dims with discrete torsion

An important special case: $[\text{point}/G]_\omega$ = (1+1)d Dijkgraaf-Witten TQFT

Decomposition implies $\text{QFT}\left([\text{point}/G]_\omega\right) = \coprod \text{QFT}(\text{point})$

(as many copies as ω-proj' irreps of G)

As a consistency check, consider the partition function.

On a genus g Riemann surface,

$$Z = \frac{1}{|G|^g} \sum_{a_i, b_i} \delta\left(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \cdots a_g^{-1} b_g^{-1}\right) \epsilon_g(a_i, b_i)$$

$$= \sum_R \left(\frac{\dim R}{\sqrt{|G|}}\right)^{2-2g}$$

= theory of as many points as (ω-proj') irreps,
each with dilaton = $\ln(\dim R/\sqrt{|G|})$

Works!

More generally, all 2d unitary TQFTs decompose….

Outline of the rest of the talk:

- Outline of decomposition in 2d & 1-form symmetries
- Examples: G gauge theories with trivially-acting $K \subset G$
- Overview of literature
- Examples: orbifolds

Next:
- Examples: unitary TQFTs
- Original motivation: string compactifications on stacks & gerbes
- Outline of application to anomaly resolution

(1+1)d unitary semisimple topological & near-topological field theories

These are all the same as (decompose into) disjoint unions of invertible field theories
(= QFT(point) w/ dilaton shifts).

Formal reason: semisimplicity of the Frobenius algebra,
which implies not only that projectors exist,
but that all local operators are linear comb's of projectors.

Ex: 2d Dijkgraaf-Witten

$$\text{2d DW} = [\text{point}/G]_\omega = \coprod_R \text{point} \ \ \text{(with dilaton shifts)}$$

Ex: Abelian BF at level k (Hellerman, ES, 1012.5999)

Ex: G/G model (Komargodski et al 2008.07567)

Ex: 2d pure Yang-Mills (Nguyen, Tanizaki, Unsal 2104.01824)

Wilson lines =
defects joining universes

All cases: $(1+1)\text{d unitary TQFT} = \coprod_R \text{Inv}(\ln(\dim R))$ (in top' limit)

(1+1)d unitary semisimple topological & near-topological field theories

Ex: Abelian BF at level k (Hellerman, ES 1012.5999)

U(1) gauge theory, Action: $S = k\int BF_{z\bar{z}}$

$B \sim B + 2\pi$ a scalar

F the gauge field curvature

Local operators: $\mathcal{O}_p =: \exp(ipB(x)) :$, independent of x $\qquad p \sim p \mod k$

Wilson lines: $W_q =: \exp\left(iq\oint A\right):$ $\qquad q \sim q \mod k$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q O_p$ $\qquad \xi = \exp(2\pi i/k)$

Projectors: $\Pi_m = \frac{1}{k}\sum_{n=0}^{k-1}\xi^{nm}\mathcal{O}_n$ $\qquad$ which obey $\Pi_m\Pi_n = \delta_{mn}\Pi_m, \ \sum_m \Pi_m = 1$

(1+1)d unitary semisimple topological & near-topological field theories

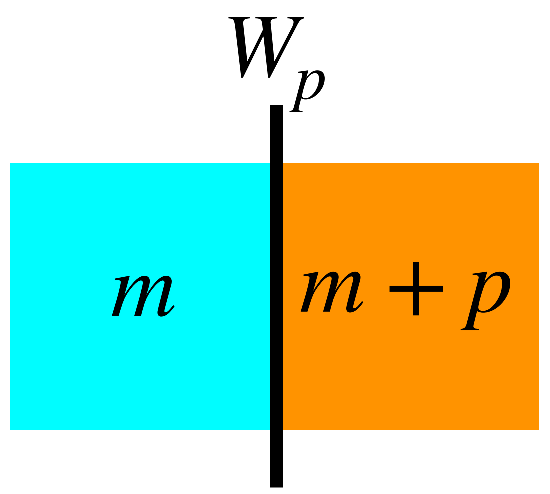
Ex: Abelian BF at level k (Hellerman, ES 1012.5999)

Clock-shift commutation relations: $\mathcal{O}_p W_q \; = \; \xi^{pq} \, W_q O_p$ $\qquad\qquad \xi = \exp(2\pi i/k)$

Projectors: $\quad \Pi_m \; = \; \dfrac{1}{k} \displaystyle\sum_{n=0}^{k-1} \xi^{nm} \mathcal{O}_n$ $\qquad$ which obey $\Pi_m \Pi_n \; = \; \delta_{mn} \Pi_m, \quad \displaystyle\sum_m \Pi_m \; = \; 1$

The clock-shift commutation relations imply

$$\Pi_m \, W_p \; = \; W_p \, \Pi_{m+p \mod k}$$



Interpretation: *The Wilson lines act as defects connecting different universes.*

That's a general feature of decomposition.

(1+1)d unitary semisimple topological & near-topological field theories

These are all the same as (decompose into) disjoint unions of invertible field theories
(= QFT(point) w/ dilaton shifts).

Formal reason: semisimplicity of the Frobenius algebra,
which implies not only that projectors exist,
but that all local operators are linear comb's of projectors.

Another reason: they all possess (noninvertible) 1-form symmetries,
defined by their (topological) operators and their OPE algebra.

Hence, as theories in (1+1)-dimensions w/ 1-form symmetry,
they decompose.

Decomposition in (1+1)-d gauge theories

Outline of the rest of the talk:

- Outline of decomposition in 2d & 1-form symmetries
- Examples: G gauge theories with trivially-acting $K \subset G$
- Overview of literature
- Examples: orbifolds
- Examples: unitary TQFTs

Next:
- Original motivation: string compactifications on stacks & gerbes
- Outline of application to anomaly resolution

Mathematical interpretation:

So far I've just talked abstractly about 2d theories & 1-form symmetries.

This has a mathematical interpretation: "gerbes"

A G–gerbe is a fiber bundle whose fibers are copies of BG.

A sigma model on a G–gerbe has a global BG symmetry,

just as a sigma model on a G–bundle has a global G symmetry,

from translations on the fibers.

Furthermore, $BG = [\mathrm{point}/G]$

so whenever a group acts trivially,

you should expect a gerbe structure (1-form symmetry) somewhere.

Mathematical interpretation:

Twenty years ago, I was interested in studying
\`sigma models on gerbes' as possible sources of new string compactifications.

Potential issues, since solved:

construction of QFT; cluster decomposition; moduli;
mod' invariance & unitarity in orbifolds; potential presentation-dependence.

What we eventually learned was that these theories are well-defined,
but,
are disjoint unions of ordinary theories, at least in (2,2) susy cases,
because of decomposition.

Not really new compactifications, but instead: GW predictions, GLSM phases.

Outline of the rest of the talk:

✓ • Outline of decomposition in 2d & 1-form symmetries

✓ • Examples: G gauge theories with trivially-acting $K \subset G$

✓ • Overview of literature

✓ • Examples: orbifolds

✓ • Examples: unitary TQFTs

✓ • Original motivation: string compactifications on stacks & gerbes

Next: • Outline of application to anomaly resolution

So far I've given a general overview of decomposition,
a property of d-dim'l QFTs with $(d-1)$-form symmetries.

Next week, at the workshop,
I'll discuss how to apply decomposition to resolve anomalies in orbifolds in 2d.

(Robbins, Vandermeulen, ES '21)

Over the next few slides I'll give a quick overview of how that works.

Anomaly resolution in orbifolds

We use decomposition to interpret a proposal of (Wang-Wen-Witten '17).

WWW: given an anomalous (ill-defined) orbifold $[X/G]$, (G finite,)
replace G by a larger finite group Γ obeying certain properties,

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

and add phases.

Because Γ has a subgroup K that acts trivially,
orbifolds $[X/\Gamma]$ will decompose,
into copies & covers of $[X/G]$.

However, just getting copies of $[X/G]$ won't help.
We also need to add certain new modular-inv't phases,
generalizing quantum symmetries....

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in orbifolds in which a subgroup K acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

It acts on twisted sector states by phases. Schematically:

$$
gz \underset{h}{\blacksquare} \; = \; B(\pi(h), z) \left(g \underset{h}{\blacksquare} \right)
$$

where

$z \in K \qquad g, h \in \Gamma$

$B \in H^1(G, H^1(K, U(1)))$

Decomposition in the presence of a quantum symmetry

Basic case:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,B}}{\mathrm{Ker}\,B}\right]_{\hat{\omega}}\right)$$

$$\text{where } B \;\in\; H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$$

This is more or less uniquely determined by consistency with previous results.

Recall for discrete torsion $\omega \in \mathrm{Ker}\ \iota^* \subset H^2(\Gamma, U(1))$, with $\beta(\omega) \neq 0$,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \;=\; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right)$$

The result at top needs to include this as a special case, and it does.

Application to anomaly resolution

Procedure: replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$

where $d_2 B \; = \; \alpha \in H^3(G, U(1))$, the anomaly of the G orbifold.

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B \right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B} \right]_{\hat{\omega}} \right)$$

— using earlier results for decomp' in orb' w/ quantum symmetry

Note that since $d_2 B = \alpha, \quad \alpha|_{\mathrm{Ker}\, B} = 0$

So, $\mathrm{Ker}\, B \subset G$ is automatically anomaly-free!

Summary: $[X/\Gamma]_B =$ copies of orbifold by anomaly-free subgroup.

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension: Define $\Gamma = \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	Result
1	1	—	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
-1	1	$\langle a \rangle, \langle ab \rangle$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

I'll discuss this more next week.

Decomposition in (1+1)-d gauge theories

- Outline of decomposition in 2d & 1-form symmetries
- Examples: G gauge theories with trivially-acting $K \subset G$
- Overview of literature
- Examples: orbifolds
- Examples: unitary TQFTs
- Original motivation: string compactifications on stacks & gerbes
- Outline of application to anomaly resolution

Summary

Decomposition: \`one' QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories
with $n-$form symmetries.

(I've focused on examples in 2d,
but examples exist in other dim's too.)

Can be used to understand anomaly-resolution procedure of www:

replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$,
but decomposition implies
$\mathrm{QFT}\left([X/\Gamma]_B\right) = \text{copies of } \mathrm{QFT}\,([X/\mathrm{Ker}\, B \subset G])$,
which is explicitly non-anomalous.

Thank you for your time !