A survey of recent developments in GLSMs

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It's been 30 years since Witten first published his "Phases…" paper,
roughly the same time that the Web came into existence.

However, work continues to this day on GLSMs,
as we shall hear this week!

We have an audience full of experts,
and the organizers would love to hear talks from everyone here,
but time does not permit — I apologize to everyone who won't get to speak.

However, we're talking about organizing a proceedings volume,
which would welcome contributions from
both speakers **and** participants
— expect to hear more about this later this summer.

Now,
we have here people with a wide range of backgrounds and approaches to GLSMs,
including both mathematicians and physicists.

The organizers thought it would be useful to have an introductory survey,
to remind everyone of the proverbial big picture,
how everything fits together,
to aid both in understanding the talks and to foster collaborations.

That's the goal of this talk.

So: The goal of this talk is to briefly outline, to survey,
developments and research areas in GLSMs.

To be clear, a one-hour talk simply is not nearly sufficient to describe,
much less give justice to,
everything going on,

but I do hope to outline an overall picture of how everything is related,
and links between different areas.

I apologize in advance to the many people whose work I won't be
able to talk about here in detail
— it's not lack of interest, it's merely lack of time!

This survey talk has four parts:

- Constructions of geometries
- Quantum cohomology and 2d mirrors
- Quantum sheaf cohomology
- Quantum K theory

I'll try to use these four broad topics to link the various talks we'll have this week.

Part 1: Constructions of geometries

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Originally, GLSMs were used to give physical realizations (as 2d QFTs) of geometries of the form $\mathbb{C}^n//G$,
and complete intersections therein.

Briefly: realize the symplectic quotient $\mathbb{C}^n//G$
as a 2d supersymmetric G-gauge theory with matter fields corresponding to $\mathbb{C}^n$,
plus additional matter and a superpotential to realize a complete intersection.

Ex: to describe hypersurface $\{G = 0\}$, add field p & $W = pG$

I'll refer to this as a "perturbative" description.

However, nowadays we know of more subtle ways to realize geometries....

Part 1: Constructions of geometries

However, nowadays we know of more subtle ways to realize geometries....

Two basic alternative phenomena:

- Strong coupling effects in 2d gauge theory restrict space of vacua.

  Prototype: Grassmannian-Pfaffian (Hori, Tong, hep-th/0609032)

- Decomposition locally realizes a branched cover

  Prototype: CI quadrics - branched covers (Caldararu, Distler, Hellerman, Pantev, ES arXiv: 0709.3855)

We also know of dualities that can turn these into perturbative descriptions.

I'll quickly walk through these in turn....

Nonperturbative construction of Pfaffians

(Hori, Tong hep-th/0609032)

Prototypical example: GLSM for $G(2,7)[1^7]$

$U(2)$ gauge theory with 7 fundamentals plus 7 p_α charged under det $U(2)$

$$W = \sum_a p_\alpha G_\alpha \left(\epsilon_{ab} \phi_i^a \phi_j^b \right) = \sum_{ij} \epsilon_{ab} \phi_i^a \phi_j^b A^{ij}(p)$$

$r \gg 0$: $G(2,7)[1^7]$ by standard analysis

$r \ll 0$: utilizes a strong-coupling analysis

Loci with 1 massless doublet (generic case): 0 susy vacua

Loci with 3 massless doublets: 1 susy vacuum

This describes a Pfaffian variety.

Nonperturbative construction of branched covers

(Caldararu, Distler, Hellerman, Pantev, ES
arXiv: 0709.3855)

Prototypical example: GLSM for $\mathbb{P}^3[2,2]$

$U(1)$ gauge theory, 4 ϕ_i chiral mult's of charge $+1$, 2 p_a chiral mult's of charge -2

$$W = \sum_a p_a G_a(\phi) = \sum_{ij} S^{ij}(p) \phi_i \phi_j$$

$r \gg 0$ phase: $\mathbb{P}^3[2,2]$, standard analysis

$r \ll 0$ phase: S^{ij} is a mass matrix; looks like $\mathbb{P}^1 = \text{Proj}\, \mathbb{C}[p_a]$?????

Correct analysis:

In a Born-Oppenheimer approx' over the base $\mathbb{P}^1$,
the 2d theory generically has a $\mathbb{Z}_2$ 1-form symmetry:
decomposes into double cover, branched over deg 4 locus $\{\det A = 0\}$

Both phases are T^2.

Nonperturbative construction of branched covers

(Caldararu, Distler, Hellerman, Pantev, ES
arXiv: 0709.3855)

Very similar example: $\mathbb{P}^5[2,2,2] \leftrightarrow$ branched double cover of $\mathbb{P}^2$, branched over deg 6 curve

Both phases are K3s.

Starting in 3-folds, the examples become more interesting.

$\mathbb{P}^7[2,2,2,2] \leftrightarrow$ *nc res'n* of branched double cover

I'll outline what that means next....

One property of these constructions is that sometimes,
what appears is actually a *noncommutative resolution* of a branched cover.

(Defined by Kuznetsov, van den Bergh, ... in terms of derived category.)

In other words, the GLSM gives a UV representation of a closed string CFT for a nc res'n.

Example: The $r \ll 0$ phase of the GLSM for $\mathbb{P}^7[2,2,2,2]$
is a noncommutative resolution
of a branched double cover of $\mathbb{P}^3$.

They are detected physically by studying matrix factorizations in (hybrid) LG phase,
ie, by examining the D-branes in the theory.

There isn't time here to explore the meaning of these nc resolutions.
However, they will instead be discussed later this week
in talks of Katz, Schimannek, and also Romo, Guo.

Another property of these 3-fold examples (both Grassmannian/Pfaffian & br' covers)
is that the different GLSM phases are *not* birational.

This contradicted folklore of the time, which said that all GLSM phases were birational.

Instead, these phases are related by homological projective duality.

(Kuznetsov, math.AG/ 0507292, 0510670, 0610957)

Homological projective duality has been studied in this context in mathematics,
in variations of GIT quotients,
see e.g. work of Ballard, Favero, Deliu, Katzarkov, Halpern-Leistner, & others.

This is beyond the scope of today's talk, but I believe will be explored further
in later talks this week by Guo, Romo.

We can also realize similar effects perturbatively.

Example:

Pfaffians via PAX, PAXY models of (Jockers, Kumar, Lapan, Morrison, Romo arXiv: 1205.3192)

Nowadays, we can use dualities to exchange perturbative and nonperturbative constructions of geometry, see work by e.g. Knapp & collaborators.

Part 2: Quantum cohomology and 2d mirrors

Quantum cohomology

(Morrison, Plesser hep-th/9412236)

One of the original applications of GLSMs was to make predictions
for quantum cohomology rings of Fano toric varieties.

In particular, quantum cohomology can be seen in a Coulomb branch computation.

Under RG flow, the GLSM for $\mathbb{P}^n$ describes a space that shrinks,
to (classical) zero-size, and then onto the Coulomb branch,
where QH is described as classical critical locus of a (twisted one-loop effective) superpotential,
instead of a sum over rat'l curves.

In other words, we can use the GLSM to replace counting rational curves (on Higgs)
with an algebraic computation (on Coulomb branch),
that encodes the same result.

Quantum cohomology

(Morrison, Plesser hep-th/9412236)

For $\mathbb{C}^n//G$ for $G = (\mathbb{C}^*)^k$, the superpotential is of the form

$$\widetilde{W}(\sigma) = \sum_{a=1}^{k} \sigma_a \left[\tau_a + \sum_i Q_i^a \left(\ln \left(\sum_{b=1}^{k} Q_i^b \sigma_b \right) - 1 \right) \right]$$

and assuming that the theory flows in the IR to the Coulomb branch
(eg, geometry is Fano $\Leftrightarrow$ QFT is asymptotically free),
then the resulting critical locus $\partial \widetilde{W} / \partial \sigma_a = 0$ is given by

$$\prod_i \left(\sum_b Q_i^b \sigma_b \right)^{Q_i^a} = \exp(2\pi i \tau_a) = q_a$$

Example: $\mathbb{P}^n$

Under RG flow, the GLSM for $\mathbb{P}^n$ describes a space that shrinks, to (classical) zero-size, and then onto the Coulomb branch.

One-loop twisted effective superpotential

$$\widetilde{W} = \sigma \left[\tau + \sum_{i=1}^{n+1} \left(\ln \sigma - 1 \right) \right]$$

Critical locus: $$\frac{\partial \widetilde{W}}{\partial \sigma} = \tau + \ln \left(\sigma^{n+1} \right)$$

$$= 0 \qquad \Rightarrow \sigma^{n+1} = \exp(-\tau) = q$$

The QH ring rel'n!

Quantum cohomology

The same ideas apply to nonabelian GLSMs,
meaning, GLSMs describing spaces of the form $\mathbb{C}^n//G$ for nonabelian G
(and subvarieties thereof).

For Fano $\mathbb{C}^n//G$,
RG flow again drives the GLSM out of geometry and onto the Coulomb branch.

Again the ring arises as the critical locus of a superpotential,
albeit with two subtleties:

- The Coulomb branch is a Weyl-group orbifold of the σ's
- The Coulomb branch is an open subset (remove 'excluded loci')

Example: $G(k,n)$

The Grassmannian $G(k,n) = \mathbb{C}^{kn}//GL(k)$, acting as n copies of fundamental rep'.

Here, the twisted one-loop effective superpotential is

$$\widetilde{W} = \sum_{a=1}^{k} \sigma_a \left[-\ln((-)^{k-1}q) + \sum_{i,b} Q^a_{ib} \left(\ln \left(\sum_{c=1}^{k} Q^c_{ib} \sigma_c \right) - 1 \right) \right] \qquad \text{for } Q^a_{ib} = \delta^a_b$$

$$= \sum_{a=1}^{k} \left[-\ln((-)^{k-1}q) + \sum_{i=1}^{n} \left(\ln \sigma_a - 1 \right) \right]$$

Critical locus:
$$\frac{\partial \widetilde{W}}{\partial \sigma_a} = -\ln\left((-)^{k-1}q\right) + \ln(\sigma_a)^n$$

$$= 0 \qquad \text{implies} \qquad (\sigma_a)^n = (-)^{k-1}q$$

Example: $G(k, n)$

Critical locus: $(\sigma_a)^n = (-)^{k-1} q$

It may not look like it yet, but this is the QH ring relation for the Grassmannian.

Quick check: number of vacua

Order n polynomial, so n solutions for each σ_a so kn possible values, but...

Recall the Weyl group of $U(k)$, namely S_k, interchanges the σ_a,
plus we avoid the excluded locus (where σ_a collide),
hence number of solutions to critical locus equation is

$$\binom{n}{k} = \chi(G(k, n)) \quad \text{as expected}$$

How to write the result more symmetrically?

Example: $G(k,n)$

Critical locus: $(\sigma_a)^n = (-)^{k-1} q$

It may not look like it yet, but this is the QH ring relation for the Grassmannian.

Briefly: The σ_a are k distinct roots of the polynomial $\xi^n + (-)^k q = 0$

Let $\overline{\sigma}_{a'}$ denote the remaining $n-k$ roots.

From Vieta's theorem (algebra), the elementary symm' polynomials in $\sigma_a, \overline{\sigma}_{a'}$ are

$$\sum_{r=0}^{n-k} e_{\ell-r}(\sigma) e_r(\overline{\sigma}) = (-)^{n-k} q \delta_{\ell,n} + \delta_{\ell,0}$$

Define $c_t(\sigma) = \sum_{\ell=0}^{k} t^\ell e_\ell(\sigma)$ then this is $c_t(\sigma)\, c_t(\overline{\sigma}) = 1 + (-)^{n-k} q t^n$

which is a standard expression for $QH^\bullet(G(k,n))$

(Witten, "Verlinde algebra..." equ'n (3.16))

So far I've reviewed Coulomb-branch-based quantum cohomology computations in GLSMs.

Another approach to these & related questions is to use mirror symmetry,
which I'll review next….

Hori-Vafa mirror

(Hori, Vafa, hep-th/0002222; Morrison-Plesser hep-th/9508107)

$U(1)^r$ gauge theory with matter multiplets of charges ρ_i^a

Mirror:

Fields: σ_a $\quad a \in \{1, \cdots, r\}, \ \ \sigma_a = \overline{D}_+ D_- V_a$

Y^i $\quad$ mirror to matter fields

Superpotential

$$W = \sum_a \sigma_a \left(\sum_i \rho_i^a Y^i - t_a \right) + \sum_i \exp\left(-Y^i\right)$$

Periodicities $\quad Y^i \sim Y^i + 2\pi i$

After all, in 2d, the theta angle acts like an electric field, and periodicity on a noncompact space is determined by screening by matter fields.

Nonabelian mirror

(Gu, ES, arXiv:1806.04678)

For a G gauge theory, pick a Cartan torus $U(1)^r \subseteq G$,

matter multiplets in representation ρ.

Mirror: Weyl-group orbifold of the following LG model

Fields:

σ_a — $a \in \{1, \cdots, r\}$, $\sigma_a = \overline{D}_+ D_- V_a$

Y^i — mirror to matter fields

$X_{\tilde{\mu}}$ — correspond to nonzero roots of $\mathfrak{g}$

Superpotential

$$W = \sum_a \sigma_a \left(\sum_i \rho_i^a Y^i - \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) + \sum_i \exp\left(-Y^i\right) + \sum_{\tilde{\mu}} X_{\tilde{\mu}}$$

ρ_i = weight vector, $\alpha_{\tilde{\mu}}$ = root vector

Idea: "Abelian duality in Cartan torus, at generic pt on Coulomb branch"

Operator mirror map

In principle, both these mirrors have the property that
correlation functions in the original A-twisted GLSM
are the same as
correlation functions in the B-twisted LG mirror.

We can derive a mirror map for operators from the critical loci of the superpotential.

$$W = \sum_a \sigma_a \left(\sum_i]rho_i^a Y^i - \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t \right) + \sum_i \exp(-Y^i) + \sum_{\tilde{\mu}} X_{\tilde{\mu}}$$

$\frac{\partial W}{\partial X_{\tilde{\mu}}} = 0$ implies $X_{\tilde{\mu}} = \sum_a \sigma_a \alpha_{\tilde{\mu}}^a$

$\frac{\partial W}{\partial Y^i} = 0$ implies $\exp(-Y^i) = \sum_a \sigma_a \rho_i^a$

B A

Example: $\mathbb{P}^n$

Mirror is LG model with superpotential

$$W \;=\; \sigma\left(\sum_i Y_i - t \right) \;+\; \exp(-Y_1) \;+\; \cdots \; \exp(-Y_{n+1})$$

Integrate out σ, Y_{n+1}:

$$W \;=\; \exp(-Y_1) \;+\; \cdots \;+\; \exp(-Y_n) \;+\; q \exp\left(Y_1 + \cdots + Y_n\right)$$

where $q = \exp(-t)$

Critical locus:

$$\frac{\partial W}{\partial Y_i} \;= -\exp(-Y_i) \;+\; q \exp\left(Y_1 + \cdots + Y_n\right)$$

$$=\; 0 \qquad \text{implies} \qquad \exp(-Y_i) \;=\; q \prod_j \exp(+Y_j)$$

so if we define $X = \exp(-Y_i)$ then $X^{n+1} = q$ Standard ring rel'n !

Example: $G(k, n)$

Mirror is S_k orbifold of LG model with superpotential

$$W = \sum_{a=1}^{k} \sigma_a \left(\sum_{ib} \rho_{ib}^a Y^{ib} - \sum_{\mu \neq \nu} \alpha_{\mu\nu}^a \ln X_{\mu\nu} - t \right) + \sum_{ia} \exp\left(-Y^{ia}\right) + \sum_{\mu \neq \nu} X_{\mu\nu}$$

for $\rho_{ib}^a = \delta_b^a, \quad \alpha_{\mu\nu}^a = -\delta_\mu^a + \delta_\nu^a$

$$= \sum_{a=1}^{k} \sigma_a \left(\sum_{a} Y^{ia} + \sum_{\nu \neq a} \left(\frac{X_{a\nu}}{X_{\nu a}} \right) - t \right) + \sum_{ia} \exp\left(-Y^{ia}\right) + \sum_{\mu \neq \nu} X_{\mu\nu}$$

Integrate out σ_a, Y^{na}, then

$$W = \sum_{i=1}^{n-1} \sum_{a=1}^{k} \exp\left(-Y^{ia}\right) + \sum_{\mu \neq \nu} X_{\mu\nu} + \sum_{a=1}^{k} \Pi_a$$

for $\quad \Pi_a = \exp(-Y^{na}) = q \left(\prod_{i=1}^{n-1} \exp(+Y^{ia}) \right) \left(\prod_{\nu \neq a} \frac{X_{a\nu}}{X_{\nu a}} \right)$

Example: $G(k,n)$

$$W \;=\; \sum_{i=1}^{n-1}\sum_{a=1}^{k} \exp\left(-Y^{ia}\right) \;+\; \sum_{\mu\neq\nu} X_{\mu\nu} \;+\; \sum_{a=1}^{k} \Pi_a$$

for $\Pi_a = \exp(-Y^{na}) = q\left(\prod_{i=1}^{n-1}\exp(+Y^{ia})\right)\left(\prod_{\nu\neq a}\frac{X_{a\nu}}{X_{\nu a}}\right)$

Critical locus: $\frac{\partial W}{\partial Y^{ia}} = -\exp(-Y^{ia}) + \Pi_a = 0$ so $\exp(-Y^{ia}) = \Pi_a$ for all i

$$\frac{\partial W}{\partial X_{\mu\nu}} \;=\; 1+\frac{\Pi_\mu - \Pi_\nu}{X_{\mu\nu}} \;=\; 0 \qquad \text{so} \qquad X_{\mu\nu} \;=-\Pi_\mu + \Pi_\nu$$

which implies $\prod_{\nu\neq a}\frac{X_{a\nu}}{X_{\nu a}} \;=\; (-)^{k-1}$ $\qquad (\Pi_a)^n \;=\; (-)^{k-1}q$

Example: $G(k,n)$

Critical locus: $$\prod_{\nu\neq a}\frac{X_{a\nu}}{X_{\nu a}} = (-)^{k-1} \qquad \left(\Pi_a\right)^n = (-)^{k-1}q$$

Operator mirror map: $$\exp(-Y^{ia}) = \Pi_a \leftrightarrow \sigma_a$$
$$X_{\mu\nu} \leftrightarrow -\sigma_\mu + \sigma_\nu$$

so the critical locus equation recovers the expression for $QH^\bullet(G(k,n))$ described earlier:

$$\left(\Pi_a\right)^n = (-)^{k-1}q \quad \text{becomes} \quad \left(\sigma_a\right)^n = (-)^{k-1}q$$

Also, poles in the superpotential at $X_{\mu\nu} = 0$ correspond to excluded locus:

$$\sigma_\mu \neq \sigma_\nu \quad \text{for} \quad \mu \neq \nu$$

Related: We'll have a talk later this week on nonabelian T duality by Cabo Bizet.

Shout-out: Supersymmetric localization

Susy localization was first applied to 2d GLSMs in, to my knowledge,

(Benini, Cremonesi, arXiv: 1206.2356; Doroud, Gomis, Le Floch, Lee, arXiv: 1206.2606)

and was quickly applied to give alternative
physical computations of Gromov-Witten invariants,

(Jockers, Kumar, Lapan, Morrison, Romo, arXiv: 1208.6244)

elliptic genera (Benini, Eager, Hori, Tachikawa arXiv: 1305.0533, 1308.4896),
and Gamma classes (Halverson, Jockers, Lapan, Morrison arXiv: 1308.2157; Libgober math/9803119,
Iritani arXiv: 0712.2204, 0903.1463; Katzarkov, Kontsevich, Pantev arXiv: 0806.0107).

These are important contributions, which I wanted to acknowledge,
but lack of time prevents me from going into any detail.

Shout-out: D-branes in GLSMs

GLSMs on open strings were explored in detail in (Herbst, Hori, Page arXiv: 0803.2045), which described e.g. the grade restriction rule.

There isn't time in this talk to explain any details, but we will be hearing more about this later this week in talks from Brunner, Hori, Guo, Aleshkin.

Part 3: Quantum sheaf cohomology

So far I've discussed GLSMs for 2d theories with (2,2) susy.

There also exist GLSMs for 2d theories with (0,2) susy.
Briefly, these specify a space X, along with a holomorphic vector bundle $\mathcal{E} \to X$,
obeying the constraint $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$.

(Discussed in Witten "Phases", also Distler, Kachru hep-th/9309110, 9406090, 9406091, ...)

These admit analogues of the A, B model topological twists.

A/2 model: exists when $\det \mathcal{E}^* \cong K_X$
operators ~ $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$

B/2 model: exists when $\det \mathcal{E} \cong K_X$
operators ~ $H^\bullet(X, \wedge^\bullet \mathcal{E})$

Reduce to (2,2) and the ordinary A, B models in the special case $\mathcal{E} = TX$

Quantum sheaf cohomology

Ordinary quantum cohomology arises from A-twisted (2,2) susy NLSMs.

Operators in the ordinary A model ~ $H^{\bullet,\bullet}(X) = H^{\bullet}(X, \wedge^{\bullet} T^*X)$

Correlation functions ~ intersection theory on a moduli space of curves

Quantum sheaf cohomology arises from A/2-twisted (0,2) susy NLSMs.

Operators in the A/2 model ~ $H^{\bullet}(X, \wedge^{\bullet} \mathscr{E}^*)$

Classical product: $H^{\bullet}(X, \wedge^{\bullet} \mathscr{E}^*) \times H^{\bullet}(X, \wedge^{\bullet} \mathscr{E}^*) \mapsto H^{\bullet+\bullet}(X, \wedge^{\bullet+\bullet} \mathscr{E}^*)$

Correlation functions ~ sheaf cohomology on a moduli space of curves

"quantum sheaf cohomology"

Reduces to ordinary quantum cohomology in the special case $\mathscr{E} = TX$

(S Katz, ES hep-th/0406226)

Example:

(McOrist, Melnikov arXiv: 0712.3272, 0810.0012; Donagi, Guffin, Katz, ES arXiv: 1110.3751, 1110.3752)

Ordinary quantum cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$\mathbb{C}[x, y]/(x^2 - q_1, y^2 - q_2)$$

Quantum sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$:

Define a holomorphic vector bundle $\mathscr{E}$ to be a deformation of the tangent bundle:

$$0 \longrightarrow \mathscr{O}^2 \stackrel{*}{\longrightarrow} \mathscr{O}(1,0)^2 \oplus \mathscr{O}(0,1)^2 \longrightarrow \mathscr{E} \longrightarrow 0$$

where $* = \begin{bmatrix} Ax & Bx \\ Cy & Dy \end{bmatrix}$ for A, B, C, D a set of constant 2×2 matrices and x, y column vectors of homog' coords

then the quantum sheaf cohomology ring of $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{E})$ is

$$\mathbb{C}[x, y]/\left(\det(Ax + By) - q_1, \det(Cx + Dy) - q_2\right)$$

When $\mathscr{E} = TX$ (meaning, $A = D = I, B = C = 0$) this reduces to standard $QH^\bullet$ above.

One way to compute quantum sheaf cohomology, for Fano spaces, is using GLSMs.

(McOrist, Melnikov arXiv:0712.3272, 0810.0012)

Basic idea is the same: under RG flow, the GLSM flows onto a Coulomb branch, where the OPE ring rel'ns can be computed as the critical locus of a (twisted one-loop effective) superpotential.

In abelian cases it is of the form

$$\widetilde{W} = \sum_a \Upsilon_a \ln \left(q_a^{-1} \prod_i (\det M_i)^{Q_i^a} \right)$$

where $M_i = M_i(\sigma_a)$ are matrices encoding tangent bundle deformations. and Υ_a is a (0,2) Fermi superfield (part of (2,2) σ_a).

Critical locus: $$\frac{\partial \widetilde{W}}{\partial \Upsilon_a} = 0 \quad \Longrightarrow \quad \prod_i \left(\det M_i(\sigma) \right)^{Q_i^a} = q_a$$

Critical locus: $$\frac{\partial \widetilde{W}}{\partial \Upsilon_a} = 0 \quad \Longrightarrow \quad \prod_i \left(\det M_i(\sigma)\right)^{Q_i^a} = q_a$$

Example (already mentioned): $\mathbb{P}^1 \times \mathbb{P}^1$

The q.s.c. ring relations are $\det(Ax + By) = q_1, \ \det(Cx + Dy) = q_2$

Example: $G(k,n)$

Deform tangent bundle to $\mathscr{E}$: $$0 \longrightarrow S^* \otimes S \xrightarrow{*} \mathbb{C}^n \otimes S \longrightarrow \mathscr{E} \longrightarrow 0$$

where $* : \omega_a^b \mapsto A^i_j \omega_a^b \phi_b^j + \omega_b^b B^i_j \phi_a^j$

The q.s.c relations are then $\det\left(A\sigma_a + B\mathrm{Tr}\sigma\right) = (-)^{k-1} q_a$

(Guo, Lu, ES arXiv: 1512.08586)

which for $\mathscr{E} = TX$ reduce to $\left(\sigma_a\right)^n = (-)^{k-1} q_a$,

defining $QH^\bullet(G(k,n))$ as seen previously.

Quantum sheaf cohomology is now known for

- Fano toric varieties (math = Donagi, Guffin, Katz, ES, arXiv: 1110.3751; physics = McOrist, Melnikov arXiv: 0712.3272, 0810.0012)
- Grassmannians (classical ring math = Guo, Lu, ES, arXiv: 1605.01410; physics of qsc = Guo, Lu, ES arXiv: 1512.08586)
- flag manifolds (physics = Guo arXiv: 1808.00716)

all with bundle = deformation of tangent bundle.

More general cases are all open.

There is also a notion of mirror symmetry for (0,2) supersymmetric theories, known as (0,2) mirror symmetry.

Just as ordinary mirror symmetry relates pairs of CYs X, Y, (0,2) mirror symmetry relates pairs $(X, \mathscr{E})$, $(Y, \mathscr{F})$, where X, Y are CYs and $\mathscr{E} \to X$, $\mathscr{F} \to Y$ are holomorphic bundles such that

$$\mathrm{ch}_2(\mathscr{E}) = \mathrm{ch}_2(TX), \, \mathrm{ch}_2(\mathscr{F}) = \mathrm{ch}_2(TY)$$

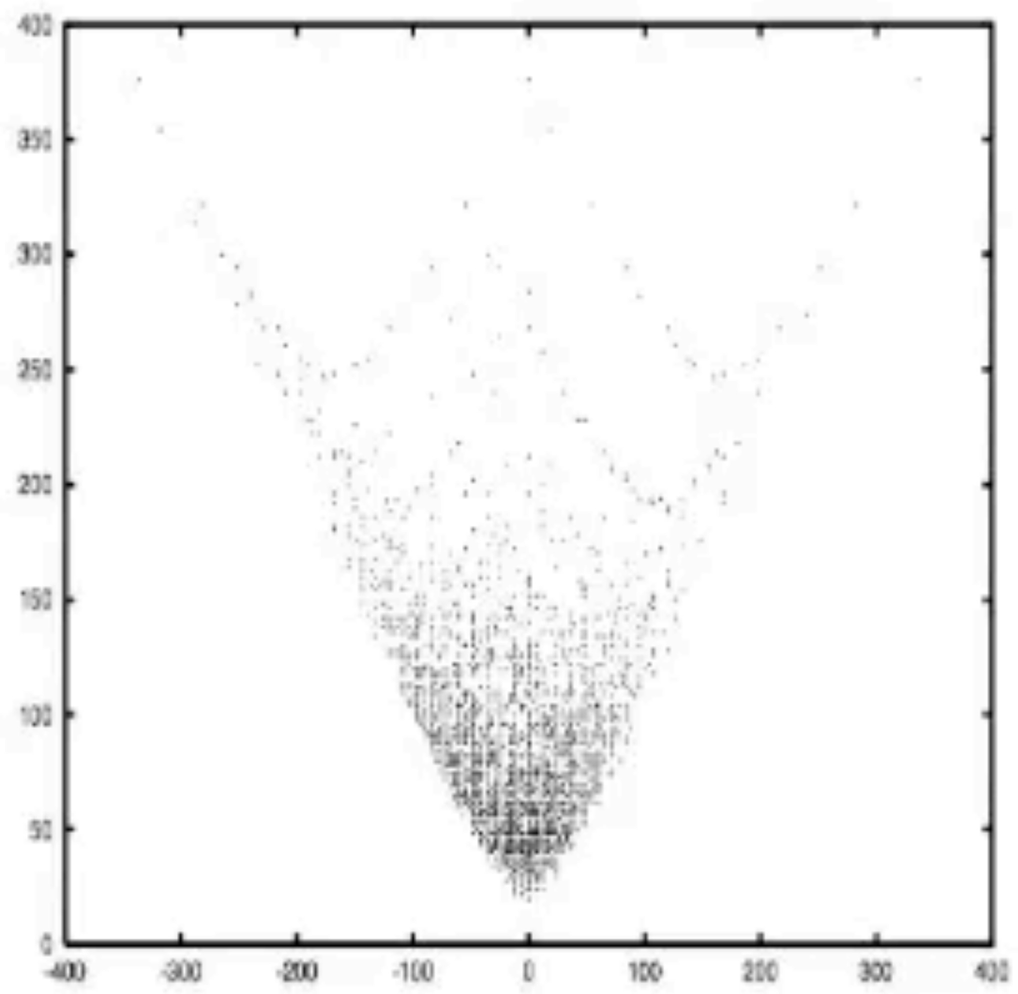
Relation: A/2 on $(X, \mathscr{E})$ = B/2 on $(Y, \mathscr{F})$

$$H^{\bullet}(X, \wedge^{\bullet} \mathscr{E}^*) = H^{\bullet}(Y, \wedge^{\bullet} \mathscr{F})$$

which for $\mathscr{E} = TX, \mathscr{F} = TY$ reduces to standard Hodge diamond flip.

(0,2) mirror symmetry:

Some numerical evidence for (0,2) mirrors:



Horizontal axis: $h^1(\mathscr{E}) - h^1(\mathscr{E}^*)$

Vertical axis: $h^1(\mathscr{E}) + h^1(\mathscr{E}^*)$

for examples with $\mathscr{E}$ of rank 4

(Blumenhagen, Schimmrigk, Wisskirchen, hep-th/9609167)

Nowadays there exist (limited) proposals for mirror constructions for 2d (0,2) theories.

(see e.g. Blumenhagen, Schimmrigk, Wisskirchen, hep-th/9609167, Blumenhagen, Sethi hep-th/9611172, Adams, Basu, Sethi hep-th/039226, Melnikov, Plesser arXiv: 1003.1303; Gu, Guo, ES arXiv: 1908.06036)

For (0,2) GLSMs describing Fano spaces,
(limited) proposals exist for (0,2) mirrors as (0,2) Landau-Ginzburg models.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Recall we deform the tangent bundle to $\mathscr{E}$ defined by

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathscr{E} \longrightarrow 0$$

where $* = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ for A, B, C, D a set of constant 2×2 matrices

If restrict to diagonal A, B, C, D, then can write a mirror (0,2) LG model (Gu, Guo, ES arXiv: 1908.06036)

$$W = \Upsilon \left(Y_0 + Y_1 - t_0 \right) + \tilde{\Upsilon} \left(\tilde{Y}_0 + \tilde{Y}_1 - t_1 \right) + \sum_{i=0}^{1} F_i \left(E_i(\sigma, \tilde{\sigma}) - \exp(-Y_i) \right) + \sum_{i=0}^{1} \tilde{F}_j \left(E_j(\sigma, \tilde{\sigma}) - \exp(-\tilde{Y}_j) \right)$$

where $E_i(\sigma, \tilde{\sigma}) = a_i \sigma + b_i \tilde{\sigma}$, $\tilde{E}_i(\sigma, \tilde{\sigma}) = c_i \sigma + d_i \tilde{\sigma}$, $A = \mathrm{diag}(a_0, a_1)$, etc

Υ_i, F_i are (0,2) Fermi superfields $\sigma_{(2,2)} \mapsto \sigma_{(0,2)}, \Upsilon_{(0,2)}$ $Y_{(2,2)} \mapsto Y_{(0,2)}, F_{(0,2)}$

We have several talks at this meeting on various aspects of 2d (0,2) theories, including talks of Gukov, Litvinov, Franco.

Shout-out: Trialities

(Gadde, Gukov, Putrov arXiv: 1306.4320)

Briefly:

A (0,2) theory on $G(k, n)$ with bundle $S^{\oplus N} \oplus (Q^*)^{2k+N-n} \oplus (\det S^*)^{\oplus 2}$

is IR equivalent to

a (0,2) theory on $G(n-k, N)$ with bundle $S^{\oplus 2k+N-n} \oplus (Q^*)^n \oplus (\det S^*)^{\oplus 2}$

and also to

a (0,2) theory on $G(N-n+k, 2k+N-n)$ with bundle $S^{\oplus n} \oplus (Q^*)^N \oplus (\det S^*)^{\oplus 2}$

for suitable values of k, n, N,
where S is the universal subbundle and Q the univ' quotient bundle.

I believe we'll hear more about triality in S. Franco's talk.

Shout-out: GLSMs with H flux, non-Kahler heterotic compactifications

GLSMs, esp (0,2) susy GLSMs, describing backgrounds with H flux have a long history,
(see e.g. Adams, Ernebjerg, Lapan hep-th/0611084, Adams, Guarrera arXiv: 0902.4440, Adams, Lapan arXiv: 0908.4294, Adams, Dyer, Lee, arXiv: 1206.5815, Quigley, Sethi, arXiv: 1107.0714, Melnikov, Quigley, Sethi, Stern, arXiv: 1212.1212, Caldeira, Maxfield, Sethi arXiv: 1810.01388)

The details are well beyond the scope of this talk,
but definitely deserve mention.

Part 4: Quantum K theory

For me personally, quantum K theory first came to my attention in GLSMs
through the work of (Jockers, Mayr arXiv: 1808.02040, 1905.03548).

(I'm under the impression that it has also appeared elsewhere in physics
— I apologize to everyone this survey is glossing over.)

Analogous to other examples in this survey,
in many cases quantum K theory can be computed using Coulomb branch techniques.

Basic idea:

Consider a 3d GLSM, on a 3-mfld of the form $S^1 \times \Sigma_2$.
quantum K theory arises as OPEs of Wilson lines around the S^1,
moving parallel to one another along the base Σ_2.

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moving parallel to one another along the base Σ_2.

To compute those OPEs,
one does a Kaluza-Klein reduction along the S^1.
One gets an effective low-energy 2d theory,
with an infinite tower of fields.

Regularizing the sum of their contributions to the
2d twisted one-loop effective superpotential
effectively changes ordinary log's to dilogarithms Li_2.

The Wilson line OPE relations are the critical loci of that 2d twisted superpotential.

Example: $\mathbb{P}^n$

$$\widetilde{W} \;=\; (\ln q)(\ln x) \;+\; \sum_{i=1}^{n+1} \mathrm{Li}_2(x)$$ for pertinent Chern-Simons level

where $x = \exp(2\pi i R\sigma)$ for R = radius of S^1

Critical locus of W: $(1-x)^{n+1} = q$

This is the QK ring relation for $\mathbb{P}^n$. Identify $x \sim S \sim \mathcal{O}(-1)$.

Take the limit $R \to 0$:

$$x = \exp(2\pi i R\sigma) \mapsto 1 + 2\pi i R\sigma$$

$$q = R^{n+1} q_{2d}$$

so $(1-x)^{n+1} = q$ becomes $\sigma^{n+1} \propto q_{2d}$

which is the QH ring relation.

Example: $G(k, n)$

$$\widetilde{W} = \frac{k}{2}\sum_{a=1}^{k} (\ln x_a)^2 - \frac{1}{2}\left(\sum_{a=1}^{k} \ln x_a\right)^2 + \left(\ln(-)^{k-1}q\right)\sum_{a=1}^{k} \ln x_a + n\sum_{a=1}^{k} \mathrm{Li}_2(x_a)$$

for pertinent Chern-Simons level

where $x_a = \exp(2\pi i R\sigma_a)$

(Still present: Weyl group (S_k) orbifold, excluded locus $\sigma_a \neq \sigma_b$ for $a \neq b$)

Critical locus:

$$(-)^{k-1}q(x_a)^k = (1 - x_a)^n\left(\prod_{b+1}^{k} x_b\right)$$

Symmetrize using Vieta:

$$\sum_{r=0}^{n-i} e_{\ell-r}(x)\, e_r(\overline{x}) = e_\ell(T) + q e_{n-k}(\overline{x})\delta_{\ell,n-k}$$

where T are $(\mathbb{C}^*)^n$-equivariant parameters

Example: $G(k,n)$

$$\sum_{r=0}^{n-i} e_{\ell-r}(x)\, e_r(\bar{x}) \;=\; e_\ell(T) \;+\; q e_{n-k}(\bar{x})\delta_{\ell,n-k}$$

where T are $(\mathbb{C}^*)^n$-equivariant parameters

Interpretation:

$$e_\ell(\bar{x}) \;=\; \begin{cases} \wedge^\ell (\mathbb{C}^n/S) & \ell < n-k \\ (1-q)^{-1} \wedge^\ell (\mathbb{C}^n/S) & \ell = n-k \end{cases}$$

so the ring relations become

$$\sum_{r=0}^{n-k-1} \wedge^{\ell-r}(S) \star \wedge^r (\mathbb{C}^n/S) \;+\; \frac{1}{1-q} \wedge^{\ell-(n-k)} S \star \det(\mathbb{C}^n/S) \;= \wedge^\ell \mathbb{C}^n \;+\; \frac{1}{1-q}\det(\mathbb{C}^n/S)\delta_{\ell,n-k}$$

or after simplification

$$\lambda_y(S) \star \lambda_y(\mathbb{C}^n/S) \;=\; \lambda_y(\mathbb{C}^n) - y^{n-k}\frac{q}{1-q}\det(\mathbb{C}^n/S) \star \left(\lambda_y(S) - 1\right)$$

where $\lambda_y(S) = 1 + yS + y^2 \wedge^2 S + \cdots$

Example: $G(k,n)$

To summarize: the QK ring of the Grassmannian $G(k,n)$ can be presented in terms of the relation

$$\lambda_y(S) \star \lambda_y(\mathbb{C}^n/S) \;=\; \lambda_y(\mathbb{C}^n) - y^{n-k}\frac{q}{1-q}\det(\mathbb{C}^n/S) \star \left(\lambda_y(S) - 1\right)$$

(Gu, Mihalcea, ES, Zou, arXiv: 2008.04909, 2208.01091)

There exists an analogous presentation of QK for partial flag manifolds:

$$\lambda_y(S_i) \star \lambda_y(S_{i+1}/S_i) \;=\; \lambda_y(S_{i+1}) \;-\; y^{k_{i+1}-k_i}\frac{q_i}{1-q_i}\det(S_{i+1}/S_i) \star \left(\lambda_y(S_i) - \lambda_y(S_{i-1})\right)$$

where S_i is a universal subbundle of rank k_i

(Gu, Mihalcea, ES, Xu, Zhang, Zou, to appear)

You'll hear more about this in Weihong Xu's talk.

One detail I've glossed over: Chern-Simons levels

There exists a Chern-Simons level for which the theory duplicates standard QK.
It's generally suspected that other choices of CS levels correspond to
level structures in the sense of (Ruan, Zhang arXiv: 1804.06552),
but I for one don't know the precise dictionary.

Another detail I've glossed over: Wilson line OPEs

There has also been a great deal of work on this subject,
independent of quantum K theory per se.
See for example work of Closset, Kim, and others.

In 2d, we've discussed ordinary mirror symmetry & (0,2) mirror symmetry, and how they can be used to compute quantum cohomology.

In 3d there are also pertinent dualities, such as 3d mirror symmetry.

The details are, unfortunately, beyond the scope of this survey.

Other speakers on various aspects of quantum K theory include
Koroteev, Lee, Xu,
and related work in 3d GLSMs will be discussed by
Closset, Jockers, Litvinov.

We'll also hear about related notions in integrable systems
in talks by Koroteev, Gu.

Questions for the future:

Can quantum K theory and quantum sheaf cohomology be linked?

The boundary of a 3d N=2 theory hosts a 2d (0,2) theory.

One could imagine moving bulk operators to the boundary and using
bulk/boundary correspondence to describe
quantum sheaf cohomology (of the 2d (0,2) susy boundary)
as a module over
quantum K theory (of the 3d N=2 susy theory).

Problem: the bulk operators are Wilson lines, not local operators (unlike boundary).

Math interpretation of 2d (0,2) Wilson lines in terms of q.s.c. ?

One last shout-out: rigorous approaches to GLSMs

I've focused on physics in this talk,
but there have also been mathematically rigorous approaches to GLSMs,
see e.g. (Fan, Jarvis, Ruan arXiv: 0712.4021, 0712.4025, 1506.02109, 1603.02666, ...)

We will hear more about such constructions in talks by
Fan, Segal, Liu, Favero.

Thank you for your time,
and enjoy the workshop!