Recent developments in 2d (0,2) theories

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Over the last half dozen years, there's been a *tremendous* amount of progress in perturbative string compactifications.

A few of my favorite examples:

- nonpert' realizations of geometry (Pfaffians, double covers)
  (Hori-Tong '06, Caldararu et al '07,…)
- perturbative GLSM's for Pfaffians (Hori '11, Jockers et al '12,…)
- non-birational GLSM phases - physical realization of homological projective duality (Hori-Tong '06, Caldararu et al '07, Ballard et al '12; Kuznetsov '05-'06,…)
- examples of closed strings on noncommutative res'ns
  (Caldararu et al '07, Addington et al '12, ES '13)
- localization techniques: new GW & elliptic genus computations, role of Gamma classes, …
  (Benini-Cremonesi '12, Doroud et al '12; Jockers et al '12, Halverson et al '13, Hori-Romo '13, Benini et al '13, ….)
- heterotic strings: nonpert' corrections, 2d dualities, non-Kahler moduli (many)

I'll focus on just the last one….

My goal today: an overview of some work on perturbative heterotic string compactifications.

Briefly, we would like to understand the low energy EFT, which means we need to understand

- massless states (inc. moduli)
- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

and of course understanding any dualities could also be helpful.

I'll briefly outline the issues next….

First, some background.

In 10d, a heterotic string describes metric & gauge field.

To compactify, must specify not only a space X,
but also a bundle $\mathcal{E}$ on that space,
satisfying consistency conditions

$$[\mathrm{tr}\, F \wedge F] \;=\; [\mathrm{tr}\, R \wedge R]$$

Described on worldsheet by 2d (0,2) susy theory.

Simplest case: $\mathcal{E} = TX$, corresponding to (2,2) susy.
"embed the spin connection in gauge connection"

Simplest case: compactification on a Calabi-Yau with gauge bundle = tangent bundle (`embedding the spin connection' = (2,2) locus)

In this case, we know basics:

- massless states (inc. moduli)

— counted by cohomology of the CY; `chiral ring'

- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

Nonperturbative corrections = GW inv'ts

$\overline{\mathbf{27}}^3$ = A model TFT computation

$\mathbf{27}^3$ = B model TFT computation

More gen'l case: compactification on a Calabi-Yau with gauge bundle $\neq$ tangent bundle

- massless states (inc. moduli)

— counted by sheaf cohomology of the CY

- Yukawa couplings, superpotentials (inc. nonperturbative corrections)

Nonperturbative corrections $\neq$ GW inv'ts

$\overline{\mathbf{27}}^3$ = A/2 model computation

$\mathbf{27}^3$ = B/2 model computation

Heterotic compactifications on non-Kahler manifolds have also been studied, but far less is known.

— we currently have a partial grasp on moduli

(Svanes-de la Ossa, Anderson-Gray-Sharpe '14; Melnikov-Sharpe '11)

— other massless states, couplings, are unknown

My goal today is to give a survey of some of the results in heterotic strings & (0,2) supersymmetric worldsheets over the last few years, through the lens of chiral rings.

Outline:

- Chiral states in 2d (0,2) NLSM's
- Product structures in chiral rings in A/2, B/2 twists:

  quantum sheaf cohomology

- Nonabelian GLSMs:
  - Dualities in 2d and their geometry
  - Tests of Gadde-Gukov-Putrov triality
- Survey of moduli in non-Kahler cases

Review: chiral rings in 2d (2,2) NLSM's

Consists of states annihilated by
1 of left-moving & 1 of right-moving supercharges.

4 distinct possibilities, labelled (c,c), (a,c), (c,a), (a,a)

In a NLSM on a complex Kahler manifold X,
all correspond to cohomology of X.

More explicitly…

Review: chiral rings in 2d (2,2) NLSM's

In a (R,R) sector, in a NLSM on a space X, states have the schematic form

$$b^{j_1 \cdots j_p}_{\overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \psi_{-,j_1} \cdots \psi_{-,j_p} |0\rangle$$

$\psi_\pm$ worldsheet fermions, $\sim \; TX$

$$Q \; = \; Q_+ + Q_- \; \leftrightarrow \; d$$

Q-cohomology classes, counted by $H^{p,q}(X)$

Sit in a topologically protected subsector.

For a (0,2) NLSM, on space X with bundle $\mathcal{E}$, we'll again look at (R,R) sector states….

For 2d (0,2) NLSM's on Calabi-Yau's (CY's), Distler-Greene ('88) worked out the analogue:

In a (R,R) sector, zero-energy Q_+-closed states of form

$$b^{a_1 \cdots a_p}_{\overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,a_1} \cdots \lambda_{-,a_p} |0\rangle$$

close to large radius.

ψ_+, λ_- worldsheet fermions, $\sim TX, \mathcal{E}$

$$Q_+ \leftrightarrow \overline{\partial}$$

States counted by Q_+-cohomology = $H^q(X, \wedge^p \mathcal{E}^*)$

= $H^{p,q}(X)$ when $\mathcal{E} \cong TX$ ((2,2) locus)

Assumed $K_X, \det \mathcal{E}$ trivial

Q_+-cohomology no longer in a topological subsector, but should be protected from *perturbative* corrections.

So, for large-radius CY, should be reliable.

Consider a more general 2d (0,2) NLSM near large-radius:

$$K_X, \ \det \mathcal{E} \quad \text{need not be trivial}$$

The zero-energy Q_+-closed states again of the form

$$b^{a_1 \cdots a_p}_{\overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,a_1} \cdots \lambda_{-,a_p} |0\rangle$$

but now $\quad |0\rangle \ \sim \ (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}$

for the Fock vacuum $\quad \psi_+^i |0\rangle \ = \ 0 \ = \ \lambda_{-,\overline{a}} |0\rangle$

States counted by

$$H^q \left(X, (\wedge^p \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right)$$

Different Fock vacua choices give equivalent results….

If instead we'd worked with a Fock vacuum defined by

$$\psi_+^i |0\rangle' \; = \; 0 \; = \; \lambda_{-,a} |0\rangle'$$

then this one related to last one by

$$|0\rangle' \; = \; \left(\prod_a \lambda_{-,\overline{a}} \right) |0\rangle \qquad \begin{array}{l} |0\rangle \; \sim \; (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \\ |0\rangle' \; \sim \; (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \end{array}$$

and states of the form

$$b_{\overline{\imath}_1 \cdots \overline{\imath}_q}^{\overline{a}_1 \cdots \overline{a}_p}(\phi) \psi_+^{\overline{\imath}_1} \cdots \psi_+^{\overline{\imath}_q} \lambda_{-,\overline{a}_1} \cdots \lambda_{-,\overline{a}_p} |0\rangle'$$

Counted by $\quad H^q \left(X, (\wedge^p \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right)$

$$= \; H^q \left(X, (\wedge^{r-p} \mathcal{E}^*) \otimes (\det \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right)$$

$$= \; H^q \left(X, (\wedge^{r-p} \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right)$$

(matching previous counting)

States:

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right)$$

Special case: (2,2) locus

$$\mathcal{E} \;=\; TX$$

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, \Omega_X^\bullet\right) \;=\; H^{\bullet,\bullet}(X)$$

as expected

On a Calabi-Yau, or if $K_X^{\otimes 2} \;\cong\; \mathcal{O}_X$

$$H^\bullet\left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right) \;=\; H^\bullet\left(X, \wedge^\bullet TX\right)$$

Sometimes we can perform a (pseudo-) topological twist.

These NLSM's have two anomalous global U(1)'s:

- a right-moving $U(1)_R$
- a canonical left-moving U(1), rotating the phase of all left fermions, which becomes $U(1)_R$ on (2,2) locus

If $\det \mathcal{E}^{\pm 1} \cong K_X$, then a nonanomalous U(1) exists along which we can twist right & left moving fermions.

Possible twists….

A/2 model: Exists when $(\det \mathcal{E})^{-1} \cong K_X$

(on (2,2) locus, always possible; reduces to A model)

States: $H^\bullet \left(X, \wedge^\bullet \mathcal{E}^* \right)$

B/2 model: Exists when $\det \mathcal{E} \cong K_X$

(on (2,2) locus, requires $K_X^{\otimes 2} \cong \mathcal{O}_X$; reduces to B model)

States: $H^\bullet \left(X, \wedge^\bullet \mathcal{E} \right)$

Exchanging $\mathcal{E} \leftrightarrow \mathcal{E}^*$ swaps the A/2, B/2 models.

(Physically, just a complex conjugation of left movers.)

Product structures

So far we've just counted states.
However, at least sometimes, OPE's also known.

(2,2) locus: OPE's = `quantum cohomology'

In a compactification on a CY 3-fold,
compute $\overline{\mathbf{27}}^3$ couplings

— Gromov-Witten inv'ts; well-established.

(0,2): OPE's = `quantum sheaf cohomology'

In compactification, compute couplings as above

— not Gromov-Witten inv'ts, but a generalization

New methods needed… and a few have been developed.

(A Adams, J Distler, R Donagi, J Guffin, S Katz, J McOrist, I Melnikov, R Plesser, ES, ….)

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

Ex: ordinary quantum cohomology of $\mathbb{P}^n$

$$\mathbb{C}[x]/(x^{n+1}-q)$$

Compare: quantum sheaf cohomology of $\mathbb{P}^n \times \mathbb{P}^n$

with bundle

$$0 \to O \oplus O \xrightarrow{*} O(1,0)^{n+1} \oplus O(0,1)^{n+1} \to E \to 0$$

where

$$* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$$

$x, \tilde{x}$ homog' coord's on $\mathbb{P}^n$'s

is given by $\mathbb{C}[x,y]/(\det(Ax+By)-q_1, \det(Cx+Dy)-q_2)$

Check: When E=T, this becomes $\mathbb{C}[x,y]/(x^{n+1}-q_1, \mathrm{y}^{n+1}-q_2)$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Schematically:

A model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n = \int_X (\text{top-form})$$

$$\omega_i \in H^{p_i, q_i}(X)$$

A/2 model: Classical contribution:

$$\langle O_1 \cdots O_n \rangle = \int_X \omega_1 \wedge \cdots \wedge \omega_n$$

Now, $\omega_1 \wedge \cdots \wedge \omega_n \in H^{\text{top}}(X, \wedge^{\text{top}} E^*) = \mathrm{H}^{\text{top}}(X, K_X)$

using the anomaly constraint $\det E^* \cong K_X$

Again, a top form, so get a number.

Review of quantum sheaf cohomology

General result:

(Math: Donagi, Guffin, Katz, ES, '11; Phys: McOrist, Melnikov, '07, '08)

For any toric variety, and any def' E of its tangent bundle,

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \to E \to 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q_\alpha^a} = q_a$$

where the M's are matrices of chiral operators built from *.

Review of quantum sheaf cohomology

State of the art: computations on toric varieties

To do: compact CY's

Intermediate step: Grassmannians (work in progress)

Briefly, what we need are better computational methods.

Conventional GW tricks seem to revolve around idea that A model is independent of complex structure,
not necessarily true for A/2.

- McOrist-Melnikov '08 have argued an analogue for A/2
- Despite attempts to check (Garavuso-ES '13),
  still not well-understood

So far: counted states, outlined OPE's

Next:

- Dualities & trialities in 2d and their geometry
- Tests of Gadde-Gukov-Putrov triality

(via comparing chiral states)

In 2d theories, dualities often have a purely geometric understanding.

Trivial example:

U(k) gauge theory,
n chiral multiplets

RG

NLSM on G(k,n)

=

U(n-k) gauge theory,
n chiral multiplets

RG

NLSM on G(n-k,n)

But G(k,n) = G(n-k,n),
so IR limits equivalent.

Can check chiral rings, elliptic genera, etc.

Another example, in 2d, (2,2) susy:

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' **k***, A<n

Seiberg ⟷ Benini-Cremonesi '12

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' **k***,
nA neutral chirals M,
superpotential: W = M Φ P

↓ RG

NLSM on $\mathrm{Tot}\left(S^{A} \rightarrow G(k,n)\right)$ $= \mathrm{Tot}\left((Q^{*})^{A} \rightarrow G(n\text{-}k,n)\right)$

$$= (\mathbb{C}^{kn} \times \mathbb{C}^{kA}) // GL(k)$$

↓ RG

. . .

Build physics for RHS using $0 \rightarrow S \xrightarrow{\Phi} O^{n} \rightarrow Q \rightarrow 0$
& discover the upper RHS.

So, 2d analogue of Seiberg duality has geometric description.

Another example, in 2d, (2,2) susy:

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' **k***, A<n

Seiberg ⟷ dual

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' **k***,
nA neutral chirals M,
superpotential: $W = M \Phi P$

To be fair, I've glossed over something….

To play this game in (2,2), I want the geometry to be either Fano or CY, to avoid `discrete Coulomb vacua.'

If the geometry is, say, negatively curved, then the correct intermediate scale description has extra `dust',
and the correct mathematical application is more complicated.

I'll suppress this level of detail in what follows.

Another example, in 2d, (2,2) susy:

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

↔

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

↓ RG ↓ RG

G(2,4) = degree 2 hypersurface in $\mathbb{P}^5$

$= \mathbb{C}^{2\cdot 4} // GL(2)$

$= \{z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}\} \subset \mathbb{C}^6 // \mathbb{C}^\times$

↓ RG

…

The physical duality implied at top relates abelian & nonabelian gauge theories, which in 4d for ex would be surprising.

Another example **((0,2) susy)**

U(2) gauge theory	⟷	U(1) gauge theory
4 chirals in fundamental		6 chirals charge +1
1 Fermi in (-4,-4) (hypersurface)		2 Fermi's charge -2, -4
8 Fermi's in (1,1) (gauge bundle E)		8 Fermi's charge +1
1 chiral in (-2,-2) (gauge bundle E)		1 chiral charge -2
2 chirals in (-3,-3) (gauge bundle E)		2 chirals charge -3
plus superpotential		plus superpotential
↓ RG		↓ RG
Bundle		Bundle
$0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$	=	$0 \to E \to \oplus^8 O(1) \to O(2) \oplus^2 O(3) \to 0$
on the CY G(2,4)[4].		on the CY $\mathbb{P}^5[2,4]$

- both satisfy anomaly cancellation
- elliptic genera match

Further examples ((2,2) susy)

U(2) gauge theory,
n chirals in fundamental

⟷

U(n-2)xU(1) gauge theory,
n chirals X in fundamental of U(n-2),
n chirals P in antifundamental of U(n-2)
(n choose 2) chirals $z_{ij} = - z_{ji}$
each of charge +1 under U(1),
W = tr PAX

↓ RG ↓ RG

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(using description of Pfaffians of
Hori '11, Jockers et al '12)

In this fashion, straightforward to generate examples;
let's move on.….

Triality ((0,2) susy)

(Gadde-Gukov-Putrov '13-'14)

GGP proposed that *triples* of (0,2) GLSM's might flow to the same IR fixed point.

In terms of lower-energy NLSM's, the theories are

Gauge bundle	$\longrightarrow$	Target space
$S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2$	$\longrightarrow$	$G(k,n)$
$S^{2k+A-n} \oplus (Q^*)^{n} \oplus (\det S^*)^2$	$\longrightarrow$	$G(n-k,A)$
$S^{n} \oplus (Q^*)^{A} \oplus (\det S^*)^2$	$\longrightarrow$	$G(A-n+k, 2k+A-n)$

related by permuting 3 of flavor symmetries.

Susy unbroken iff geometric description above valid.

Gadde-Gukov-Putrov triality ('13) ((0,2) susy)

Not merely a geometric equivalence;
instead, closest geom' realization is as follows:

bundle space

$$S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \quad \cdots\text{phase}\cdots \quad (S^*)^A \oplus (Q^*)^n \to G(k,2k+A-n)$$

$$\updownarrow =$$

$$(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \quad \cdots\text{phase}\cdots \quad (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k,A)$$

$$\updownarrow =$$

$$S^n \oplus (Q^*)^A \to G(A-n+k,2k+A-n) \cdots (S^*)^n \oplus (Q^*)^{2k+A-n} \to G(A-n+k,A)$$

$$\updownarrow =$$

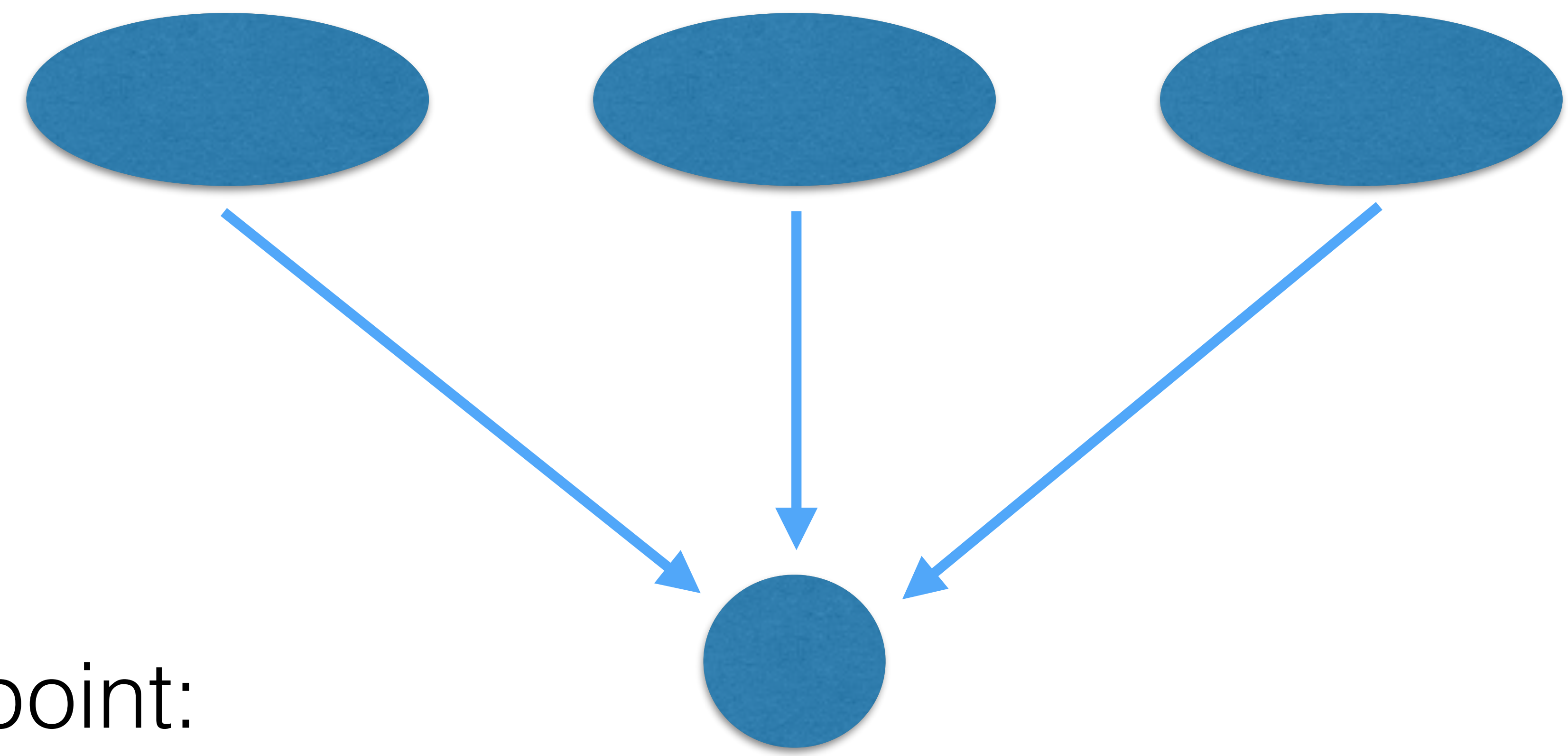
$$(Q^*)^n \oplus S^A \to G(k,2k+A-n) \quad \cdots\text{phase}\cdots \quad (Q)^{2k+A-n} \oplus S^A \to G(k,n)$$

For brevity, I've omitted writing out the (0,2) gauge theory.

Utilizes another duality: NLSM(X,E) = NLSM(X,E*)

Triality predicts

(0,2) NLSM's:



IR fixed point:

IR SCFT = (left-moving Kac-Moody) $\otimes$ (rt-moving Kazama-Suzuki)

UV global $SU(n) \times SU(A) \times SU(2k + A - n) \times SU(2)$

(present in GLSM & each NLSM)

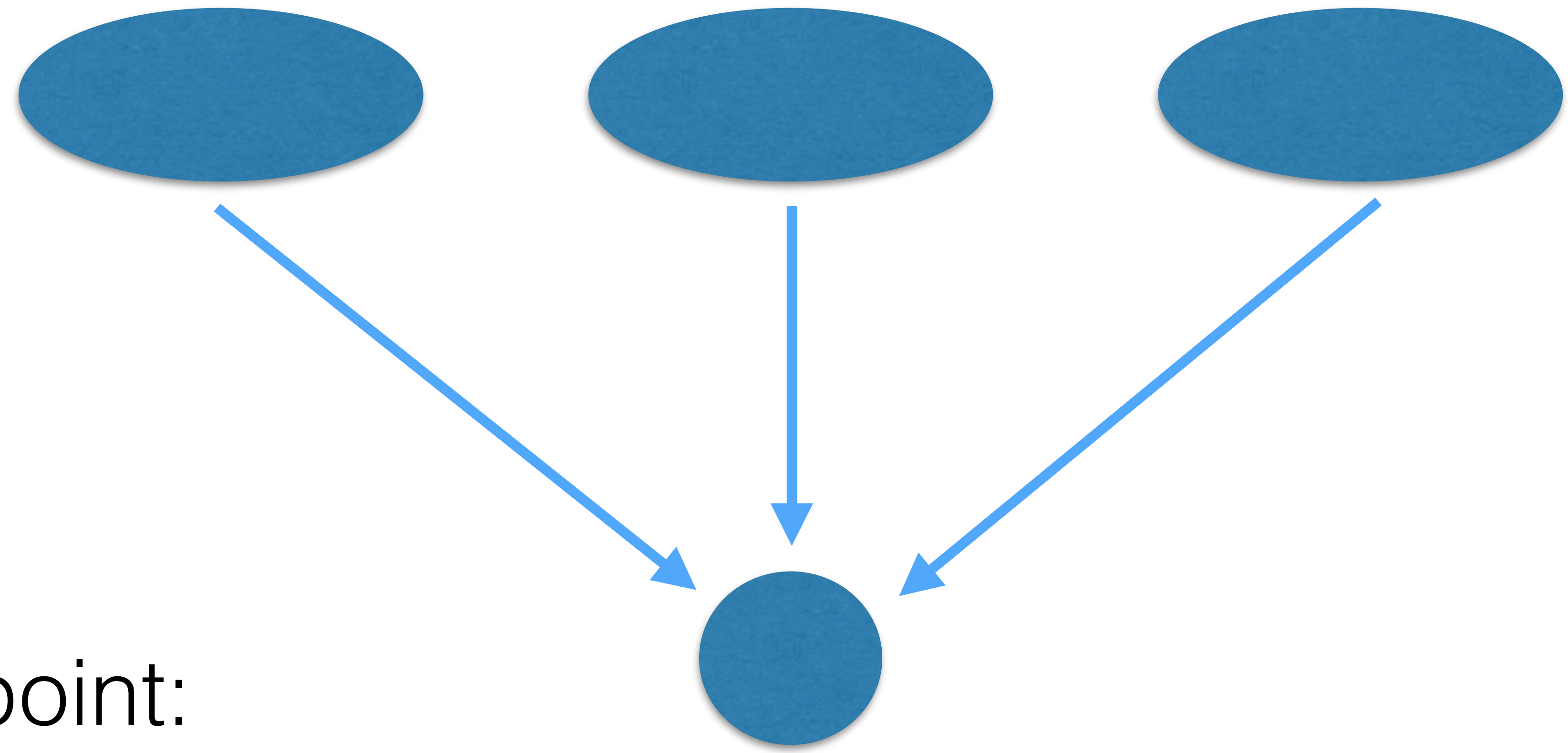
enhanced in IR to affine

$$SU(n)_{k+A-n} \times SU(A)_k \times SU(2k + A - n)_{n-k} \times SU(2)_1$$

Chiral states should live in integrable reps of affine algebras.

Let’s check triality, using chiral rings.

(0,2) NLSM’s:



IR fixed point:

Plan: Compute chiral states in each theory and compare.

Alas, not quite so simple….

Subtleties in comparing chiral states:

- Q*-cohomology in large-radius (0,2) NLSM invariant under perturbative corrections, but, here RG flow goes to strong coupling — states might enter/leave.

We'll see exactly that —
not all states will match between different presentations,
but, states that don't match, shouldn't be in IR either.

- Chiral ring computations in 2d KS models not under good control; Lie algebra cohomology is part of answer.

We'll focus on comparing states across UV presentations, then, merely outline in general terms how form of Lie algebra cohomology is appropriate.

Let's compare chiral states between 2 of 3 triality-related geometries (which will be 2 phases of a single GLSM)....

Example 1:

$r \gg 0$:

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^2$$

$r \ll 0$:

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^1$$

$$U \;=\; \mathbb{C}^3, \;\; V \;=\; \mathbb{C}^2, \;\; W \;=\; \mathbb{C}^2, \;\; \tilde{V} \;=\; \mathbb{C}^3$$

Example 1:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^2$$

Compute states:

$$H^{\bullet}(\mathbb{P}^2, (\wedge^{\bullet}\mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_{\mathbb{P}^2}^{+1/2})$$

Global symmetries:

$SU(U) \times SU(V) \times SU(W)$ manifest — acts on bundle

$SU(\tilde{V})$ also present:

Compute sheaf cohomology with Bott-Borel-Weil, which gives sheaf cohomology as reps of $U(\tilde{V})$.

These computations are an application of Bott-Borel-Weil, so, brief aside:

For a bundle $\mathcal{E}_\xi$ on G/P defined by rep' ξ of P, $H^\bullet(G/P, \mathcal{E}_\xi)$ is naturally a rep' of G.

For Grassmannians, compute $H^\bullet\left(G(k,n), K_{(a_1,\cdots,a_k)}S^* \otimes K_{(b_1,\cdots,b_{n-k})}Q^*\right)$:

$(a_1, \cdots, a_k)$ rep' of U(k) $\quad a_1 \geq a_2 \geq \cdots \geq a_k$

$(b_1, \cdots, b_{n-k})$ rep' of U(n-k) $\quad b_1 \geq b_2 \geq \cdots \geq b_{n-k}$

'Mutate' $(a_1, \cdots, a_k, b_1, \cdots, b_{n-k})$ to $(c_1, \cdots, c_n)$ rep of U(n)

$$H^\bullet\left(G(k,n), K_{(a_1,\cdots,a_k)}S^* \otimes K_{(b_1,\cdots,b_{n-k})}Q^*\right) = K_{(c_1,\cdots,c_n)}V^*$$

for $\bullet$ = number of mutations, & zero in other degrees.

Bott-Borel-Weil, cont'd

Ex: $H^\bullet(G(k,\tilde{V}^*), U \otimes S^*)$

$$= U \otimes H^\bullet(G(k,\tilde{V}^*), K_{(1,0,\cdots,0)}S^* \otimes K_{(0,0,\cdots,0)}Q^*)$$

$$= U \otimes K_{(1,0,\cdots,0)}\tilde{V}\,\delta^{\bullet,0} = U \otimes \tilde{V}\,\delta^{\bullet,0}$$

Constraints on results:

- Invariance under Serre duality

$$H^{\bullet}(X, \mathcal{E}) \; = \; H^{\dim - \bullet}(X, \mathcal{E}^* \otimes K_X)^*$$

Should map state spectrum into itself,
dualizing representation.

- Integrability of (IR) representations

GGP triality predicts that states should live
in `integrable' rep's.

$SU(n)_k$: integrable reps have Young tableaux of width $\leq$ k

- States that don't survive to IR,
  should cancel out of indices such as elliptic genera

Let's look at some states in the first example….

Examples of states shared between two phases:

SU(3)xSU(2)xSU(2)xSU(3)	U(1)³
(1,1,1,1)	(+3,0,-3)
(1,1,2,3)	(+2,-1/2,-3/2)
(1,1,1,3*)	(+1,+2,-3)
(3,2,1,1)	(+2,0,-2)
(3,1,2,1)	(-2,-3/2,-1/2)
(1,2,2,3*)	(+1,+1/2,-3/2)
(3,1,1,3)	(+1,+1,-2)
...	...
(3*,1,1,3*)	(-1,-1,+2)
(1,2,2,3)	(-1,-1/2,+3/2)
(3*,1,2,1)	(-2,+3/2,+1/2)
(3*,2,1,1)	(-2,0,+2)
(1,1,1,3)	(-1,-2,+3)
(1,1,2,3*)	(-2,+1/2,+3/2)
(1,1,1,1)	(-3,0,+3)

Serre duals

Integrable reps of $SU(3)_1 \times SU(2)_2 \times SU(2)_1 \times SU(3)_1$

Non-shared states in $r \gg 0$ phase:

wedge	coh' degree	SU(3)xSU(2)xSU(2)xSU(3)	U(1)³
2	0	**(1,1,1,6)**	(+1,-1,0)
3	0	**(1,2,1,8)**	(0,0,0)
4	0	**(1,1,1,6*)**	(-1,+1,0)
5	2	**(1,1,1,6)**	(+1,-1,0)
6	2	**(1,2,1,8)**	(0,0,0)
7	2	**(1,1,1,6*)**	(-1,+1,0)

Serre duals

Matching rep'

Non-integrable rep' of $SU(3)_1 \times SU(2)_2 \times SU(2)_1 \times SU(3)_1$

- All states come in Serre dual pairs
- Rep's are non-integrable — should not survive to IR
- States come in pairs with matching rep's (so cancel out of elliptic genera)

So far, I’ve compared chiral states in two phases of one GLSM, corresponding to 2 of the 3 geometries related by triality.

We can perform the same analysis in phases of other GLSM’s, describing geometries related by triality to the two above.

We find the same results:

- There is a set of states shared between all geometries related by triality, falling in integrable representations
- There are non-shared states, in non-integrable representations, and which cancel out of elliptic genera.

So far, only discussed one example of a triple, but the same pattern appears in other examples….

Example 2:

$r \gg 0:$

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}\tilde{V}^* \;=\; \mathbb{P}^1$$

$r \ll 0:$

$$\mathcal{E} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; \mathbb{P}V^* \;=\; \mathbb{P}^3$$

$$U \;=\; \mathbb{C}^4, \;\; V \;=\; \mathbb{C}^4, \;\; W \;=\; \mathbb{C}^2, \;\; \tilde{V} \;=\; \mathbb{C}^2$$

Let’s compare states in these two phases
(= 2 of 3 triality-related geometries).

We’ll find closely analogous results.

Examples of states shared between two phases:

SU(4)xSU(4)xSU(2)xSU(2)	U(1)^3
(1,1,1,3)	(+4,0,-4)
(4,1,1,2)	(+4,-1,-3)
(1,4,1,2)	(-3,+1,-4)
(1,1,2,4)	(+2,0,-2)
(6,1,1,1)	(+4,-2,-2)
(4,4,1,1)	(+3,0,-3)
(4,1,2,3)	(+2,-1,-1)
…	…
(4*,1,2,3)	(-2,+1,+1)
(4*,4*,1,1)	(-3,0,+3)
(6,1,1,1)	(-4,+2,+2)
(1,1,2,4)	(-2,0,+2)
(1,4*,1,2)	(-3,-1,+4)
(4*,1,1,2)	(-4,+1,+3)
(1,1,1,3)	(-4,0,+4)

Serre duals

Integrable reps of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

Non-shared states in $r \ll 0$ phase:

wedge	coh' degree	SU(4)xSU(4)xSU(2)xSU(2)	U(1)³
4	0	**(1,10,1,1)**	(+2,-2,0)
4	3	**(1,10,1,1)**	(+2,-2,0)
5	0	**(1,20,1,2)**	(+1,-1,0)
…	…	**…**	…
7	3	**(1,20,1,2)**	(-1,+1,0)
8	0	**(1,10*,1,1)**	(-2,+2,0)
8	3	**(1,10*,1,1)**	(-2,+2,0)

Serre duals (rows 1 and last); Matching rep' (rows 1–2); Matching rep' (rows 6–7)

Non-integrable rep' of $SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3$

- All states come in Serre dual pairs
- Rep's are non-integrable — should not survive to IR
- States come in pairs with matching rep's (so cancel out of elliptic genera)

So far in example 2,
we’ve compared states between 2 of 3 triality geometries.

If we compare other pairs of the 3,
we get analogous results.

We've seen that the between geometries that should flow to same fixed point, the chiral states don't all match,
but,
the ones that don't, also have nonintegrable reps,
and make no net contribution to refined elliptic genera.

We believe that they get a mass and disappear from RG flow.

The fact that the remaining states are both

- shared between phases, and
- in integrable reps of proposed IR symmetry algebras,

serves as a nontrivial check of triality.

Next: we've compared UV presentations, what about IR ?

How in principle might these UV sheaf cohomology groups relate, in general, to the IR states?

In IR, expect states ~ Lie algebra cohomology.
[*roughly* — correspondence incomplete] (W Lerche, private communication)

How is that related?

We won't pursue this in detail, but,
want to observe that another flavor of BBW provides the missing link:

sheaf cohomology

$$H^\bullet(G/P, \mathcal{E}_\xi)_\lambda \;=\; H^\bullet(\mathfrak{n}, V_\lambda)_\xi$$

Lie algebra cohomology

λ a representation of G

ξ a representation of P

$$\mathfrak{p} \;=\; (\text{Levi}) \;+\; \mathfrak{n}$$

There is an interesting lesson in this for the (0,2) community.

We have often been a bit fast about assuming that chiral state computations in UV & IR match, and here we have counterexamples.

Example 3:

$r \gg 0:$

$$\mathcal{E} = U \otimes S + V \otimes Q^* + W \otimes \det S^* \longrightarrow \mathbb{P}\tilde{V}^* = \mathbb{P}^1$$

$r \ll 0:$

$$\mathcal{E} = U \otimes S^* + \tilde{V} \otimes Q^* + W \otimes \det S \longrightarrow \mathbb{P}V^* = \mathbb{P}^1$$

$$U = \mathbb{C}^2, \quad V = \mathbb{C}^2, \quad W = \mathbb{C}^2, \quad \tilde{V} = \mathbb{C}^2$$

We can compare states as before —
the geometries are identical,
but the global symmetries have rotated.

Results follow same pattern:
matching states in integrable reps,
non-matching states cancel out of indices.

Example 3, cont'd

Something new here happens in IR:
$(SU(2)_1)^4$ believed to be enhanced to $(E_6)_1$

(GGP '13)

One can show that the matching chiral states fill out the $\mathbf{27}$, $\overline{\mathbf{27}}$ of E_6:

$$\mathbf{27} = (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + 3(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$$

consistent with GGP's predictions.

Math conjecture:

The shared states, the sheaf cohomology that survives to IR, should define some sort of `stable cohomology.’

Stable under `physics homotopy’ = RG flow

Next, let’s switch gears and turn to moduli….

Brief overview of moduli

It was known historically that for large-radius het' NLSM's on the (2,2) locus, there were three classes of infinitesimal moduli:

$H^1(X, T^*X)$ Kahler moduli

$H^1(X, TX)$ Complex moduli

$H^1(X, \mathrm{End}\,\mathrm{E})$ Bundle moduli

where, on (2,2) locus, $E = TX$

When the gauge bundle $E \neq TX$,
the correct moduli counting is more complicated….

Brief overview of moduli

For Calabi-Yau (0,2) compactifications off the (2,2) locus, moduli are as follows:

(Anderson-Gray-Lukas-Ovrut, '10)

$H^1(X, T^*X)$ Kahler moduli

$H^1(Q)$ where

$$0 \to \mathrm{End}\,\mathrm{E} \to Q \to TX \to 0 \qquad (F)$$

(Atiyah sequence)

There remained for a long time the question of moduli of non-Kahler compactifications….

Brief overview of moduli

For non-Kahler (0,2) compactifications,
in the **formal** $\alpha' \to 0$ limit,

(Melnikov-ES, '11)

$H^1(S)$ where

$$0 \to T^* X \to S \to Q \to 0 \qquad (H,\ dH = 0)$$

$$0 \to \mathrm{End}\,\mathrm{E} \to Q \to TX \to 0 \qquad (F)$$

Now, we also need α' corrections….

Brief overview of moduli

Through first order in α',
the moduli are *overcounted* by

(Anderson-Gray-ES '14; de la Ossa-Svanes '14)

$H^1(S)$ where

$$0 \to T^* X \to S \to Q \to 0 \qquad (H, \text{Green-Schwarz})$$

$$0 \to \mathrm{End}\,\mathrm{E} \oplus \mathrm{End}\, TX \to Q \to TX \to 0 \qquad (F, R)$$

on manifolds satisfying the $\partial\bar{\partial}$ lemma.

Current state-of-the-art

WIP to find correct counting, & extend to higher orders

Brief overview of moduli

So far I've outlined infinitesimal moduli — marginal operators.

These can be obstructed by eg nonperturbative effects.

Dine-Seiberg-Wen-Witten '86 observed that a single worldsheet instanton can generate a superpotential term obstructing def's off (2,2) locus….

… but then Silverstein-Witten '95, Candelas et al '95, Basu-Sethi '03, Beasley-Witten '03 observed that for polynomial moduli in GLSM's, the contributions of all pertinent worldsheet instantons cancel out.

— those moduli are unobstructed; math not well-understood.

Moduli w/o such a description can still be obstructed, see for example Aspinwall-Plesser '11, Braun-Kreuzer-Ovrut-Scheidegger '07

Summary:

- Chiral states in 2d (0,2) NLSM's
- Product structures in chiral rings in A/2, B/2 twists:

  quantum sheaf cohomology

- Nonabelian GLSMs:
  - Dualities in 2d and their geometry
  - Tests of Gadde-Gukov-Putrov triality
- Survey of moduli, esp. non-Kahler cases

Thank you for your time!