Landau-Ginzburg models, gerbes, and Kuznetsov's homological projective duality

Eric Sharpe, Virginia Tech

Topology, C* algebras, and string duality
Texas Christian University, May 18-22, 2009

Today I'm going to talk about
``Landau-Ginzburg models,"
which are examples of 2d QFT's used in string theory.

Typical application is to describe string propagation,
possibly on a space, possibly something else.

A LG model looks like,
string on a space + a potential
(which in supersymmetric theory is defined by
a holomorphic function -- ``superpotential")

Historically, most LG models considered described strings propagating on vector spaces (+ superpotential).

Reasons:

* Many such examples are closely related to strings on nontrivial spaces

* but, string on vector space + potential is much easier to analyze than a string on a nontrivial space.

LG models on vector spaces have been around for a long time, and are fairly well understood.

New ground: LG models on nontrivial spaces, and GW invariants of any LG model.

Technically more difficult, but, also get some more interesting results.

What I'll talk about today are various results concerning LG models over nontrivial spaces.

Outline:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* LG models over gerbes:

  * Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

  * Strings on gerbes: decomposition conjecture

  * Application of decomposition conj' to LG's: physical realization of Kuznetsov's homological projective duality & some nc resolutions

A **Landau-Ginzburg model** is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

$$S = \int_\Sigma d^2x \left(g_{i\bar{\jmath}} \partial \phi^i \overline{\partial} \phi^{\jmath} + i g_{i\bar{\jmath}} \psi_+^{\jmath} D_{\overline{z}} \psi_+^i + i g_{i\bar{\jmath}} \psi_-^{\jmath} D_z \psi_-^i + \cdots \right.$$
$$\left. + g^{i\bar{\jmath}} \partial_i W \partial_{\jmath} \overline{W} + \psi_+^i \psi_-^j D_i \partial_j W + \psi_+^{\bar{\imath}} \psi_-^{\bar{\jmath}} D_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W} \right)$$

The superpotential $W : X \longrightarrow \mathbf{C}$ is holomorphic, (so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....

For nonlinear sigma models,
there are 2 topological twists: the A, B models.

1) A model

$$\psi_+^i \in \Gamma(\phi^*(T^{1,0}X)) \to \chi^i \qquad \psi_-^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X)) \to \chi^{\bar{\imath}}$$

$$Q \cdot \phi^i = \chi^i, \;\; Q \cdot \phi^{\bar{\imath}} = \chi^{\bar{\imath}}, \;\; Q \cdot \chi = 0, \;\; Q^2 = 0$$

Identify $\chi^\mu \sim dx^\mu \qquad Q \sim d$

States $b_{\mu\cdots\nu}\chi^\mu \cdots \chi^\nu \leftrightarrow H^{\cdot,\cdot}(X)$

2) B model

$$\psi_{\pm}^{\bar{\imath}} \in \Gamma(\phi^*(T^{0,1}X))$$

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} \qquad \theta_i = g_{i\bar{\jmath}} \left(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}} \right)$$

$$Q \cdot \phi^i = 0, \;\; Q \cdot \phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \;\; Q \cdot \eta^{\bar{\imath}} = 0, \;\; Q \cdot \theta_j = 0, \;\; Q^2 = 0$$

Identify $\eta^{\bar{\imath}} \leftrightarrow d\overline{z}^{\bar{\imath}}$ $\qquad \theta_j \leftrightarrow \dfrac{\partial}{\partial z^j}$ $\qquad Q \leftrightarrow \overline{\partial}$

States:

$$b_{\bar{\imath}_1 \cdots \bar{\imath}_n}^{j_1 \cdots j_m} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m} \leftrightarrow H^n\left(X, \Lambda^m TX\right)$$

We can also talk about A, B twists of LG models over nontrivial spaces....

LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

$$b(\phi)^{j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where η, θ are linear comb's of ψ

$$Q \cdot \phi^i = 0, \quad Q \cdot \phi^{\bar{\imath}} = \eta^{\bar{\imath}}, \quad Q \cdot \eta^{\bar{\imath}} = 0, \quad Q \cdot \theta_j = \partial_j W, \quad Q^2 = 0$$

Identify $\quad \eta^{\bar{\imath}} \leftrightarrow d\overline{z}^{\bar{\imath}}, \quad \theta_j \leftrightarrow \frac{\partial}{\partial z^j}, \quad Q \leftrightarrow \overline{\partial}$

so the states are hypercohomology

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \stackrel{dW}{\longrightarrow} TX \stackrel{dW}{\longrightarrow} \mathcal{O}_X \right)$$

Quick checks:

1) W=0, standard B-twisted NLSM

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto H^{\cdot}\left(X, \Lambda^{\cdot} TX\right) \checkmark$$

2) X=$\mathbf{C}^n$, W = quasihomogeneous polynomial

Seq' above resolves fat point {dW=0}, so

$$\mathbf{H}^{\cdot}\left(X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X\right)$$

$$\mapsto \mathbf{C}[x_1, \cdots, x_n]/(dW) \checkmark$$

LG A model:

(Fan, Jarvis, Ruan) (Ito; J Guffin, ES)

To produce a TFT, we twist:

$$\psi \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* TX) \mapsto \Gamma(\Sigma, \phi^* TX), \Gamma(\Sigma, K_\Sigma \otimes \phi^* TX)$$

To be consistent, the action must remain well-defined after the twist.

Problem w/ a term in action: $\int_\Sigma \psi_+^i \psi_-^j D_i \partial_j W$

In standard A NLSM twist, this is a 1-form on Σ, which can't integrate over Σ.

Fix: modify the A twist.

LG A model:

There are several ways to fix the A twist,
and hence, several different notions of a LG A model.

One fix: multiply offending terms in the action
by another 1-form.

Another fix: combine twist with a U(1) action,
so fermions couple to diff' bdles.

The second is advantageous for physics, so I'll use it,
but,
disadvantage: not all LG models admit A twist
in that prescription.

To twist, need a U(1) isometry on X w.r.t. which the superpotential is quasi-homogeneous.

Twist by "R-symmetry + isometry"

Let $Q(\psi_i)$ be such that

$$W(\lambda^{Q(\psi_i)}\phi_i) \;=\; \lambda W(\phi_i)$$

then twist: $\psi \;\mapsto\; \Gamma\left(\mathrm{original} \otimes K_\Sigma^{-(1/2)Q_R} \otimes \overline{K_\Sigma}^{-(1/2)Q_L}\right)$

where $$Q_{R,L}(\psi) \;=\; Q(\psi) \;+\; \left\{ \begin{array}{ll} 1 & \psi = \psi_+^i, R \\ 1 & \psi = \psi_-^i, L \\ 0 & \mathrm{else} \end{array} \right.$$

Example: $X = \mathbf{C}^n$, W quasi-homog' polynomial

Here, to twist, need to make sense of e.g. $K_\Sigma^{1/r}$

where r = 2(degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)
-- but then usually can't make sense of $K_\Sigma^{1/r}$

I'll work with the latter case.

LG A model:

A twistable example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with $W = p\pi^* s, \; s \in \Gamma(B, \mathcal{E})$

U(1) action acts as phases on fibers

Turns out that correlation functions in this theory match those in a NLSM on $\{s = 0\} \subset B$.

LG A model, cont'd

In prototypical cases,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \underbrace{\int d\chi^p d\chi^{\overline{p}} \exp\left(-|s|^2 \,-\, \chi^p dz^i D_i s \,-\, \mathrm{c.c.} \,-\, F_{i\overline{\jmath}} dz^i d\overline{z}^{\overline{\jmath}} \chi^p \chi^{\overline{p}} \right.}_{\mathrm{Mathai-Quillen\ form}}$$

The MQ form rep's a Thom class, so

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \mathrm{Eul}(N_{\{s=0\}/\mathcal{M}}) \\ &= \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \end{aligned}$$

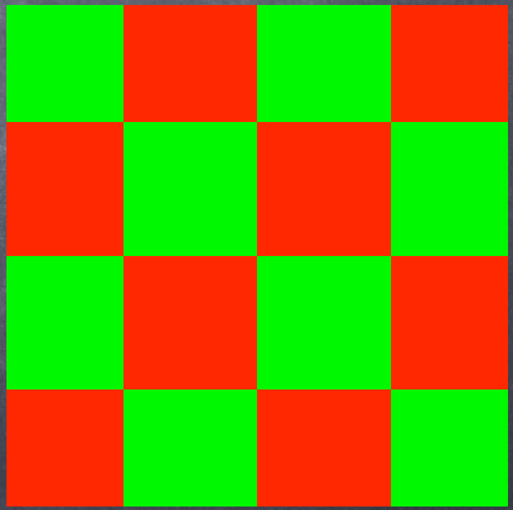
-- same as A twisted NLSM on {s=0}

Not a coincidence, as we shall see shortly....

Renormalization (semi)group flow

Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.



Problem: cannot follow it explicitly.

Renormalization group

Longer distances

Lower energies

Space of physical theories

Furthermore, RG preserves TFT's.

If two physical theories are related by RG,
then, correlation functions in a top' twist of one
=
correlation functions in corresponding twist of other.

Example:

LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

with W = p s

Renormalization group flow

NLSM on {s = 0} $\subset$ B

where $s \in \Gamma(\mathcal{E})$

This is why correlation functions match.

Computational advantage:

For example, consider curve-counting in a deg 5 (quintic) hypersurface in $\mathbf{P}^4$
-- need moduli space of curves in quintic, rather complicated

Can replace with LG model on

$$\mathrm{Tot}\left(\mathcal{O}(-5) \rightarrow \mathbf{P}^4\right)$$

and here, curve-counting involves moduli spaces of curves on $\mathbf{P}^4$, much easier

Elliptic genera:

Elliptic genus of LG model on X = Tot($\mathcal{E}^\vee \xrightarrow{\pi} B$)

$$\int_B \mathrm{Td}(TB) \wedge \mathrm{ch}\Bigg(\Lambda_{-1}(TB) \otimes \Lambda_{-1}(\mathcal{E}^\vee) \bigotimes_{n=1,2,3,\cdots} S_{q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\cdots} S_{q^n}((\mathcal{E}^\vee)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\cdots} \Lambda_{-q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\cdots} \Lambda_{q^n}((\mathcal{E}^\vee)^{\mathbf{C}}) \Bigg)$$

matches Witten genus of $\{s = 0\} \subset B$

by virtue of a Thom class computation.

(M Ando, ES, '09)

RG flow interpretation:

In the case of the A-twisted correlation f'ns, we got a Mathai-Quillen rep of a Thom form.

Something analogous happens in elliptic genera: elliptic genera of the LG & NLSM models are related by Thom forms.

Suggests: RG flow interpretation in twisted theories as Thom class.

(possibly from underlying Atiyah-Jeffrey, Baulieu-Singer description)

Next:

* decomposition conjecture for strings on gerbes

* application of gerbes to LG's as, physical realization of Kuznetsov's nc resolutions and homological projective duality

To do this, need to review how stacks appear in physics....

String compactifications on stacks

First, motivation:

-- new string compactifications

-- better understand certain existing string compactifications

Next: how to construct QFT's for strings propagating on stacks?

Stacks

How to make sense of strings on stacks concretely?

Most (smooth, Deligne-Mumford) stacks can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

(G need not be finite; need not act effectively.)

To such a presentation, associate a ``G-gauged sigma model on X.'' (T Pantev, ES)

Such presentations not unique; fix with RG flow.

Gerbes

(= quotients by noneffectively acting groups)

A problem with the physics of gerbes:

The set of massless states contains multiple dimension zero states,
which violates cluster decomposition,
one of the foundational axioms of quantum field theory.

There is a single known loophole: if the target space is disconnected. We **think** that's what's going on....

Decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

Claim

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on\ } X) \, = \, \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \stackrel{Z(G) \rightarrow U(1)}{\longrightarrow} H^2(X, U(1))$$

Checks:

* For global quotients by finite groups, can compute partition f'ns exactly at arb' genus

* Implies $K_H(X) = \text{twisted}\, K_K(X \times \hat{G})$ which can be checked independently

* Implies known facts about sheaf theory on gerbes

* Implications for Gromov-Witten theory

(Andreini, Jiang, Tseng, 0812.4477, 0905.2258, and to appear)

Apply decomp' conjecture to some examples of Landau-Ginzburg models.

Let $$X = \mathrm{Tot}\left(\mathcal{O}(-1)^{\oplus 8} \longrightarrow \mathbf{P}^3_{[2,2,2,2]} \right)$$

with $$W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

where p's are homog' coord's on $\mathbf{P}^3_{[2,2,2,2]}$

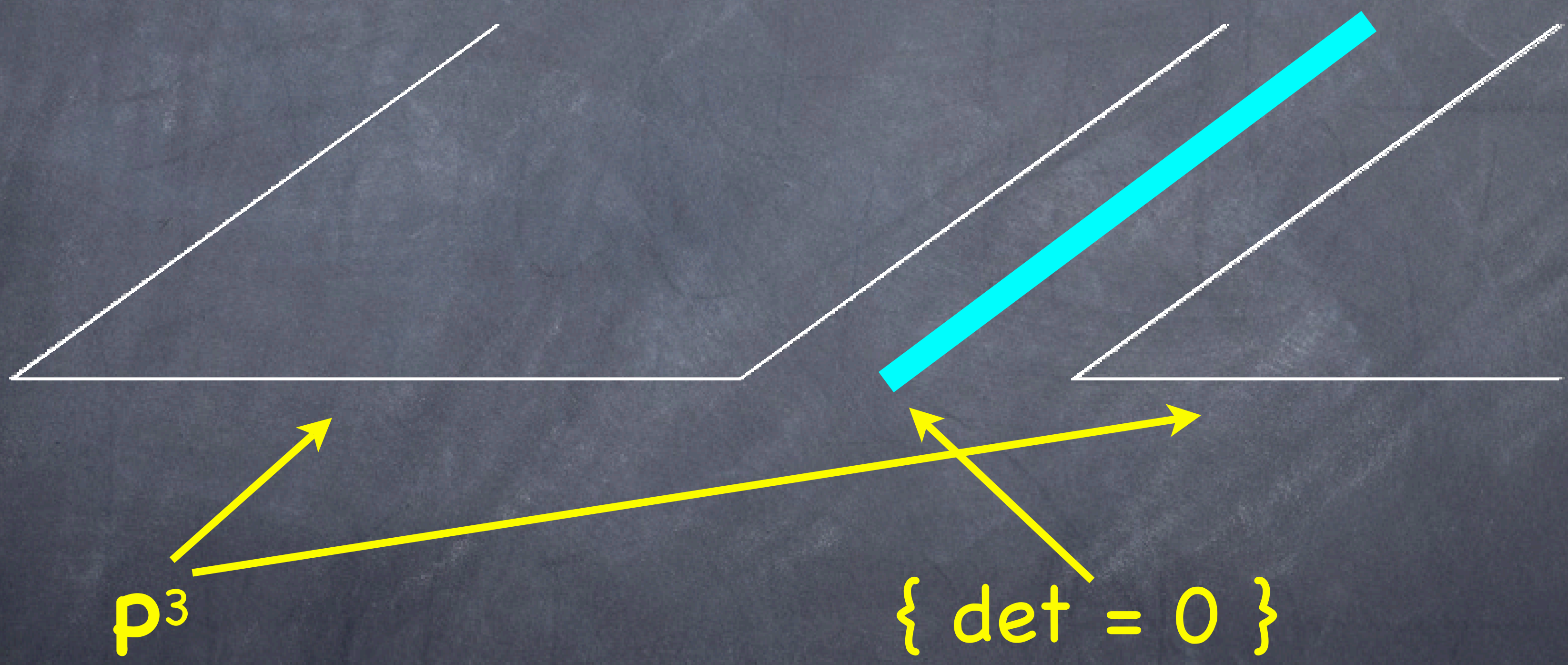
ϕs are fiber coord's

and G's are a set of quadrics in $\mathbf{P}^7$

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

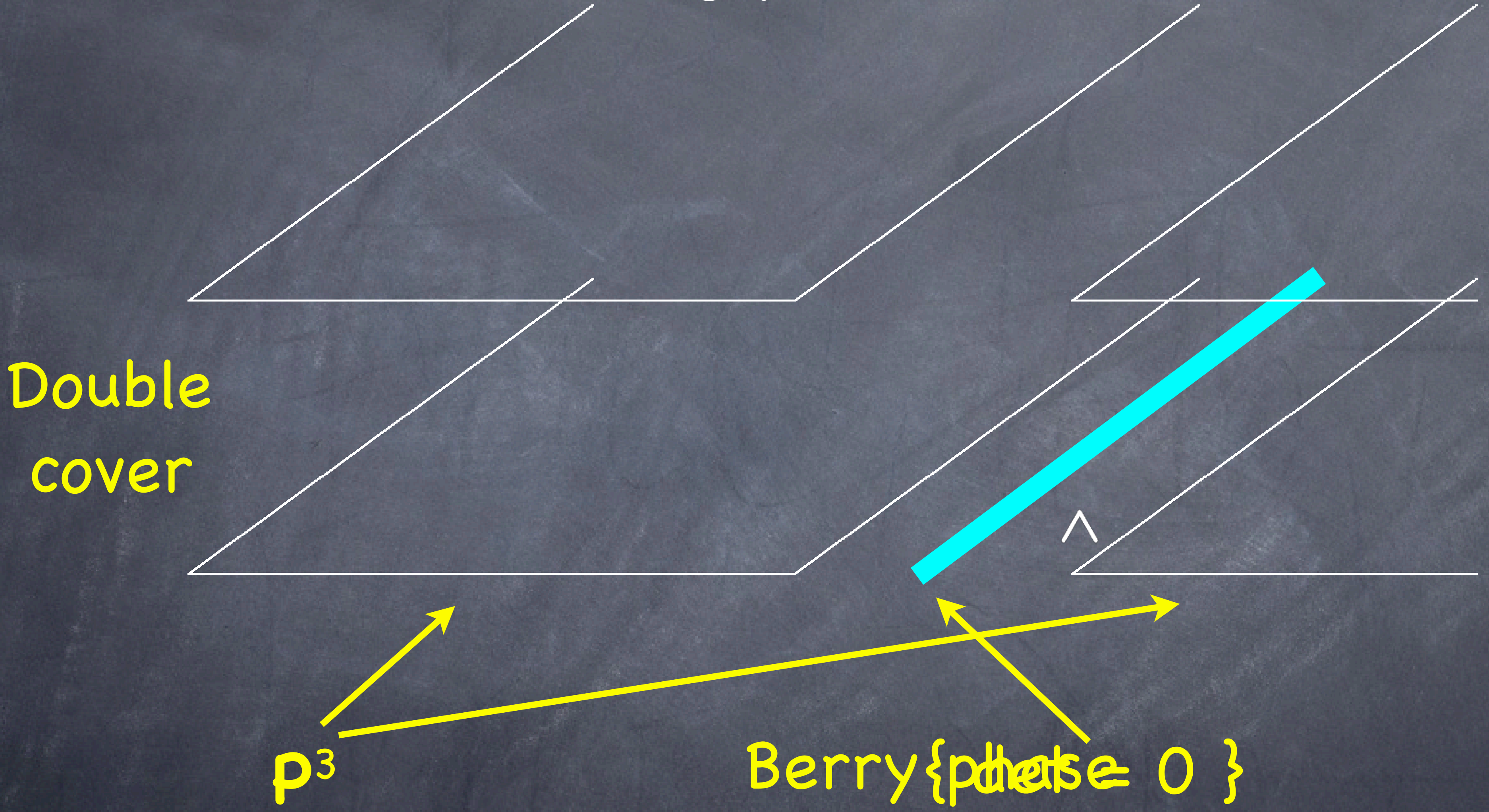
* $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg model:



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{P}^3$ - det....

The Landau-Ginzburg point:



Result: branched double cover of $\mathbf{P}^3$

So far:

The LG realizes:

branched double cover

of $\mathbf{P}^3$

(Clemens' octic double solid)

realized via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07;
A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Unusual physical realization of geometry

Puzzle:

the branched double cover will be singular, but the physics behaves as if smooth at those singularities.

Solution?....

We believe the LG is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Check that we are seeing K's noncomm' resolution:

K (+Kontsevich, Kapranov, Costello, van den Bergh,..) define a \`noncommutative space' via its sheaves

Here, K's noncomm' res'n = ($\mathbf{P}^3$,B) where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the LG superpotential.

B ~ structure sheaf; other sheaves ~ B-modules.

Physics?......

Physics:

Claim: D-branes (``matrix factorizations'') in LG
= Kuznetsov's B-modules

K has a rigorous proof of this;
D-branes = Kuznetsov's nc res'n sheaves.

Intuition....

Local picture:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a LG model fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / superpotential) and modules thereof.

So: D-branes duplicate Kuznetsov's def'n.

Summary so far:

The LG realizes:

nc res'n of

branched double cover

of $\mathbf{P}^3$

realized via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Unusual physical realization of geometry

+ physical realization of nc res'n

Topology change:

This particular example of a LG model arose as one limit of a `GLSM,' the name for a family of theories linking typically topologically-distinct spaces.

In particular, this nc res'n of a branched double cover arose in a GLSM describing, at the other limit,

$$\mathbf{P}^7[2,2,2,2].$$

-- Kuznetsov's ``homological projective duality"

We conjecture all GLSM phases are related by h.p.d.

Outline:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* LG models over gerbes:

  * Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

  * Strings on gerbes: decomposition conjecture

  * Application of decomposition conj' to LG's: physical realization of Kuznetsov's homological projective duality & some nc resolutions

Mathematics		Physics
Geometry: Gromov-Witten, Donaldson-Thomas, quantum cohomology, etc	⟺	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⟺	Renormalization group