An introduction to heterotic mirror symmetry

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I’ll begin today by reminding us all of ordinary mirror symmetry.

Most basic incarnation:

String theory on a Calabi-Yau X

= String theory on a Calabi-Yau Y

Ex: X = quintic threefold, $\mathbb{P}^4[5]$ Y = $\widetilde{\mathbb{P}^4[5]/\mathbb{Z}_5^3}$

dim(X) = dim(Y)

Relates Hodge numbers: $h^{p,q}(X) = h^{p,n-q}(Y)$

Also swaps perturbative & nonpert’ corrections:
made computing GW invariants easy.

Plan for today:

Outline a generalization of mirror symmetry,
(involving heterotic strings,)
that is perhaps not so well-known.

- Brief review of ordinary mirrors,
  then heterotic analogues

- Some other more exotic dualities

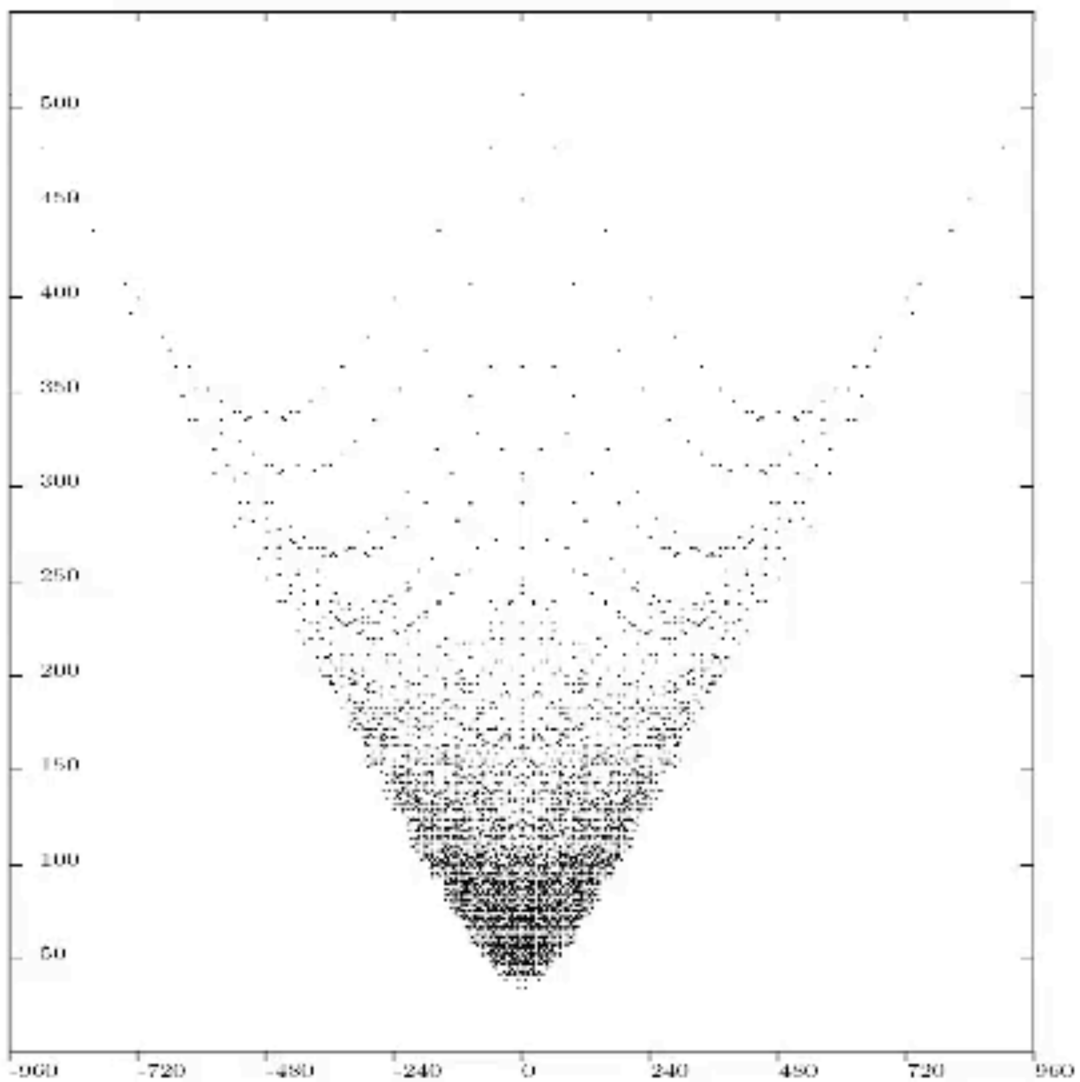
- Heterotic version of quantum cohomology:
  quantum sheaf cohomology

Let’s quickly review some of the reasons physicists believe in and think about mirror symmetry,
en route to talking about the `heterotic’ generalization.

Some of the original checks….

Numerical checks of mirror symmetry

Plotted below are data for a large number of Calabi-Yau 3-folds.



Vertical axis: $h^{1,1} + h^{2,1}$

Horizontal axis: $2(h^{1,1} - h^{2,1})$
= 2 (# Kahler - # cpx def's)

Mirror symmetry exchanges $h^{1,1} \longleftrightarrow h^{2,1}$

==> symm' across vert' axis

(Klemm, Schimmrigk, NPB 411 ('94) 559-583)

Constructions of mirror pairs

One of the original methods:
in special cases, can quotient by a symmetry group.
"Greene-Plesser orbifold construction"

(Greene-Plesser '90)

Example: quintic

$$Q_5 \subset \mathbb{P}^4 \quad \xleftrightarrow{\text{mirror}} \quad \widetilde{Q_5/\mathbb{Z}_5^3}$$

More general methods exist….

Constructions of mirror pairs

Batyrev's construction:

For a hypersurface in a toric variety,
mirror symmetry exchanges

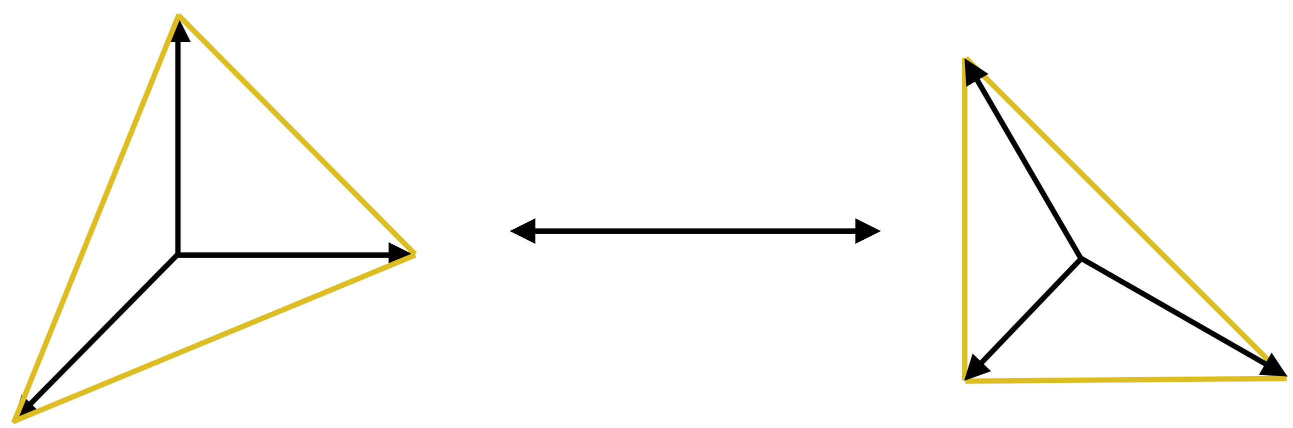
polytope of ambient toric variety	↔	dual polytope for ambient t.v. of mirror

Constructions of mirror pairs

Example of Batyrev's construction:

T^2 as degree 3 hypersurface in $\mathbb{P}^2$

$\mathbb{P}^2 =$



$= \mathbb{P}^2/\mathbb{Z}_3$

$$P^0 = \{y | \langle x, y \rangle \geq -1 \; \forall x \in P\}$$

Result:
degree 3 hypersurface in $\mathbb{P}^2$,
mirror to
$\mathbb{Z}_3$ quotient of degree 3 hypersurface

(matching Greene-Plesser '90)

Ordinary mirror symmetry is pretty well understood nowadays.

- lots of constructions
- both physics and math proofs

  Givental / Yau et al in math

  Morrison-Plesser / Hori-Vafa in physics

However, there are some extensions of mirror symmetry that are still being actively studied….

Ordinary mirror symmetry is a property of type II strings, or worldsheets with "(2,2) supersymmetry."

It is also believed to apply to **heterotic** strings, whose worldsheets have "(0,2) supersymmetry."

(2,2): specified, in simple cases, by a Kahler mfld X

(0,2): specified, in the same simple cases,
by a Kahler manifold X
together with a holomorphic bundle $\mathcal{E} \rightarrow X$
such that

$$\mathrm{ch}_2(\mathcal{E}) \, = \, \mathrm{ch}_2(TX)$$

(Recover (2,2) in special case that $\mathcal{E} = TX$.)

Heterotic aka (0,2) mirror symmetry involves bundles + spaces.

Analogues of topological field theories:

True TFT's based on (0,2) theories do not exist,
but,
there do exist pseudo-topological field theories with closely related properties, at least in special cases.

A/2 model: Exists when $\det \mathcal{E}^* \cong K_X$

States counted by $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$

Reduces to A model on (2,2) locus ($\mathcal{E} = TX$)

B/2 model: Exists when $\det \mathcal{E} \cong K_X$

States counted by $H^\bullet(X, \wedge^\bullet \mathcal{E})$

Reduces to B model on (2,2) locus ($\mathcal{E} = TX$)

$$\mathrm{A}/2(X, \mathcal{E}) \ \cong \ \mathrm{B}/2(X, \mathcal{E}^*)$$

(0,2) mirror symmetry **((0,2) susy)**

How should this work?

Nonlinear sigma models with (0,2) susy defined by space X, with hol' vector bundle $E \longrightarrow X$

(0,2) mirror defined by space Y, w/ bundle F.

dim X = dim Y

rk E = rk F

A/2(X, E) = B/2(Y, F)

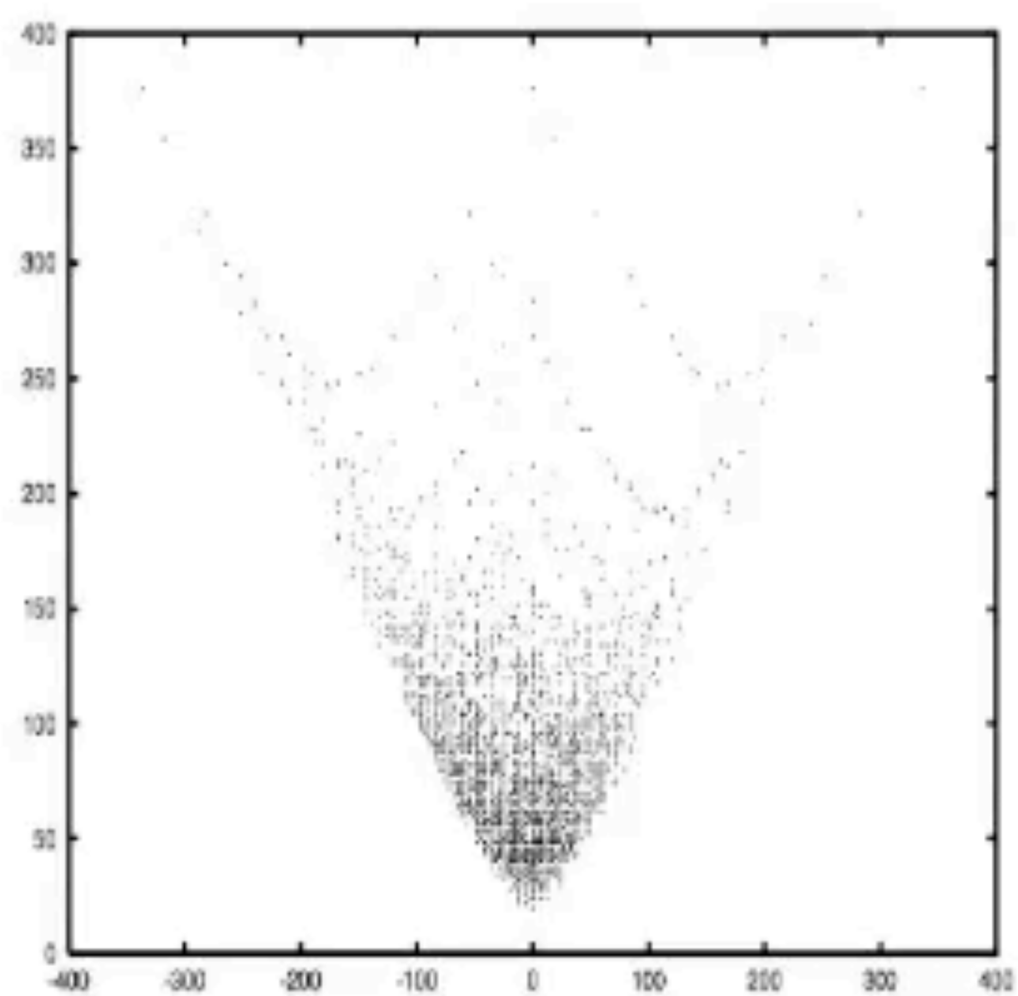
$$H^p(X, \wedge^q E^*) = H^p(Y, \wedge^q F)$$

(moduli) = (moduli)

When E=TX, should reduce to ordinary mirror symmetry.

(0,2) mirror symmetry

Not as much known about heterotic/(0,2) mirror symm',
though a few basics have been worked out.



(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

Example: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

(0,2) mirror symmetry ((0,2) susy)

Constructions include:

- Blumenhagen-Sethi '96 extended Greene-Plesser orbifold construction to (0,2) models — handy but only gives special cases

- Adams-Basu-Sethi '03 repeated Hori-Vafa-Morrison-Plesser-style GLSM duality in (0,2)

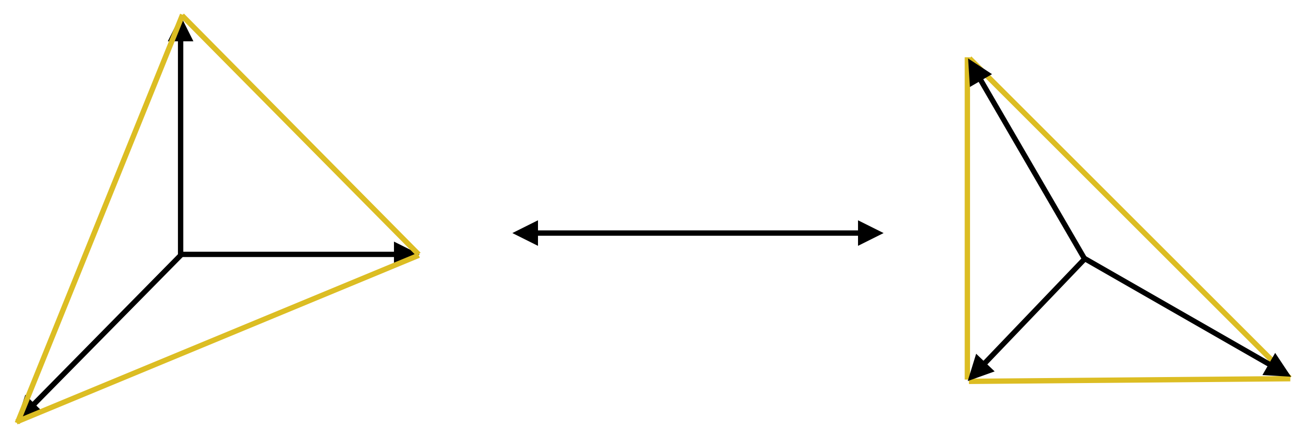
— but results must be supplemented by manual computations;
(0,2) version does not straightforwardly generate examples

More recent progress includes a version of Batyrev's construction….

(0,2) mirror symmetry **((0,2) susy)**

- Melnikov-Plesser '10 extended Batyrev's construction & monomial-divisor mirror map to include def's of tangent bundle, for special ('reflexively plain') polytopes

Dualize polytopes as before:



$$P^0 = \{y | \langle x, y \rangle \geq -1 \, \forall x \in P\}$$

& encode tangent bdle def's in a matrix:

$$A \longleftrightarrow A^T$$

Progress, but still don't have a general construction.

Now let's turn to a few other dualities,
which may or may not be related….

Gauge bundle dualization duality ((0,2) susy)

(Nope, not a typo….)

Nonlinear sigma models with (0,2) susy defined by space X, with hol' vector bundle $E \to X$

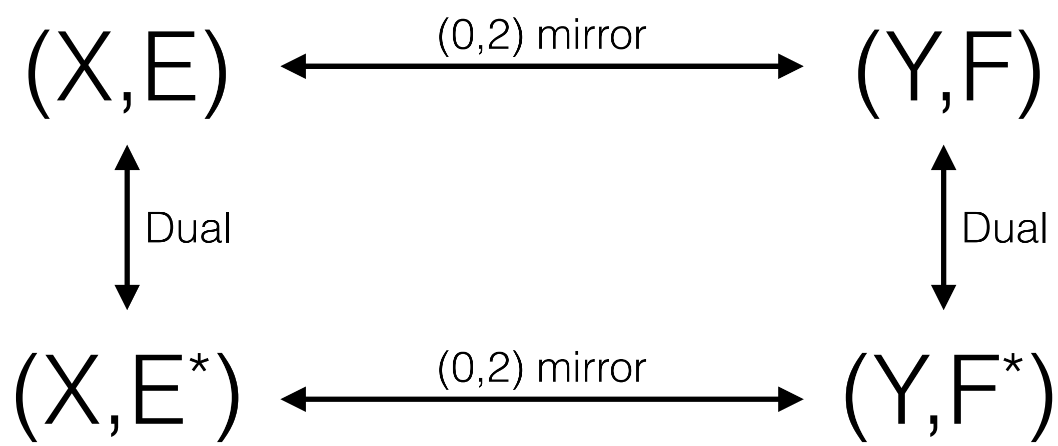
Duality: CFT(X, E) = CFT(X, E^*)

ie, replacing the bundle with its dual is an invariance of the theory.

Gauge bundle dualization duality ((0,2) susy)

How is this related to (0,2) mirrors?

Maybe orthogonal:



On the other hand,

both exchange A/2, B/2 models, both flip sign of left U(1)…

…maybe it's also a sort of (0,2) mirror.

More exotic variations….

Triality ((0,2) susy)

(Gadde-Gukov-Putrov '13-'14)

It has been proposed that *triples* of certain (0,2) theories might be equivalent.

Gauge bundle	$\longrightarrow$	Target space
$S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2$	$\longrightarrow$	$G(k,n)$
$S^{2k+A-n} \oplus (Q^*)^n \oplus (\det S^*)^2$	$\longrightarrow$	$G(n-k,A)$
$S^n \oplus (Q^*)^A \oplus (\det S^*)^2$	$\longrightarrow$	$G(A-n+k, 2k+A-n)$

are conjectured to all be equivalent, for n, k, A such that the geometries above are all sensible.

Moving on….

Triality ((0,2) susy)

How is this related to (0,2) mirrors?

Maybe notion of (0,2) mirrors is richer,
& more variations exist to be found:

$$\longleftrightarrow (X_1,E_1) \longleftrightarrow (X_2,E_2) \longleftrightarrow (X_3,E_3) \longleftrightarrow (X_4,E_4) \longleftrightarrow$$

Triality seems to be in this spirit.

So far I've outlined (0,2) mirrors and some possibly related dualities.

Next: analogue of curve counting, Gromov-Witten….

Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle.

(Katz-ES '04, ES '06, Guffin-Katz '07, ….)

On the (2,2) locus, where bundle = tangent bundle, encodes Gromov-Witten invariants.

Off the (2,2) locus, Gromov-Witten inv'ts no longer relevant.

Mathematical GW computational tricks no longer apply.

No known analogue of periods, Picard-Fuchs equations.

New methods needed….

… and a few have been developed.

(A Adams, J Distler, R Donagi, J Guffin, S Katz, J McOrist, I Melnikov, R Plesser, ES, ….)

Minimal area surfaces:

standard case ("type II strings")

Schematically: For X a space,

$\mathcal{M}$ a space of holomorphic S^2 —> X

we compute a "correlation function" in A model TFT

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_k$$

where $\mathcal{O}_i \sim \omega_i \in H^{p_i, q_i}(\mathcal{M})$

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

which encodes minimal area surface information.

Such computations are at the heart of Gromov-Witten theory.

Minimal area surfaces:
heterotic case

Schematically: For X a space, $\mathcal{E}$ a bundle on X,
$\mathcal{M}$ a space of holomorphic S^2 —> X

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k$$

where $\mathcal{O}_i \sim \tilde{\omega}_i \in H^{q_i}\left(\mathcal{M}, \wedge^{p_i} \mathcal{F}^*\right)$

$\mathcal{F}$ = sheaf of 2d fermi zero modes over $\mathcal{M}$

anomaly cancellation $\stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} \mathcal{F}^* \cong K_{\mathcal{M}}$

hence, again,

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

(S Katz, ES, 2004)

This computation takes place in "A/2 model," a pseudo-topological field theory.

Correlation functions are often usefully encoded in \`operator products' (OPE's).

Physics: Say $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$ ("operator product")

if all correlation functions preserved:

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_C \cdots \rangle$$

Math: if interpret correlation functions as maps

$$\mathrm{Sym}^\bullet W \longrightarrow \mathbb{C}$$

(where W is the space of $\mathcal{O}$'s)

then OPE's are the kernel, of form $\mathcal{O}_A \mathcal{O}_B - \sum_i \mathcal{O}_i$

Examples:

Ordinary ("type II") case:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

where $q, \tilde{q} \sim \exp(-\mathrm{area})$

$\longrightarrow 0$ in classical limit

Looks like a deformation of cohomology ring, hence called "quantum cohomology"

Examples:

Ordinary ("type II") case: $X = \mathbb{P}^1 \times \mathbb{P}^1$

OPE's: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Heterotic case:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \mathcal{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathcal{E}$: $0 \longrightarrow W^* \otimes \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$

where $* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ $\quad A, B, C, D$ const' 2x2 matrices

$x, \tilde{x}$ vectors of homog' coord's

Here, $W = H^1(X, \mathcal{E}^*) = \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$

OPE's: $\det\left(A\psi + B\tilde{\psi}\right) = q, \quad \det\left(C\psi + D\tilde{\psi}\right) = \tilde{q}$

Check: $\mathcal{E} = TX$ when $A = D = I_{2\times 2}, \quad B = C = 0$

& in this limit, OPE's reduce to those of ordinary case

quantum sheaf cohomology

Review of quantum sheaf cohomology

To make this more clear, let's consider an

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \rightarrow W^* \otimes O \stackrel{*}{\rightarrow} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \rightarrow E \rightarrow 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map $\mathrm{Sym}^n \mathrm{H}^1(\mathrm{E}^*) = \mathrm{Sym}^n \mathrm{W} \rightarrow H^n(\wedge^n E^*)$

OPE's = kernel

Plan: study map corresponding to classical corr' f'n

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle E a deformation of the tangent bundle:

$$0 \to W^* \otimes O \overset{*}{\to} \underbrace{O(1,0)^2 \oplus O(0,1)^2}_{Z^*} \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*) = H^0(W \otimes O) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*)$ = corr' f'n

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $\cancelto{0}{H^0(Z \otimes W)} \to H^0(\mathrm{Sym}^2 W \otimes O) \stackrel{\delta}{\to} H^1(S_1) \to \cancelto{0}{H^1(Z \otimes W)}$

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes O) \stackrel{\sim}{\to} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to S_1 \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes O) \stackrel{\sim}{\to} H^1(S_1)$$

Examine other sequence:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \to H^1(S_1) \stackrel{\delta}{\to} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)$ ↘ 0

Since Z is a sum of *O*(-1,0)'s, *O*(0,-1)'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 E^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes O) \underset{\delta}{\overset{\sim}{\to}} H^1(S_1) \underset{\delta}{\to} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}) \right)$$

Review of quantum sheaf cohomology

Quantum sheaf cohomology

= OPE ring of the A/2 model

Instanton sectors have the same form,
except X replaced by moduli space M of instantons,
E replaced by induced sheaf F over moduli space M.

Must compactify M,
and extend F over compactification divisor.

$$\left.\begin{array}{l} \wedge^{\text{top}} E^* \cong K_X \\ \text{ch}_2(\text{E}) = \text{ch}_2(\text{TX}) \end{array}\right\} \overset{\text{GRR}}{\Longrightarrow} \wedge^{\text{top}} F^* \cong K_M$$

Within any one sector, can follow the same method just outlined….

Review of quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the `classical' ring in that sector is of the form

$$\mathrm{Sym}^{\bullet} \mathrm{W} / (Q^{a+1}, \tilde{Q}^{b+1})$$

where $Q = \det(A\psi + B\tilde{\psi})$, $\tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, OPE's can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => OPE's $Q = q$, $\tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Review of quantum sheaf cohomology

General result: (Donagi, Guffin, Katz, ES, '11)

For any toric variety, and any def' E of its tangent bundle,

$$0 \to W^* \otimes O \xrightarrow{*} \underbrace{\oplus O(\vec{q}_i)}_{Z^*} \to \mathrm{E} \to 0$$

the chiral ring is

$$\prod_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} = \mathrm{q}_a$$

where the M's are matrices of chiral operators built from *.

Review of quantum sheaf cohomology

So far, I've outlined mathematical computations of quantum sheaf cohomology, but GLSM-based methods also exist:

- Quantum cohomology ((2,2)): Morrison-Plesser '94
- Quantum sheaf cohomology ((0,2)): McOrist-Melnikov '07, '08

Briefly, for (0,2) case:

One computes quantum corrections to effective action of form

$$L_{\mathrm{eff}} = \int d\theta^+ \sum_a \Upsilon_a \log\left[\prod\nolimits_\alpha (\det M_{(\alpha)})^{Q^a_\alpha} / q_a \right]$$

from which one derives $\prod_\alpha \left(\det M_{(\alpha)} \right)^{Q^a_\alpha} = q_a$

— these are q.s.c. rel'ns — match math' computations


The Future
NEXT EXIT

Long-term

More general constructions of (0,2) mirrors, & related duals, as current methods are limited

Generalize quantum sheaf cohomology computations to arbitrary compact Calabi-Yau manifolds

Generalize quantum sheaf cohomology…

State of the art: computations on toric varieties

To do: compact CY's

Intermediate step: Grassmannians (work in progress)

Briefly, what we need are better computational methods.

Conventional GW tricks seem to revolve around idea that A model is independent of complex structure, not necessarily true for A/2.

- McOrist-Melnikov '08 have argued an analogue for A/2
- Despite attempts to check (Garavuso-ES '13), still not well-understood

Mathematics	**Physics**
Geometry: Gromov-Witten, Donaldson-Thomas, quantum cohomology, etc	Supersymmetric, topological quantum field theories
Homotopy, categories:	
derived categories	D-branes
stacks	gauge theories
derived spaces	sigma models w/ potential
categorical equivalence	renormalization group flow