An introduction to decomposition

UVA

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An overview of hep-th/0502027, 0502044, 0502053, 0606034, … (many …),
& recently arXiv: 2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423

w/ D. Robbins, T. Vandermeulen

My talk today concerns the application of **decomposition**,
a new notion in quantum field theory (QFT),
to resolution of anomalies as proposed in Wang-Wen-Witten.

Briefly, decomposition is the observation that some QFTs
are secretly equivalent to
sums of other QFTs, known as 'universes.'

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one QFT to be a sum of other QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i Z_i = \sum_i \sum \exp(-\beta H_i)$$

(on a connected spacetime)

This reflects a (higher-form) symmetry....

There are many examples of decomposition ...

Finite gauge theories (orbifolds): we'll see many examples later. (T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

Common thread: a subgroup of the orbifold group acts trivially.

Example: If $K \subset \text{center}(\Gamma) \subset \Gamma$ acts trivially, then $[X/\Gamma] = \coprod_{\text{irreps } K} [X/(\Gamma/K)]_{\hat{\omega}}$

Gauge theories:

- 2d $U(1)$ gauge theory with nonmin' charges = sum of $U(1)$ theories w/ min charges (Hellerman et al '06)
- 2d G gauge theory w/ center-invt matter = sum of $G/Z(G)$ theories w/ discrete theta (ES '14)

  Ex: $SU(2)$ theory (w/ center-invt matter) = $SO(3)_+ \coprod SO(3)_-$ (w/ same matter)

- 2d pure G Yang-Mills = sum of invertibles indexed by irreps of G (Nguyen, Tanizaki, Unsal '21) (U(1): Cherman, Jacobson '20)

  Ex: pure $SU(2) = \coprod_{\text{irreps } SU(2)}$ (sigma model on pt)

- 4d Yang-Mills w/ restriction to instantons of deg' divisible by k (Tanizaki, Unsal '19)

  = union of ordinary 4d Yang-Mills w/ different θ angles

More examples ....

There are many examples of decomposition ...

More examples :

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

Examples:

(Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

(Also: Komargodski et al '20, Huang et al 2110.02958)

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)

  (Hellerman, ES, 1012.5999)

- 2d G/G model at level k = disjoint union of invertible theories as many as integrable reps of the Kac-Moody algebra

  (Komargodski et al 2008.07567)

- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps

  (In fact, is a special case of orbifolds discussed later in this talk.)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

(T Pantev, ES '05)

Solves tech issue w/ cluster decomposition.

What do these examples have in common?....

What do the examples have in common?
When is one QFT a sum of other QFTs ?

Answer: in d spacetime dimensions,
a theory decomposes when it has a $(d-1)$-form symmetry.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

Decomposition & higher-form symmetries go hand-in-hand.

Today I'm interested in the case $d=2$,
so get a decomposition if a $(d-1)=1$-form symmetry is present.

What is a 1-form symmetry?

What is a (linearly realized) one-form symmetry in 2d ?

For this talk, *intuitively*, this will be a `group' that exchanges nonperturbative sectors.

Example: G gauge theory or orbifold in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume K abelian,
but decompositions exist more generally.)

Then, at least for K in the center, nonperturbative sectors are invariant under

$$(G-\text{bundle}) \mapsto (G-\text{bundle}) \otimes (K-\text{bundle})$$

$$A \mapsto A + A'$$

This is the action of the `group' of K-bundles.

That group is denoted BK or $K^{(1)}$

(Technically, is a 2-group, only weakly associative.)

One-form symmetries can also be seen in algebra of topological local operators,
where they are often realized *non*linearly (eg 2d TFTs). (Komargodski et al '20, Huang et al 2110.02958)

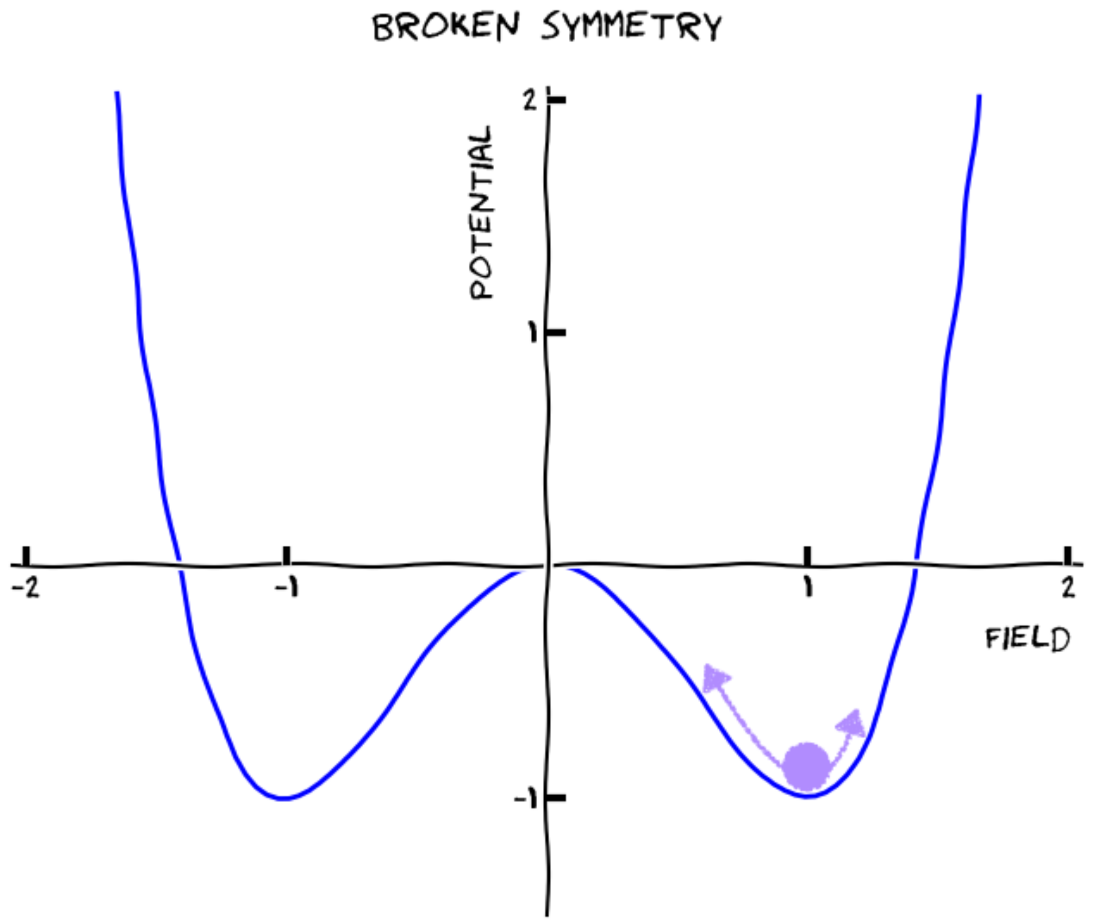
Decomposition ≠ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

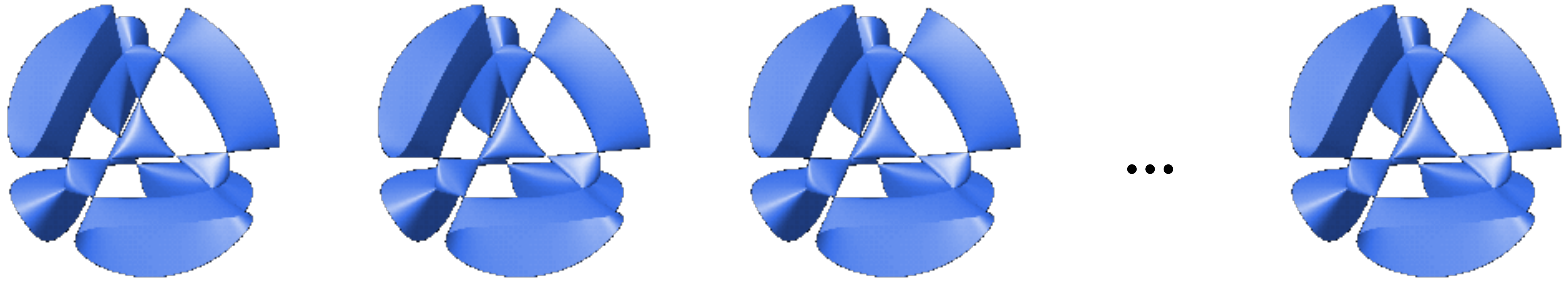


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:

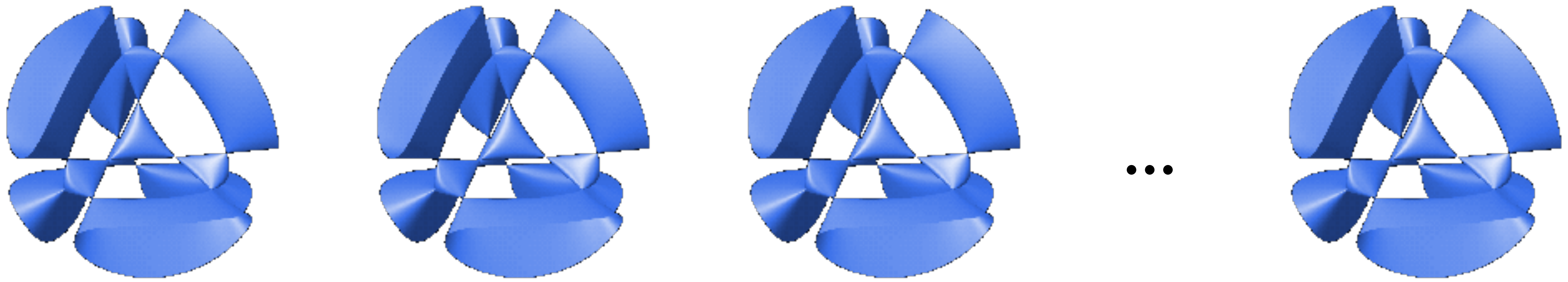


(see e.g. Tanizaki-Unsal 1912.01033)

Decomposition ≠ spontaneous symmetry breaking

Note that they both have an order parameter, so be careful when distinguishing.

Ex: sigma model on disjoint union of n spaces ('universes')



Have topological projectors Π_i $\quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1$

Have order parameter X $\quad X = \sum_{i=0}^{n-1} \xi^i \Pi_i, \quad \xi = \exp(2\pi i/n)$

Vev in ith universe: $\quad \langle \Pi_i X \rangle = \langle \xi^i \Pi_i \rangle = \xi^i$

So, could be described as spontaneously broken phase
— but that clearly does **not** capture the physics.

Sums vs products

Note: today I'm talking about sums of QFTs, not products.

Example of product: QFT of 2 free bosons = product of QFTs of each boson separately.
— that's not a decomposition.

Product:

States of $A \otimes B$ are of the form $|\psi_A\rangle \otimes |\psi_B\rangle$

Lagrangian $L(A \otimes B) \;=\; L(A) + L(B)$

Partition function $Z(A \otimes B) \;=\; Z(A)\, Z(B)$

Sum / disjoint union (as in decomposition):

States of $A \coprod B \;=\; |\psi_A\rangle \oplus |\psi_B\rangle$

Partition function $Z\left(A \coprod B\right) \;=\; Z(A) \;+\; Z(B)$

(on connected spacetime)

The particular QFTs I'm interested in today, which have a decomposition, are (1+1)-dimensional theories with global 1-form symmetries of the following form:

(Pantev, ES '05; Hellerman et al '06)

Symmetry

1-form
- Gauge theory or orbifold w/ trivially-acting subgroup (<-> non-complete charge spectrum)

$(d-1)$-form
- Theory w/ restriction on instantons

1-form
- Sigma models on gerbes = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$

$(d-1)$-form
- Algebra of topological local operators

Decomposition (into 'universes') often relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Example: Decomposition in 2d gauge theories

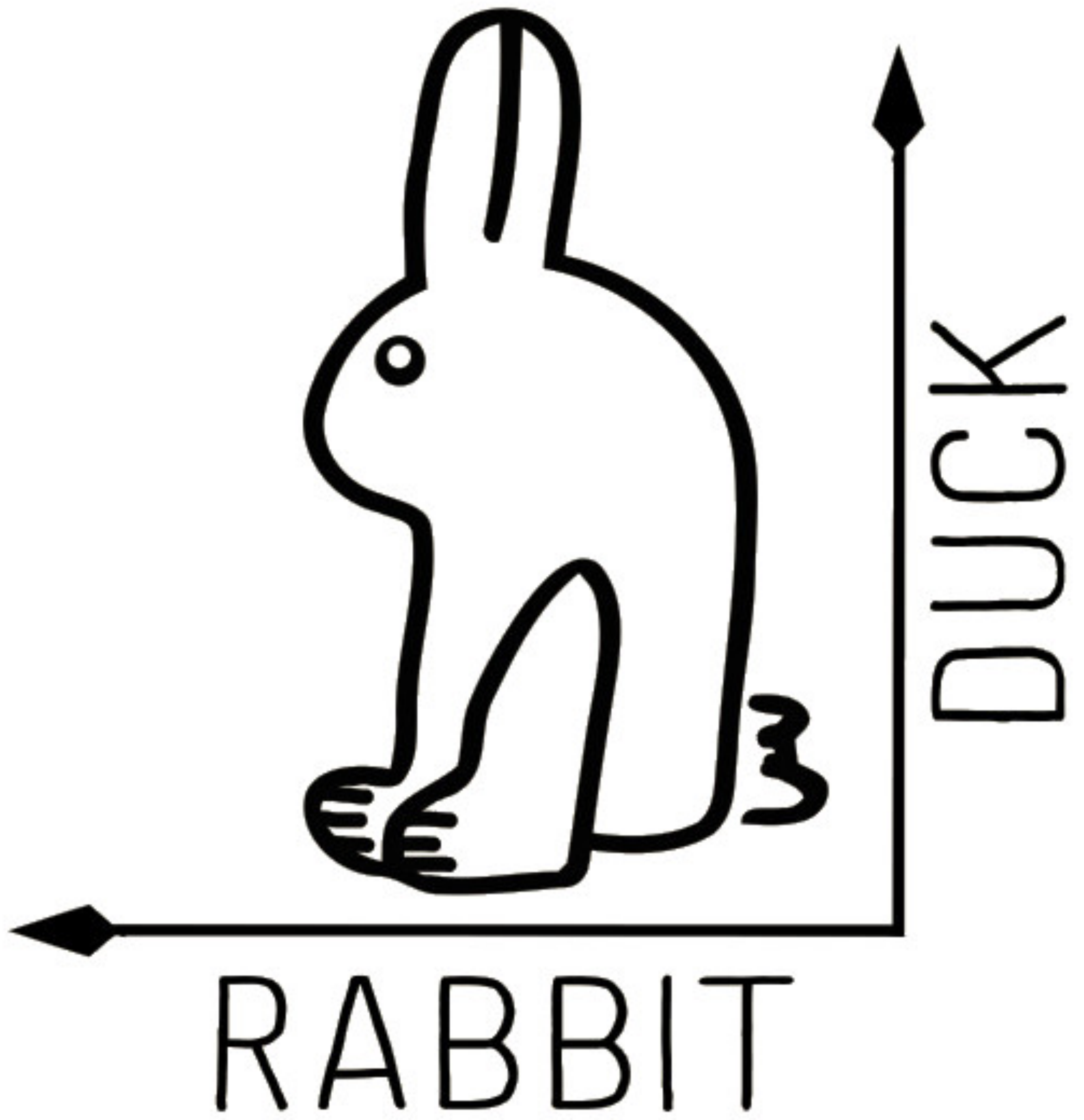
(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

So far, this sounds like just one QFT.



However, I'll outline how, from another perspective,
QFTs of this form are also each
a disjoint union of other QFTs;
they "decompose."

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Claim this theory decomposes.
Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn's theorem in math.

Universes $\longleftrightarrow$ Irreducible representations of K

Partition functions & relation of decomp' to restrictions on instantons....

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Statement of decomposition (in this example):

$$\text{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \text{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

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Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset$ $SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

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Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A) \right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A) \right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

("multiverse interference" cancels out some sectors)

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left(\overbrace{\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right]}^{\text{projection operator}} \right)$$

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

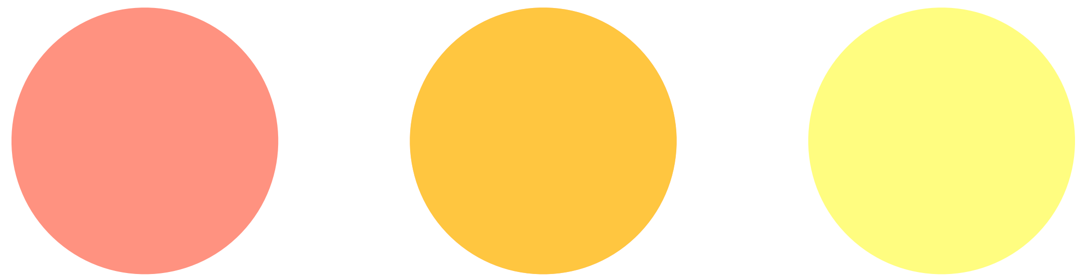
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union of several QFTs / universes = \`One' QFT with a restriction on nonperturbative sectors = \`multiverse interference'

Schematically,
two theories combine to form a distinct third:

universe ($SO(3)_+$)



universe ($SO(3)_-$)

multiverse interference effect ($SU(2)$)

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps

that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

Another feature these theories all have in common:
violation of cluster decomposition

As Weinberg taught us years ago,
restricting instantons violates cluster decomposition,
and as we'll see, instanton restriction is a common feature in these theories.

A disjoint union of QFTs also violates cluster decomposition,
but in a trivially controllable fashion.

Lesson: restricting instantons OK,
so long as one has a disjoint union.

(Hellerman, Henriques, T Pantev, ES, M Ando, hep-th/0606034)

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (lattice gauge thy) (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, … …, Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies in finite gauge theories (Robbins et al '21)

Next, I'll look at application to anomalies….

Fun features of decomposition:

Multiverse interference effects

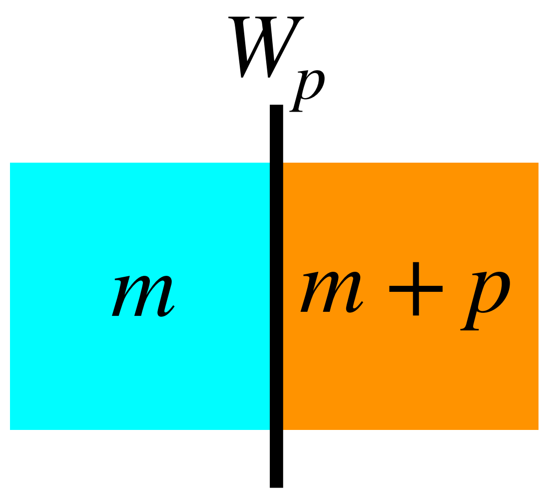
Ex: 2d $SU(2)$ gauge theory w/ center-invariant matter = $SO(3)_+ + SO(3)_-$

Summing over the two universes ($SO(3)$ gauge theories)

cancels out $SO(3)$ bundles which don't arise from $SU(2)$.

Wilson lines = defects between universes

Ex: 2d abelian BF theory at level k



Projectors: $\Pi_m = \frac{1}{k}\sum_{n=0}^{k-1} \xi^{nm}\mathcal{O}_n \qquad \xi = \exp(2\pi i/k)$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq}\, W_q \mathcal{O}_p \quad \Leftrightarrow \quad \Pi_m W_p = W_p \Pi_{m+p \mod k}$

Wormholes between universes

(GLSMs: Caldararu et al, 0709.3855)

Ex: U(1) susy gauge theory in 2d: 2 chirals p charge 2, 4 chirals ϕ charge -1, $W = \sum_{ij} \phi_i \phi_j A^{ij}(p)$

Describes double cover of $\mathbb{P}^1$ (sheets are universes), linked over locus where ϕ massless — Euclidean wormhole

Let's switch gears now.

So far, I've given a broad overview of decomposition.

Next, I'm going to discuss details in finite gauge theories (= orbifolds),
and a specific application,
namely to Wang-Wen-Witten's work on anomaly resolution.

Not only will this be an excellent example of a use of decomposition,
but we'll also see explicitly in concrete examples how decomposition works.

My goal for the rest of this talk is to apply decomposition
to an anomaly resolution procedure in finite gauge theories (orbifolds) (Wang-Wen-Witten '17).

Briefly, the idea of WWW is that if a given orbifold $[X/G]$ is ill-defined because of an anomaly (which obstructs the gauging),
then replace G with a larger group Γ whose action is anomaly-free.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

The larger group Γ has a subgroup $K \subset \Gamma$ that acts trivially on X,
and $G = \Gamma/K$.

However, orbifolds with trivially-acting subgroups are standard examples in which decomposition arises (in 1+1 dimensions), so one expects decomposition is relevant here.

(Hellerman et al '06)

Plan for the remainder of the talk:

- Describe decomposition in orbifolds with trivially-acting subgroups,
- Add a new modular invariant phase: “quantum symmetry,” in $H^1(G, H^1(K, U(1)))$,
- Review the anomaly-resolution procedure of (Wang-Wen-Witten ’17),
- and apply decomposition to that procedure.

What we’ll find is that, in (1+1)-dimensions,

$$\text{QFT}(\text{“}\widetilde{[X/G]}\text{”} = [X/\Gamma]_B) \;=\; \text{QFT}(\text{ copies and covers of } [X/(\text{nonanomalous subgp of } G] \text{ })$$

as a consequence of decomposition.
This gives a simple understanding of why the WWW procedure works,
as well as of the result.

Decomposition in finite gauge theories (orbifolds) in (1+1) dimensions

Let's begin by discussing ordinary orbifolds w/o extra phases.
(We'll need a more complicated version for anomaly resolution,
but let's start here, and build up.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

(K, Γ, G finite)

For simplicity, assume K central.

Decomposition implies

(Hellerman et al '06)

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

Note similar to gauge theories:
$SU(2) = SO(3)_+ + SO(3)_-$

$\hat{K}$ = set of iso classes of irreps of K

$\hat{\omega}$ = phases called "discrete torsion".

$= \mathrm{Image}\left(H^2(G,K) \xrightarrow{\theta \in \hat{K}} H^2(G,U(1)) \right)$

Decomposition in finite gauge theories (orbifolds) in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (assume K central)

Decomposition implies

(Hellerman et al '06)

$$\mathrm{QFT}\left([X/\Gamma]\right) \;=\; \mathrm{QFT}\left(\coprod_{\hat{K}} \left[X/G\right]_{\hat{\omega}} \right)$$

$\hat{K}$ = set of iso classes of irreps of K

Projectors: For $R \in \hat{K}$, we have the projector

$$\Pi_R \;=\; \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

(Wedderburn's theorem for center of group algebra)

which obey $\Pi_R \Pi_S = \delta_{R,S} \Pi_R$, $\displaystyle\sum_R \Pi_R = 1$ $\quad [\Pi_R, \mathcal{O}] = 0$

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \;=\; \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \;=\; 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} \;=\; \frac{1}{2}(1 \pm \hat{z})$$

(= specialization of formula given earlier)

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_{+} + \Pi_{-} = 1$$

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \, \underset{h}{\blacksquare} \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

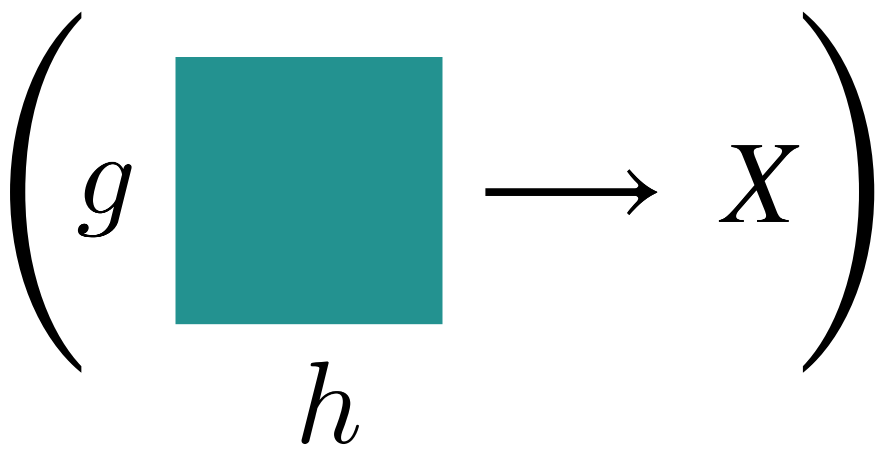
$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h\in D_4,\ gh=hg} Z_{g,h}$$

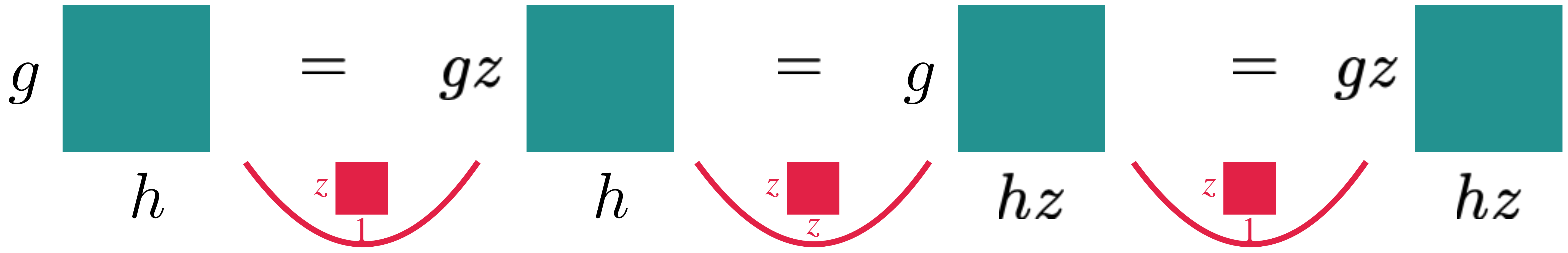
where $Z_{g,h} =$



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z

$Z_{g,h} =$



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\} \qquad \text{where } \overline{a} = \{a, az\} \text{ etc}$$

$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \underset{h}{\blacksquare} \longrightarrow X \right)$$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a} \underset{\overline{b}}{\blacksquare} \quad \overline{a} \underset{\overline{ab}}{\blacksquare} \quad \overline{b} \underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $\quad Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h) \, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

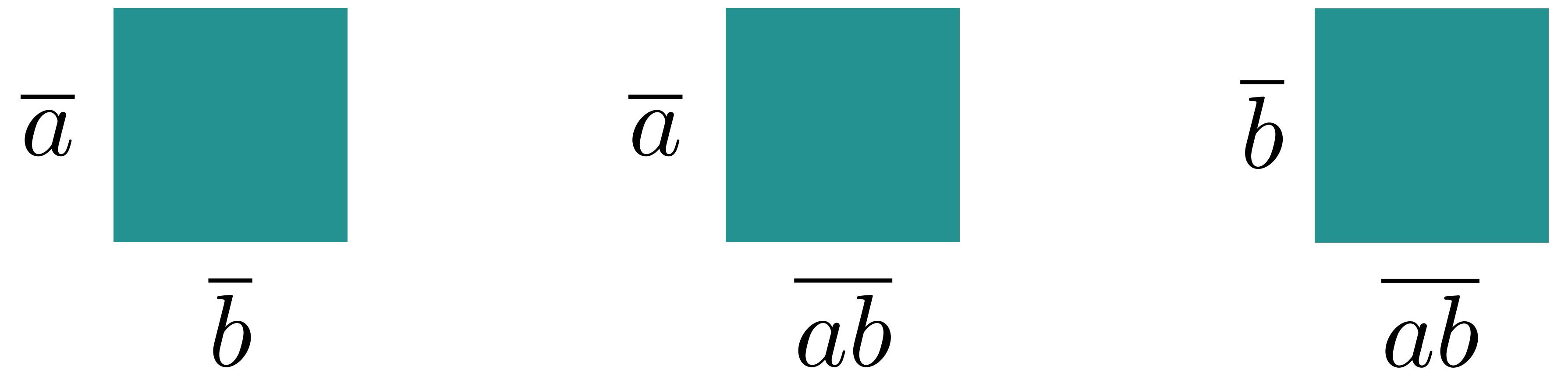
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$=\; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

In a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold, discrete torsion $\in\; H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) \;=\; \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

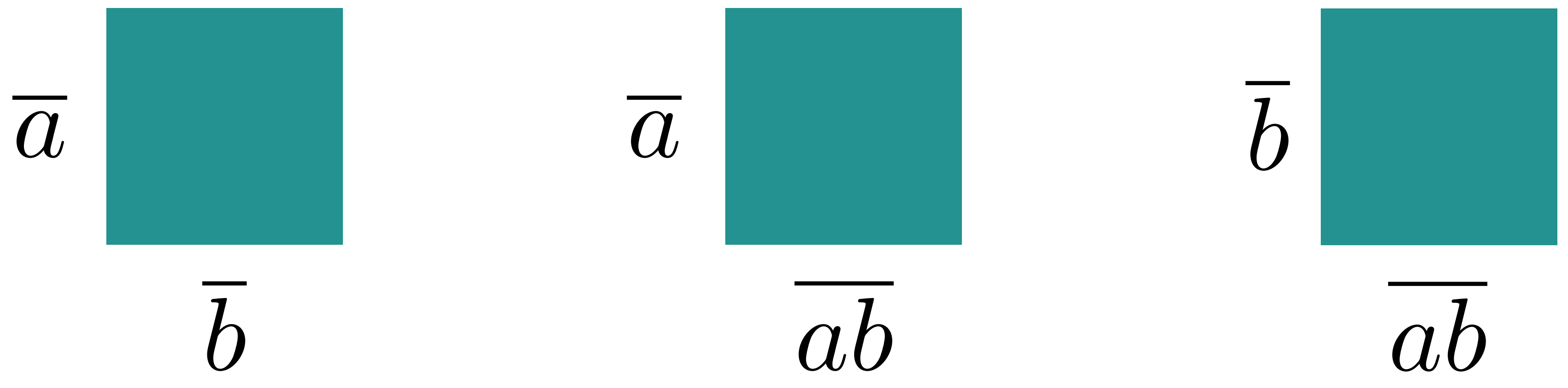
Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}([X/D_4]) = Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}) + Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}})$$

Matches prediction of decomposition

$$\text{QFT}([X/D_4]) = \text{QFT}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}) \coprod \text{QFT}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}})$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \; = \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \; + \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \; = \; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

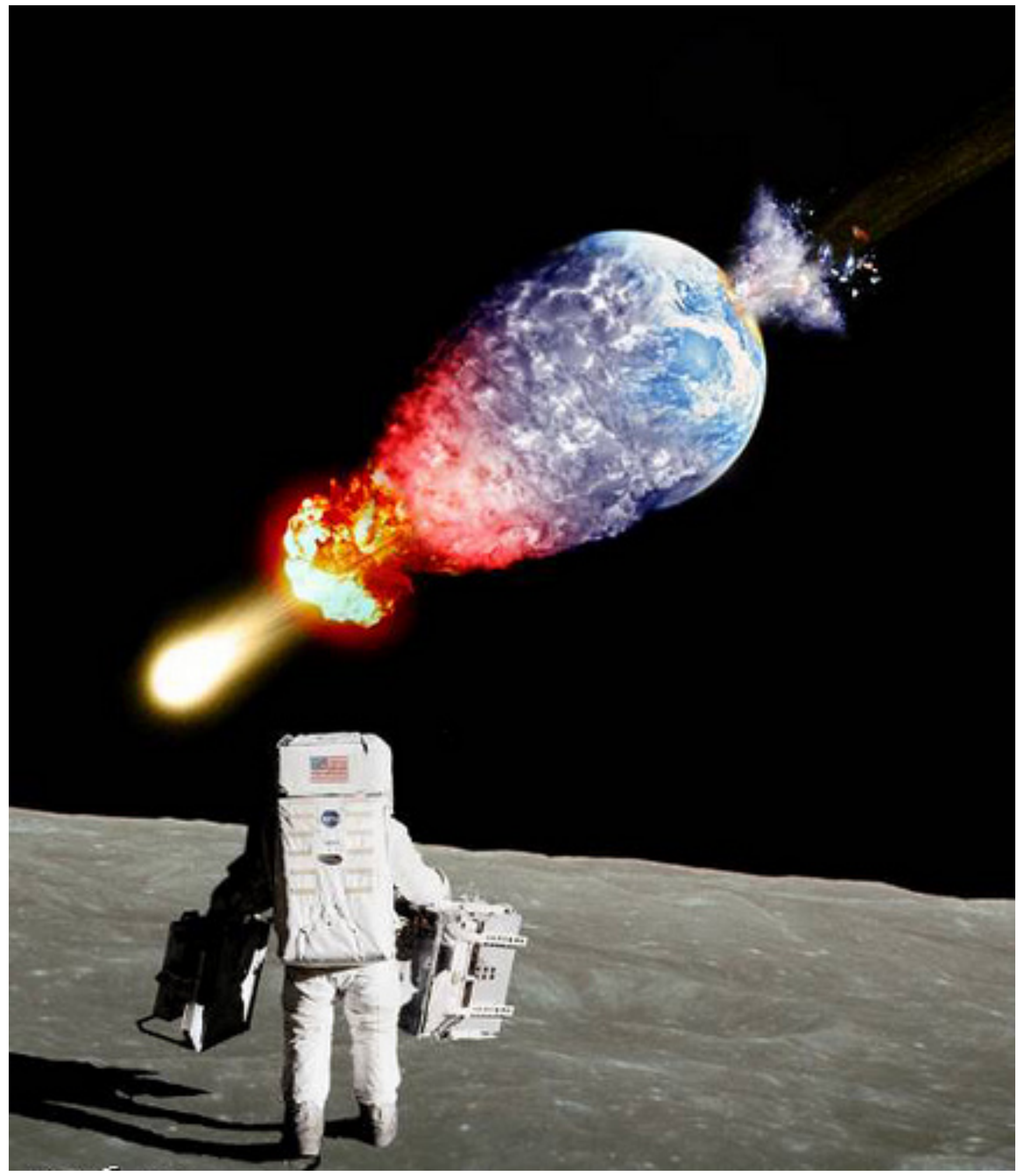
$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

If we didn't know about decomposition, the 2's in the corners would be a problem…

A big problem!

They signal a violation of cluster decomposition, the same axiom that's violated by restricting instantons.



Ordinarily, I'd assume that the computation was wrong.

However, decomposition saves the day….

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
+
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/ d.t.

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions, where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) \; = \; \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

where $\hat{K}$ = irreps of K

$\hat{\omega}$ = discrete torsion on universes

Here, $G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

so $$\mathrm{QFT}\left([X/\mathbb{H}]\right) \; = \; \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

(Hellerman et al, hep-th/0606034, sect. 5.4)

Let's get back on track.

My goal today is to talk about anomaly resolution in 1+1 dimensions.

Decomposition will play a vital role in understanding how the anomalies are resolved.

Recall the idea of www is that given an anomalous (ill-defined) $[X/G]$,

replace G by a larger finite group Γ obeying certain properties,

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

and add phases.

Because Γ has a subgroup K that acts trivially,

orbifolds $[X/\Gamma]$ will decompose,

into copies & covers of $[X/G]$.

However, just getting copies of $[X/G]$ won't help.

We also need to add certain new phases, which I will describe next....

New modular invariant phases: quantum symmetries

(Tachikawa '17; Robbins et al '21)

A quantum symmetry is a modular-invariant phase in finite gauge theories (orbifolds) in which a subgroup K acts trivially.

Classified by elements of $H^1(G, H^1(K, U(1))) \; = \; \mathrm{Hom}(G, \hat{K})$.

It acts on twisted sector states by phases. Schematically:

$$gz \, \underset{h}{\blacksquare} \;=\; B(\pi(h), z) \left(g \, \underset{h}{\blacksquare} \right)$$

where

$z \in K \qquad g, h \in \Gamma$

$B \in H^1(G, H^1(K, U(1)))$

These generalize the old notion of `quantum symmetries' in the orbifolds literature.

Decomposition in the presence of a quantum symmetry

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\coprod_{\widehat{\mathrm{Coker}\,B}} \left[X/\mathrm{Ker}\,B\right]_{\hat{\omega}} \right)$$

where $B \;\in\; H^1(G, H^1(K, U(1))) \;=\; \mathrm{Hom}(G, \hat{K})$

This is more or less uniquely determined by consistency with results for decomposition in presence of discrete torsion.

The result at top needs to include this as a special case, and it does.

Also, checked in (lots of) examples. Let's move on....

How do WWW relate quantum symmetries to anomalies?

Fact: gauge anomalies in a finite G gauge theory in $(n+1)$ dimensions are classified by $H^{n+2}(G, U(1))$.

(Reasoning from `topological defect lines')

We're going to pick a quantum symmetry B such that $d_2 B =$ anomaly:

$$\underbrace{\left(\mathrm{Ker}\,\iota^* \subset H^2(\Gamma, U(1))\right)}_{\text{(discrete torsion)}} \xrightarrow{\beta} \underbrace{H^1(G, H^1(K, U(1)))}_{\text{(quantum symmetry)}} \xrightarrow{d_2} \underbrace{H^3(G, U(1))}_{\text{(anomalies)}}$$

(Hochschild '77)

Now we're ready to walk through the WWW anomaly resolution procedure....

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
gauge anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (assumed central)

where Γ is chosen so that $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

The idea is then to replace $[X/G]$ with $[X/\Gamma]$,
but, need to describe how Γ acts on X.

If K acts triv'ly on X, and we do nothing else,
then we have accomplished nothing:

decomposition $\Rightarrow$ $\mathrm{QFT}([X/\Gamma]) = \coprod_{\hat{K}} \mathrm{QFT}([X/G])$ — still anomalous

Fix by adding quantum symmetry....

Application to anomalies

Suppose we have an orbifold $[X/G]$ in 1+1d which is anomalous,
gauge anomaly $\alpha \in H^3(G, U(1))$ (Wang-Wen-Witten '17)

Algorithm to resolve:

1) Make G bigger: replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} 1$ (assumed central)

2) Turn on quantum symmetry $B \in H^1(G, H^1(K, U(1)))$
chosen so that $d_2 B = \alpha$. This implies $\pi^* \alpha \in H^3(\Gamma, U(1))$ is trivial.

$$\underbrace{\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right)}_{\text{(discrete torsion)}} \xrightarrow{\beta} \underbrace{H^1(G, H^1(K, U(1)))}_{\text{(quantum symmetry)}} \xrightarrow{d_2} \underbrace{H^3(G, U(1))}_{\text{(anomalies)}}$$

(Hochschild '77)

K acts trivially on X, but nontrivially on twisted sector states via B

These two together — extension Γ plus B — resolve anomaly.

Decomposition explains how....

Application to anomaly resolution

Procedure: replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$

where $d_2 B \;=\; \alpha \in H^3(G, U(1))$, the anomaly of the G orbifold.

Decomposition:

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \;=\; \mathrm{QFT}\left(\coprod_{\widehat{\mathrm{Coker}\, B}} \left[X/\mathrm{Ker}\, B\right]_{\hat{\omega}} \right)$$

— using earlier results for decomp' in orb' w/ quantum symmetry

Note that since $d_2 B = \alpha$, $\quad \alpha|_{\mathrm{Ker}\, B} = 0$

So, $\mathrm{Ker}\, B \subset G$ is automatically anomaly-free!

Summary: $[X/\Gamma]_B =$ copies of orbifold by anomaly-free subgroup.

Let's see this in examples....

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 1: Define $\Gamma = D_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	**d_2(B) (anomaly)**	**w/o d.t. in D4**	**w/ d.t. in D4**
1	1	—	$[X/G] \coprod [X/G]_{\mathrm{dt}}$	$[X/\langle b \rangle]$
-1	1	—	$[X/\langle b \rangle]$	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
1	-1	$\langle b \rangle$	$[X/\langle a \rangle]$	$[X/\langle ab \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle a \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 2: Define $\Gamma = \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	Result
1	1	—	$[X/G] \coprod [X/G]_{\mathrm{dt}}$
-1	1	$\langle a \rangle, \langle ab \rangle$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

Extension 3: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$, $\quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Quantum symmetry B determined by image on $\{a, b\}$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	w/o d.t. in Z2 x Z4	w/ d.t. in Z2 x Z4
1	1	—	$[X/G] \coprod [X/G]$	$[X/G]_{\mathrm{dt}} \coprod [X/G]_{\mathrm{dt}}$
-1	1	$\langle ab \rangle$	$[X/\langle b \rangle]$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$	$[X/\langle a \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle ab \rangle]$

Get only anomaly-free subgroups, varying w/ B.

Works!

Example: Resolve an anomalous orbifold $[X/G]$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$

$$\text{Anomaly } \alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle$$

In the examples so far, we picked a `minimal' resolution Γ.

If we pick larger K, we get copies.

Extension 4: Define $\Gamma = \mathbb{Z}_2 \times \mathbb{H}$, $\quad 1 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$

Results:

$B(a)$	$B(b)$	d_2(B) (anomaly)	Result
1	1	–	$\coprod_2 \left([X/G] \coprod [X/G]_{\mathrm{dt}} \right)$
-1	1	$\langle a \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$\coprod_2 [X/\langle ab \rangle]$

Get copies of orb's w/ anomaly-free subgroups.

Works!

Summary

Decomposition: `one' QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories
with $n-$form symmetries.

(I've focused on examples in 1+1d,
but examples exist in other dim's too.)

Can be used to understand anomaly-resolution procedure of www:

replace anomalous $[X/G]$ with non-anomalous $[X/\Gamma]_B$,
but decomposition implies
$$\mathrm{QFT}\left([X/\Gamma]_B\right) = \text{copies of } \mathrm{QFT}\left([X/\mathrm{Ker}\, B \subset G]\right),$$
which is explicitly non-anomalous.

Thank you for your time!