Duality in two-dimensional nonabelian gauge theories

Eric Sharpe

B Jia, ES, R Wu, arXiv: 1401.1511

My talk today concerns analogues of Seiberg duality in two-dimensional nonabelian gauge theories with (2,2) and (0,2) supersymmetry.

These theories arise as part of recent ongoing developments in gauged linear sigma models (GLSM's).

I'll outline constructions of a number of nonabelian theories and their duals, when known.

Theme: dualities derived from geometry

Outline

- (2,2) theories:
  - review $\mathbb{CP}^N$ model, hypersurfaces, Grassmannian
  - Theories w/ both fundamentals and antifundamentals - Benini-Cremonesi duality, and first application of geometry to derive gauge theory dualities
  - Abelian/nonabelian duality: G(2,4) vs $\mathbb{P}^5[2]$
  - Pfaffian constructions and more dualities
- (0,2) theories:
  - `gauge bundle dualization duality', dynamical susy breaking
  - Gadde-Gukov-Putrov triality via geometry,
  - abelian/nonabelian examples, Pfaffian examples
- Obstructions to some dualities

Gauge duality from geometry

Two-dimensional gauge theories are very different from four-dimensional gauge theories.

Crucial difference: no gauge dynamics.

In effect, in 2d,
gauge fields = Lagrange multipliers.

As a result, all gauge effects can be understood as low-energy NLSM effects.

Ex: gauge instantons =>
worldsheet instantons in low-energy NLSM

In principle, makes 2d Seiberg duality a lot easier.

Prototypical example: $\mathbb{CP}^n$ **model** **((2,2) susy)**

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Analyze semiclassical low-energy behavior:

Potential $V = D^2$

where $$D = \sum_i |\phi_i|^2 - r$$

r = Fayet-Iliopoulos parameter

When $r \gg 0$, $\{V = 0\} = S^{2n+1}$

so semiclassical Higgs moduli space is $\{V = 0\} / U(1) = \mathbb{CP}^n$

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Gauge theory:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

Semiclassical Higgs moduli space is $\{V=0\}/U(1)=\mathbb{CP}^n$

Of course, that doesn't tell the whole story.

r is renormalized at one-loop:

$$\Delta r \propto \sum_i q_i \quad \text{here, } = n+1$$

so the $\mathbb{CP}^n$ shrinks to strong coupling under RG.

Prototypical example: $\mathbb{CP}^n$ model ((2,2) susy)

Summary:

U(1) gauge group,
matter: n+1 chiral multiplets, charge +1

↓

Nonlinear sigma model on $\mathbb{CP}^n$

Hypersurfaces ((2,2) susy)

For later use, it will be handy to describe hypersurfaces.

S'pose want NLSM on $\{G = 0\} \subset \mathbb{CP}^n$

where G is a homogeneous polynomial of degree d.

Try: U(1) gauge theory, n+1 chiral multiplets charge +1, superpotential W = G

But that superpotential is not gauge invariant, so this isn't the answer.

Correct method….

Hypersurfaces ((2,2) susy)

Want gauge theory with low energy limit
= NLSM on $\{G=0\} \subset \mathbb{CP}^n$

Answer:

U(1) gauge theory, n+1 chiral multiplets charge +1,
1 chiral multiplet P charge -d,
superpotential W = P *G*

Gauge-invariant superpotential

$r \gg 0: \quad D = \sum_i |\phi_i|^2 - d|p|^2 - \mathrm{r}$ implies ϕ_i not all zero

$G=0, \quad pdG=0$ imply, for smooth hypersurface,
$p=0, \quad G=0$

Result is desired NLSM at low energies
(modulo r renormalization)

Next example: nonabelian version **((2,2) susy)**

Gauge theory:

U(k) gauge group,
matter: n chiral multiplets in fund' **k**, n > k

Similar analysis:

Nonlinear sigma model on G(k,n)

Next example: nonabelian version ((2,2) susy)

Dualities:

Mathematically, G(k,n) = G(n-k,n)

Since IR limits are same,

U(k) gauge group,
matter: n chiral multiplets in fund' **k**, n > k

is Seiberg dual to

U(n-k) gauge group,
matter: n chiral multiplets in fund' **k**
(n > n-k trivially)

Automatic: same chiral rings, same anomalies, same Higgs moduli space

Next example: nonabelian version ((2,2) susy)

What if we add antifundamentals ?

Answer (Benini-Cremonesi, '12):

U(k) gauge group,
matter: n chirals in fund' **k**, n>k,
A chirals in antifund' $\mathbf{k}^*$, A<n

is Seiberg dual to

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**, A chirals P in antifund' $\mathbf{k}^*$,
nA neutral chirals M,
superpotential: $W = M \Phi P$

B-C justified by checking elliptic genera;
we will justify with geometry momentarily….

Next example: nonabelian version ((2,2) susy)

We can understand that case geometrically.

U(k) gauge group,
matter: n chirals in fund' **k**, A chirals in antifund' $\mathbf{k}^*$

Nonlinear sigma model on $\mathrm{Tot}\left(S^A \to G(k,n)\right)$

Duality: $= \mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k,n)\right)$

generalizing G(k,n) = G(n-k,n)

But how to realize $\mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k,n)\right)$?

Next example: nonabelian version ((2,2) susy)

How to realize $\mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k, n) \right)$ in physics?

Trick: S, Q are related:

$$0 \to S \xrightarrow{\Phi} O^n \to Q \to 0$$

so we build Q using S, $\mathcal{O}^n$,
and a superpotential realizing the map.

Here: A antifundamentals P, to realize S^A

nA neutrals M, to realize A copies of $\mathcal{O}^n$

superpotential $W = M \Phi P$

— matching B-C dual

Next example: nonabelian version ((2,2) susy)

U(k) gauge group,
matter: n chirals in fund' **k**, $n>k$,
A chirals in antifund' $\mathbf{k}^*$, $A<n$

Seiberg ⟷ dual

U(n-k) gauge group,
matter: n chirals Φ in fund' **k**,
A chirals P in antifund' $\mathbf{k}^*$,
nA neutral chirals M,
superpotential: $W = M \Phi P$

$$\mathrm{Tot}\left(S^A \to G(k,n)\right) = \mathrm{Tot}\left((Q^*)^A \to G(n\text{-}k,n)\right)$$

In this fashion, we can understand this 2d version of Seiberg duality purely geometrically.

Next example: nonabelian version ((2,2) susy)

What about more general matter representations?
Adjoints, higher tensors, etc?

In 4d, demanding asymptotic freedom would exclude most arbitrarily complicated matter representations.

In 2d, no such constraint in principle.

However, we will argue later that there may be different constraints in 2d that make dualities for more complicated matter representations, rare.

Abelian/nonabelian dualities **((2,2) susy)**

In 2d there are also Seiberg-like dualities between abelian and nonabelian theories.

Simple example: G(1,n) = G(n-1,n)

LHS = U(1) gauge theory, n chiral multiplets

RHS = U(n-1) gauge theory, n chiral multiplets

More fun example next….

Abelian/nonabelian dualities ((2,2) susy)

A more interesting example is motivated by the geometry
G(2,4) = degree 2 hypersurface in $\mathbb{CP}^5$

Result:

U(2) gauge theory,
matter: 4 chirals ϕ_i in fundamental **2**

is Seiberg dual to

U(1) gauge theory,
matter: 6 chirals $z_{ij} = - z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential
$W = P(z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23})$

Abelian/nonabelian dualities ((2,2) susy)

G(2,4) = degree 2 hypersurface in $\mathbb{CP}^5$

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**

⟷

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
superpotential

Relation: $z_{ij} = \epsilon_{\alpha\beta}\phi_i^\alpha \phi_j^\beta$

Compare symmetries: GL(4) action

$\phi_i^\alpha \mapsto V_i^{\,j} \phi_j^\alpha$

$z_{ij} \mapsto V_i^{\,k} V_j^{\,\ell} z_{k\ell}$

Chiral rings, anomalies, Higgs moduli space match automatically.

Can also show elliptic genera match.

Abelian/nonabelian dualities ((2,2) susy)

Brief outline of elliptic genus of $\mathbb{P}^5[2]$:

By applying susy localization, can derive exact expressions in terms of iterated residues.

(Benini, Eager, Hori, Tachikawa '13; Gadde, Gukov '13)

Here,

$$Z = \frac{2\pi\eta(q)^3}{\theta_1(q,y^{-1})} \oint du \left(\prod_{i,j} \frac{\theta_1(q, y^{-1} x e^{2\pi i(\zeta_i+\zeta_j)})}{\theta_1(q, x e^{2\pi i(\zeta_i+\zeta_j)})} \right) \frac{\theta_1(q, x^{-2} e^{2\pi i(-\zeta_1-\zeta_2-\zeta_3-\zeta_4)})}{\theta_1(q, y x^{-2} e^{2\pi i(-\zeta_1-\zeta_2-\zeta_3-\zeta_4)})}$$

where the ζ_i are fugacities for $(\mathbb{C}^\times)^4 \subset GL(4)$ symmetry

Can show with eg Mathematica that the residues match those of corresponding flavored elliptic genus of G(2,4).

Abelian/nonabelian dualities ((2,2) susy)

This extends to hypersurfaces & complete intersections:

$$G(2,4)[d_1,d_2,\cdots]=\mathbb{P}^5[2,d_1,d_2,\cdots]$$

U(2) gauge theory,
matter: 4 chirals ϕ_i in **2**
chirals p_a of charge $-d_a$
under det U(2)
superpotential

$$W=\sum_a p_a f_a(\epsilon_{\alpha\beta}\phi_i^\alpha\phi_j^\beta)$$

U(1) gauge theory,
6 chirals $z_{ij} = -z_{ji}$, i,j=1…4, of charge +1,
one chiral P of charge -2,
chirals P_a of charge $-d_a$,
superpotential

$$W=P(z_{12}z_{34}-z_{13}z_{24}+z_{14}z_{23})+\sum_a P_a f_a(z_{ij})$$

$$\epsilon_{\alpha\beta}\phi_i^\alpha\phi_j^\beta = z_{ij}$$

Straightforward extrapolation of previous duality,
as one might hope.

Abelian/nonabelian dualities ((2,2) susy)

$$G(2,4)[d_1, d_2, \cdots] = \mathbb{P}^5[2, d_1, d_2, \cdots]$$

Examples of this form illustrate the usefulness of geometry, in those cases in which it's applicable.

Specifically: **superpotentials break global symmetries**.

W/o global symmetries as a guide, finding dualities would be far more difficult.

In these examples, can instead use geometry to locate otherwise obscure dualities.

Pfaffians ((2,2) susy)

There exist more exotic dualities implied by geometry.
To justify them, need to outline construction of Pfaffians.

Let A be an nxn matrix,
each entry a homogeneous poly' over a proj' space
(or other toric variety), call it V.

A Pfaffian variety is defined by the locus on V where
$\mathrm{rank}\, A \leq k$ for some k.

- Not a hypersurface or a complete intersection in general.
- Only recently has anyone figured out how to describe such spaces with GLSM's. **(Hori-Tong '06, Hori '11, Jockers et al '12)**

Pfaffians **((2,2) susy)**

Two constructions of Pfaff = $\{\mathrm{rank}\, A \leq k\}$

- PAX model

U(n-k) gauge theory,
chirals X_a in n copies of fundamental,
chirals P_a in n copies of antifundamental,
$W = \mathrm{tr}\, P A(\Phi) X$ (plus data for V)

- PAXY model

U(k) gauge theory,
chirals $\tilde{X}_a$ in n copies of fundamental,
chirals $\tilde{Y}^a$ in n copies of antifundamental,
nxn matrix of neutral chirals $\tilde{P}^a_b$,
$W = \mathrm{tr}\, \tilde{P}\left(A(\Phi) - \tilde{Y}\tilde{X} \right)$ (plus data for V)

These two constructions are dual to one another….

Pfaffians ((2,2) susy)

Duality between PAX, PAXY constructions:

Start with PAX model:

U(n-k) gauge theory,
chirals X_a in n copies of fundamental,
chirals P_a in n copies of antifundamental,

$W = \mathrm{tr}\, PA(\Phi)X$ (plus data for V)

Apply B-C duality:

U(k) gauge theory,
chirals $\tilde{X}_a$ in n copies of fundamental,
chirals $\tilde{Y}^a$ in n copies of antifundamentals
n^2 neutral chirals $\tilde{P}^a_b = (XP)^a_b$,
plus a new superpotential term for total

$W = \mathrm{tr}\left(A\tilde{P} + \tilde{P}\tilde{Y}\tilde{X} \right)$ (plus data for V)

original term new term

Result = PAXY model

Pfaffians ((2,2) susy)

Duality: G(2,n) = Pfaffian

Mathematically,

G(2,n) = rank 2 locus of nxn matrix A over $\mathbb{P}^{\binom{n}{2}-1}$

$$A(z_{ij}) = \begin{bmatrix} z_{11}=0 & z_{12} & z_{13} & \cdots \\ z_{21}=-z_{12} & z_{22}=0 & z_{23} & \cdots \\ z_{31}=-z_{13} & z_{32}=-z_{23} & z_{33}=0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Proposal:

U(2) gauge theory, n chirals in fundamental

dual to

U(n-2)xU(1) gauge theory,
n chirals X in fundamental of U(n-2),
n chirals P in antifundamental of U(n-2),
(n choose 2) chirals $z_{ij} = - z_{ji}$ of charge +1 under U(1),
W = tr PAX

Even more complicated possibilities exist.

So far, I've outlined various dualities in 2d (2,2) susy theories.

Next: 2d (0,2)

I'll begin by describing

- a frequently occurring duality
- dynamical susy breaking in (0,2)
- (0,2) superspace

and then discuss various gauge-theoretic dualities.

Gauge bundle dualization duality ((0,2) susy)

(Nope, not a typo….)

Nonlinear sigma models with (0,2) susy defined by space X, with gauge bundle $E \to X$

Duality: CFT(X, E) = CFT(X, E^*)

ie, replacing the gauge bundle with its dual seems to be an invariance of the theory.

(Parts in ES ’06, complete in Gadde-Gukov-Putrov ’13, Jia-ES-Wu ’14)

We’ll use this duality, but first, some checks….

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Action invariant:

$$L = \frac{1}{2} g_{\mu\nu} \partial\phi^\mu \overline{\partial}\phi^\nu + \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\overline{z}} \psi_+^\nu + \frac{i}{2} h_{\alpha\beta} \lambda_-^\alpha D_z \lambda_-^\beta + F_{i\overline{j}a\overline{b}} \psi_+^i \psi_+^{\overline{j}} \lambda_-^a \lambda_-^{\overline{b}}$$

Under $E \leftrightarrow E^*$, $\lambda_-^a \leftrightarrow \lambda_-^{\overline{b}}$ & $F \leftrightarrow -F$

so we see the Lagrangian is invariant.

Consistency conditions:

$\mathrm{ch}_2(\mathrm{E}) = \mathrm{ch}_2(\mathrm{TX})$ invariant under $E \leftrightarrow E^*$

Massless spectra:

$h^\bullet(X, \wedge^\bullet E)$, $h^\bullet(X, \mathrm{End}\, E)$ invariant under $E \leftrightarrow E^*$

using $h^p(X, \wedge^q E^*) \cong h^{n-p}(X, \wedge^{r-q} E)$ (Serre duality on CY)

Gauge bundle dualization duality **((0,2) susy)**

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Bundle must be `stable': $g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0$

Math result: a bundle is stable iff its dual is stable.

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Elliptic genera:

Ex: (left movers in NS, + further twist)

$$\mathrm{Tr}\, (-)^{F_R} z^{J_L} q^{L_0} \overline{q}^{\overline{L}_0}$$

$$= q^{-(2n+r)/24} \int_X \mathrm{Td}(TX) \wedge \mathrm{ch}\Big(\bigotimes_{k=1,2,3,\cdots} S_{q^k}((TX)^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,5/2,\cdots} \wedge_{q^k} ((zE)^{\mathbb{C}}) \Big)$$

where $(zE)^{\mathbb{C}} = zE \oplus \overline{z}E^*$

Manifestly invariant under $E \leftrightarrow E^*$, so long as also $z \leftrightarrow \overline{z}$.

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Worldsheet instantons:

Consider the A/2, B/2 pseudo-topological twists.

It has been argued elsewhere that

$$A/2(X, E) = B/2(X, E^*)$$

so exchanging $E \leftrightarrow E^*$ merely exchanges A/2, B/2 models.

(ES '06)

Consistent w/ invariance of physical theory.

Gauge bundle dualization duality ((0,2) susy)

Check that (0,2) theory invariant under $E \leftrightarrow E^*$:

Reducible bundles: $E = A_1 \oplus A_2 \oplus \cdots \oplus A_n$

Consider dualizing $A_i \leftrightarrow A_i^*$ for just one (or a few) i

- action, massless spectra, Green-Schwarz condition all invariant under dualizing factors separately
- each A_i must be stable, and since for vacua slope=0, dualizing any one factor maintains stability
- elliptic genera invariant under dualizing factors separately
- now several U(1)'s hence several versions of A/2, B/2. Essentially as before, this duality should exchange them.

Duality holds.

Dynamical susy breaking ((0,2) susy)

Now that we've established a fairly common duality, let's turn to understanding when dynamical susy breaking will happen in 2d (0,2) theories.

Necessary condition: Witten index $\mathrm{Tr}(-)^{F_R}$ vanishes.

$\mathrm{Tr}(-)^{F_R}$ by itself is somewhat ill-defined, as it is a sum over infinitely many states.

We can refine, by adding into the trace any operators that commute with $(-)^{F_R}$

Easy choice: $\mathrm{Tr}\,(-)^{F_R}(-)^{F_L} q^{L_0} \overline{q}^{\overline{L}_0}$ = elliptic genus

So: vanishing of elliptic genus is necessary condition for (0,2) susy breaking

Dynamical susy breaking ((0,2) susy)

Example: (2,2) locus

Start with the elliptic genus for left-movers in R sector:

$$(-)^{r/2} q^{+(r-n)/12} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det E)^{+1/2} \wedge_{-1} E^* \bigotimes_{k=1,2,3,\cdots} S_{q^k}\big((TX)^{\mathbb{C}}\big) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k}\big(E^{\mathbb{C}}\big)\Big)$$

where $S_q(TX) = q + qTX + q^2 \mathrm{Sym}^2 TX + q^3 \mathrm{Sym}^3 (TX) + \cdots$

$$\wedge_q(E) = 1 + qE + q^2 \wedge^2 (E) + \mathrm{q}^3 \wedge^3 (E) + \cdots$$

Set $E = TX$ and use $S_q(E) = \wedge_{-q}(E)^{-1}$

The expression above reduces to

$$(-)^{r/2} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det TX)^{+1/2} \wedge_{-1} (T^* X)\Big) = (-)^{r/2} \chi(X)$$

In this case the (refined) Witten index reduces to the Euler characteristic of the space.

Dynamical susy breaking ((0,2) susy)

For general (0,2) theories, the higher q terms won't cancel.

However, we can get part of the elliptic genus just by truncating them.

Result: something whose vanishing is necessary (but not sufficient) for (0,2) susy breaking.

Dynamical susy breaking ((0,2) susy)

Let's apply that idea, to get a partial index for susy breaking.

Start with elliptic genus:

$$(-)^{r/2} q^{+(r-n)/12} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det E)^{+1/2} \wedge_{-1} E^* \bigotimes_{k=1,2,3,\cdots} S_{q^k}\left((TX)^{\mathbb{C}} \right) \bigotimes_{k=1,2,3,\cdots} \wedge_{-q^k}\left(E^{\mathbb{C}} \right) \Big)$$

Truncate to leading term:

$$(-)^{r/2} q^{+(r-n)/12} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det E)^{+1/2} \wedge_{-1} E^* \Big)$$

Can show $= (-)^{r/2} q^{+(r-n)/12} \begin{cases} 0 & r > n, \\ \int_X c_r(E) & r = n, \\ \cdots & r < n. \end{cases}$

Implications:

If rank=dim, susy breaking impossible unless $c_r(E)$ vanishes.

In a rank-increasing transition, danger of susy breaking.

Dynamical susy breaking **((0,2) susy)**

Let's apply these ideas to a def' of the $\mathbb{CP}^N$ model.

In (0,2) superspace, if omit the twisted chiral multiplet that corresponds to part of (2,2) vector multiplet, result is a sigma model on $\mathbb{CP}^N$ with gauge bundle $\oplus^{N+1} O(1)$, related to tangent bundle by

$$0 \to O \to \bigoplus^{N+1} O(1) \to T\mathbb{P}^N \to 0$$

This model is known to break (0,2) susy dynamically.

Here, $r = N+1$ so our primitive index vanishes, consistent with susy breaking.

Dynamical susy breaking ((0,2) susy)

Deformation of $\mathbb{CP}^N$ model, cont'd

The anomalous axial U(1) in this theory
has a nonanomalous $\mathbb{Z}_{N+1}$ subgroup,
so we can refine the index further by taking that into account.

$$\mathrm{Tr}\ (-)^{F_R}(-)^{F_L} y^A q^{L_0} \overline{q}^{\bar{L}_0}$$

$$= (-)^{r/2} q^{+(r-n)/12} y^{+r/2} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det E)^{+1/2} \wedge_{-1} (y^{-1} E^*) \Big) + \cdots$$

for y an (N+1)th root of unity.

Can show 1st term above $= (-)^{r/2} q^{+(r-n)/12} y^{+r/2} \sum_{s=0}^{n} (-y^{-1})^s \chi\left(\wedge^s E^* \right)$

$$\propto y^{+r/2}\left(1 - y^{-N-1}\right)$$

and so vanishes for y an (N+1)th root of unity.

Dynamical susy breaking ((0,2) susy)

Calabi-Yau's play a special role in this story.

The reason is that their NLSM's have extra nonanomalous symmetries — left and right $U(1)_R$'s — which can further refine the elliptic genus, hence provide extra necessary conditions for susy breaking.

As a result, dynamical susy breaking more difficult to achieve in Calabi-Yau theories.

Let's look at this in more detail…..

Dynamical susy breaking ((0,2) susy)

For Calabi-Yau's, with $c_1(E)=0$,
the elliptic genus of the NLSM can be refined:

$$\mathrm{Tr}\, (-)^{F_R} (-)^{F_L} y^A q^{L_0} \overline{q}^{\overline{L}_0}$$

$$= (-)^{r/2} q^{+(r-n)/12} y^{+r/2} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Big((\det E)^{+1/2} \wedge_{-1} (y^{-1} E^*) \Big) + \cdots$$

Leading term $= (-)^{r/2} q^{+(r-n)/12} y^{+r/2} \sum_{s=0}^{n} (-y^{-1})^s \chi\left(\wedge^s E^* \right)$

For CY 3-fold, $= -(-)^{r/2} q^{+(r-3)/12} y(1+y)(1-y)^{r-3} \frac{1}{2} \int_X c_3(E)$

for $r \geq 3$

Note for y=1, this vanishes for r>3, matching earlier results.

However, b/c general y allowed,
susy breaking not possible for r>3 unless $c_3(E)=0$.

Review of (0,2) multiplets

Next I'll describe some (0,2) gauge theories,
so let me here briefly review (0,2) susy multiplets:

(2,2) chiral: $(\phi, \psi_+, \psi_-, F)$

(0,2) chiral: (ϕ, ψ_+)

(0,2) Fermi: (ψ_-, F)

(2,2) vector: $(A_\mu, \sigma, \lambda_+, \lambda_-, D)$ (WZ gauge)

(0,2) vector: (A_μ, λ_-, D)

(0,2) twisted chiral: (σ, λ_+)

Gadde-Gukov-Putrov triality **((0,2) susy)**

This is a Seiberg-like duality,
that closes after 3 steps instead of 2.

Let's walk through it.

Start: U(k) gauge theory,

matter: n chirals Φ in fund' **k**, n>k,
A Fermi's in antifund' $\mathbf{k}^*$,
B chirals P in antifund' $\mathbf{k}^*$,
nB neutral Fermi's Γ,

$$W = \Gamma\Phi P$$

There is a potential gauge anomaly,
which can be cancelled if $B = 2k - n + A$.

Let's analyze the geometry….

Gadde-Gukov-Putrov triality ((0,2) susy)

U(k) gauge theory,

matter: n chirals Φ in fund' **k**, n>k,

A Fermi's in antifund' **k***,

B chirals P in antifund' **k***,

nB neutral Fermi's Γ,

$$W = \Gamma \Phi P$$

$r \gg 0:$	$\overset{r}{\longleftrightarrow}$	$r \ll 0:$
Space: $G(k,n)$		Space: $G(k,B)$
Bundle: $S^A \oplus (Q^*)^B$		Bundle: $(S^*)^A \oplus (Q^*)^n$

Can apply duality to either side....

Gadde-Gukov-Putrov triality ((0,2) susy)

$r \gg 0:$

Space: $G(k,n)$

Bundle: $S^A \oplus (Q^*)^B$

$\overset{r}{\longleftrightarrow}$

$r \ll 0:$

Space: $G(k,B)$

Bundle: $(S^*)^A \oplus (Q^*)^n$

Let's look at geometric equivalences, on LHS:

$$G(k,n) = G(n-k,n)$$
$$S_k = Q^*_{n-k}$$
$$Q^*_k = S_{n-k}$$

So:

Space: $G(k,n)$ = Space: $G(n-k,n)$

Bundle: $S^A \oplus (Q^*)^B$ = Bundle: $(Q^*)^A \oplus S^B$

which implies a statement about (0,2) gauge theories.

Gadde-Gukov-Putrov triality **((0,2) susy)**

In this fashion, we get a chain of dualities:

$$S^A \oplus (Q^*)^{2k+A-n} \to G(k,n) \quad \overset{r}{\cdots\cdots} \quad (S^*)^A \oplus (Q^*)^n \to G(k,2k+A-n)$$

$$\Big\updownarrow =$$

$$(Q^*)^A \oplus S^{2k+A-n} \to G(n-k,n) \quad \overset{r}{\cdots\cdots} \quad (Q^*)^n \oplus (S^*)^{2k+A-n} \to G(n-k,A)$$

$$\Big\updownarrow =$$

$$(S^*)^n \oplus Q^A \to G(A-n+k,2k+A-n) \quad \overset{r}{\cdots\cdots} \quad S^n \oplus Q^{2k+A-n} \to G(A-n+k,A)$$

$$\Big\updownarrow =$$

$$Q^n \oplus (S^*)^A \to G(k,2k+A-n) \quad \overset{r}{\cdots\cdots} \quad Q^{2k+A-n} \oplus (S^*)^A \to G(k,n)$$

But applying gauge bundle dualization duality,
last line = first line,
so there is a 3-step sequence.

Triality

Prelude to other examples: **((0,2) susy)**

U(2) representation conventions

Our next examples will involve gauge bundles defined by more general representations of U(2), so let me take just a moment to outline conventions.

Will describe an irrep of U(2) by $(a,b), \ a \geq b$

(0,0) = trivial

(1,0) = **2**

(0,-1) = $\mathbf{2}^*$

(1,-1) + (0,0) = ad

(a,a) = rep of det U(2)

dim (a,b) = a - b + 1, Cas_1(a,b) = a + b

Abelian/nonabelian dualities ((0,2) susy)

Let's build on our previous duality

$$G(2,4)[d_1, d_2, \cdots] = \mathbb{P}^5[2, d_1, d_2, \cdots]$$

by extending to heterotic cases.

Example:

Bundle $0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$

on the CY G(2,4)[4].

Described by U(2) gauge theory

4 chirals in fundamental

1 Fermi in (-4,-4) (hypersurface)

8 Fermi's in (1,1) (gauge bundle E)

1 chiral in (-2,-2) (gauge bundle E)

2 chirals in (-3,-3) (gauge bundle E)

plus superpotential

Abelian/nonabelian dualities ((0,2) susy)

Example:

Bundle $0 \to E \to \oplus^8 O(1,1) \to O(2,2) \oplus^2 O(3,3) \to 0$

on the CY G(2,4)[4].

Geometry predicts this is dual to

Bundle $0 \to E \to \oplus^8 O(1) \to O(2) \oplus^2 O(3) \to 0$

on the CY $\mathbb{P}^5[2,4]$

Checks:
- both satisfy anomaly cancellation
- elliptic genera match

To make the duality work, we used the fact that reps defining bundle all lives in det U(2)….

Abelian/nonabelian dualities ((0,2) susy)

Another example:

Bundle $0 \to E \to O(1,0) \oplus^5 O(2,1) \to O(3,1) \oplus^2 O(3,2) \to 0$

on the CY G(2,4)[4].

- Satisfies anomaly cancellation.
- No idea if there's an abelian dual on $\mathbb{P}^5[2,4]$.

Pfaffians ((0,2) susy)

It's also possible to build (0,2) models on Pfaffians.

Deformations off (2,2) locus:

PAX:

$$W = \mathrm{tr}\left(\Lambda_P A(\Phi) X + P A(\Phi) \Lambda_X + P \left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} + G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha X \right)$$

PAXY:

$$W = \mathrm{tr}\left(\Lambda_{\tilde{P}} A(\Phi) + \tilde{P} \left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} + G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha + \Lambda_{\tilde{P}} \tilde{X} \tilde{Y} + \tilde{P} \Lambda_{\tilde{X}} \tilde{Y} + \tilde{P} \tilde{X} \Lambda_{\tilde{Y}} \right)$$

In both cases, $G_\alpha(\Phi)$ (satisfying certain conditions) define deformations off (2,2) locus.

These (0,2) PAX, PAXY models are related by Seiberg / B-C-like gauge duality.

Pfaffians ((0,2) susy)

More (0,2) models on Pfaffians.

Example: PAX model, Pfaffian $\{\mathrm{rank}\, A \leq 2\} \subset \mathbb{P}^7$

Bundle

$$0 \rightarrow E \rightarrow \oplus^5 O((0,0)_{-1}) \oplus^2 O((2,2)_0) \rightarrow \oplus^2 O((2,2)_{-1}) \oplus O((1,-1)_{-1}) \rightarrow 0$$

Described by

U(2)xU(1) gauge theory

4 chirals in $(0,-1)_0$ 4 Fermi's in $(1,0)_{-1}$

8 chirals in $(0,0)_{+1}$ (defining $\mathbb{P}^7$)

5 Fermi's in $(0,0)_{-1}$ 2 chirals in dual of $(2,2)_{-1}$

2 Fermi's in $(2,2)_0$ 1 chiral in dual of $(1,-1)_{-1}$

defines E

+ superpotential

- anomaly free
- dual not known

Possible obstructions to duality **((0,2) susy)**

So far we have discussed dualities in two-dimensional gauge theories with (anti)fundamentals.

What about more general matter representations?

From a geometric perspective,
our dualities have all boiled down to exchanging

$$G(k,n) \leftrightarrow G(n-k,n)$$

$$(S \to G(k,n)) \leftrightarrow (Q^* \to G(n-k,n))$$

What would be the analogue for more general matter reps ?

Possible obstructions to duality **((0,2) susy)**

Geometrically, to dualize more general rep's,
must construct resolutions of corresponding bundles.

Example: U(k) gauge theory, Fermi's in $\wedge^2 \overline{\mathbf{k}}$

(plus fundamental chirals….)

Pertinent bundle: $\wedge^2 S \to G(k,n)$

Dual: $\wedge^2 Q^* \to G(n-k,n)$

Q* cannot be realized directly, only indirectly w/ sequence.

To realize $\wedge^2 Q^*$ use

$$0 \to \wedge^2 Q^* \to \wedge^2 O^n \to S^* \otimes O^n \to \mathrm{Sym}^2 S^* \to 0$$

Potential Problem: how to realize that sequence physically.

Possible obstructions to duality ((0,2) susy)

Example, cont'd

To realize dual $\wedge^2 Q^*$ use

$$0 \to \wedge^2 Q^* \to \wedge^2 O^n \to S^* \otimes O^n \to \mathrm{Sym}^2 S^* \to 0$$

In open strings, this is easy, and implicitly I'm describing a prescription for dualizing arbitrary matter reps on boundaries.

But in (0,2) , we only know how to realize 3-term sequences.
To realize the dual above,
I'd need to realize a 4-term sequence,
and no one knows how to do that in (0,2).

Possible obstructions to duality ((0,2) susy)

This analysis suggests that it may be difficult to find a Seiberg-like gauge theoretic dual to a (0,2) theory with random matter representations.

Basic obstruction: we only know how to realize 3-term sequences in (0,2); we'd need to realize longer sequences.

However, for open string boundaries, no such obstruction, this gives instead a prescription for construction of duals.

Summary: duality from geometry

- (2,2) theories:
    - review $\mathbb{CP}^N$ model, hypersurfaces, Grassmannian
    - Theories w/ both fundamentals and antifundamentals - Benini-Cremonesi duality, and first application of geometry to derive gauge theory dualities
    - Abelian/nonabelian duality: G(2,4) vs $\mathbb{P}^5[2]$
    - Pfaffian constructions and more dualities
- (0,2) theories:
    - `gauge bundle dualization duality', dynamical susy breaking
    - Gadde-Gukov-Putrov triality via geometry,
    - abelian/nonabelian examples, Pfaffian examples
- Obstructions to some dualities