Examples of homological projective duality in physics

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T Pantev, ES, hepth/0502027, 0502044, 0502053
S Hellerman, A Henriques, T Pantev, ES, M Ando, hepth/0606034
R Donagi, ES, arxiv: 0704.1761
A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855
N Addington, E Segal, ES, arXiv: 1211.2446
ES, arXiv: 1212.5322

In this talk, I'm going to describe how some examples of Kuznetov's homological projective duality (hpd) (for complete intersections of quadrics) are realized physically, as phases of abelian GLSM's.

GLSM = `gauged linear sigma model'

These are the bread-and-butter tools used by physicists to describe families of spaces and related aspects of string compactifications.

Hpd taught us a great deal about GLSM's, as we'll see today.

WARNING: physics talk

What did hpd teach us?

Prior to ~ 2006, it was (falsely) believed that:

* GLSM's could only describe global complete intersections,
* which could only arise physically as critical locus of a superpotential, and
* GLSM Kahler `phases' are all birational to one another

The papers

Hori-Tong hep-th/0609032, Hellerman et al hep-th/0606034,
Donagi-ES 0704.1761, Caldararu et al 0709.3855, ...

provided counterexamples to each statement above, all special cases of hpd.

Since ~ 2006, many of us have come to believe that all geometries realized by GLSM's are related by hpd.

This now seems to be (close to) proven:
Ballard, Favero, Katzarkov, 1203.6643 & to appear

Today, I'll discuss some (old) examples of such exotic GLSM's, and some (newer) analyses.

Matrix factorizations will play an important role.

Outline:

* physical realization of hpd
  as phases of abelian GLSM, i.e. $V // \mathbf{C}^*$

* detour through physics of $\mathbf{Z}_2$ gerbes

* some phases will be CFT's for nc resolutions

* D-brane probes of nc res'ns (w/ Nick, Ed)

* analogues of GW invt's for nc res'ns

Which GLSM's will I describe?

You've heard (E Segal's talk, J Knapp's talk) about how Hori-Tong describe an exotic GLSM interpolating between two smooth non-birational spaces with the same mirror.

Singular examples of the same form also exist, and also have (exotic) GLSM descriptions.

My goal today is to describe examples of this form (as further examples of hpd).
In fact, the singularities are nc-resolved, so we'll see nc res'ns in physics, and some of their properties.

Prototype for the exotic GLSM's I'll discuss today:

A complete intersection of k quadrics in $\mathbf{P}^n$,

$$\{Q_1 = \cdots = Q_k = 0\}$$

is hpd to

a (nc resolution of a) branched double cover of $\mathbf{P}^{k-1}$, branched over the locus

$$\{\det A = 0\}$$

where $$\sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

I'll describe how this arises in physics.

We'll begin with the easiest possible example:

the GLSM for $\mathbf{P}^3[2,2]$ (=T^2):

GLSM's are families of 2d gauge theories that RG flow to families of CFT's.

In this case:

$r \gg 0$ one-parameter Kahler moduli space $r \ll 0$

$\longleftrightarrow$

NLSM on $\mathbf{P}^3[2,2]$ r LG point = branched double cover

GLSM for $\mathbf{P}^3[2,2]$ ($=T^2$):

Looks like a GIT quotient $V//\mathbf{C}^\times$

where $V = \mathbf{C}^4 \oplus \mathbf{C}^2$

(ϕ_i) weight 1

(p_a) weight -2

Idea: ϕ_i ~ homogeneous coordinates on $\mathbf{P}^3$

p_a count quadric hypersurfaces

and there's a superpotential:

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

We physicists speak of Kahler phases, which are the chambers in VGIT.

One description of result is as LG models:

$$V//\mathbf{C}^\times \; = \; \left(\mathbf{C}^4_{(1)} \oplus \mathbf{C}^2_{(-2)} \right) //\mathbf{C}^\times$$

$V//_+\mathbf{C}^\times$: $\xleftrightarrow[\text{r}]{\text{Kahler}}$ $V//_-\mathbf{C}^\times$:

LG model on Tot($O(-2)^2$ --> $\mathbf{P}^3$)

LG model on Tot($O(-1)^4$ --> $\mathbf{P}^1_{[2,2]}$)

both with

$$W \; = \; \sum_a p_a Q_a(\phi) \; = \; \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

All phases of GLSM's can be described as LG models, and ultimately this is why work such as BFK relates GLSM phases to hpd, via matrix factorizations.

That said, we usually don't stop at LG.
We can often use the `renormalization group' to give an alternative description of the low-energy dynamics of the theory
(which for some purposes may be more complicated than a LG model).

One phase:

$$V//_{+}\mathbf{C}^{\times}:$$

LG model on

Tot(O(-2)^2 --> **P**^{3})

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p)\phi_j$$

$$\Big\downarrow \text{RG}$$

NLSM on {Q_1 = Q_2 = 0} in **P**^{3}

= T^2

(This is the easy example after all.)

The other phase is more exciting:

$$V//_{-}\mathbf{C}^{\times}:$$

LG model on

Tot(O(-1)^4 --> $\mathbf{P}^1_{[2,2]}$)

$$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

Here, the p's not all zero (describing $\mathbf{P}^1$)

so W looks like it's giving a mass to the ϕ_i

which would mean this RG flows to

NLSM on $\mathbf{P}^1$

Can't be right! ....

The correct analysis of the //- phase is more subtle.

One subtlety is that the ϕ_i are not massive everywhere.

Write $$W = \sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p)\phi_j$$

then they are only massive away from the locus

$$\{\det A = 0\} \subset \mathbf{P}^1$$

But that just makes things more confusing....

A more important subtlety is the fact that the p's are coordinates on the $\mathbf{Z}_2$ gerbe $\mathbf{P}^1_{[2,2]}$, so over most of the $\mathbf{P}^1$ of p vevs, there is a trivially-acting $\mathbf{Z}_2$.

Physics sees this as a double cover.

Let's quickly review how this works....

Strings on gerbes:

Present a (smooth DM) stack as [X/H].

String on stack = H-gauged sigma model on X.

(presentation-dependence washed out w/ renormalization group)

If a subgroup G acts trivially, then this is a G-gerbe.

Physics questions:

* Does physics know about G?

(Yes, via nonperturbative effects -- Adams, Plesser, Distler.)

* The result violates cluster decomposition;
why consistent?

(B/c equiv to a string on a disjoint union....)

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

gerbe

We now believe, for (2,2) CFT's,

disjoint union of spaces

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G-\mathrm{gerbe\ on\ }Y) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (Y,B)\right)$$

$$(Y \;=\; [X/K])$$

where the B field is determined by the image of

$$H^2(Y, Z(G)) \;\stackrel{Z(G)\rightarrow U(1)}{\longrightarrow}\; H^2(Y, U(1))$$

Quick consistency check:

A sheaf on a banded G-gerbe
is the same thing as
a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

This implies a decomposition of D-branes (~ sheaves),
which is precisely consistent with the decomposition
conjecture.

Another quick consistency check:

Prediction:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K\right]$

and this has been checked in

H-H Tseng, Y Jiang, et al,
0812.4477, 0905.2258, 0907.2087, 0912.3580, 1001.0435, 1004.1376, ...

GLSM's

Let's now return to our analysis of GLSM's.

Example: $\mathbf{P}^3[2,2]$

Superpotential: $$\sum_a p_a Q_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

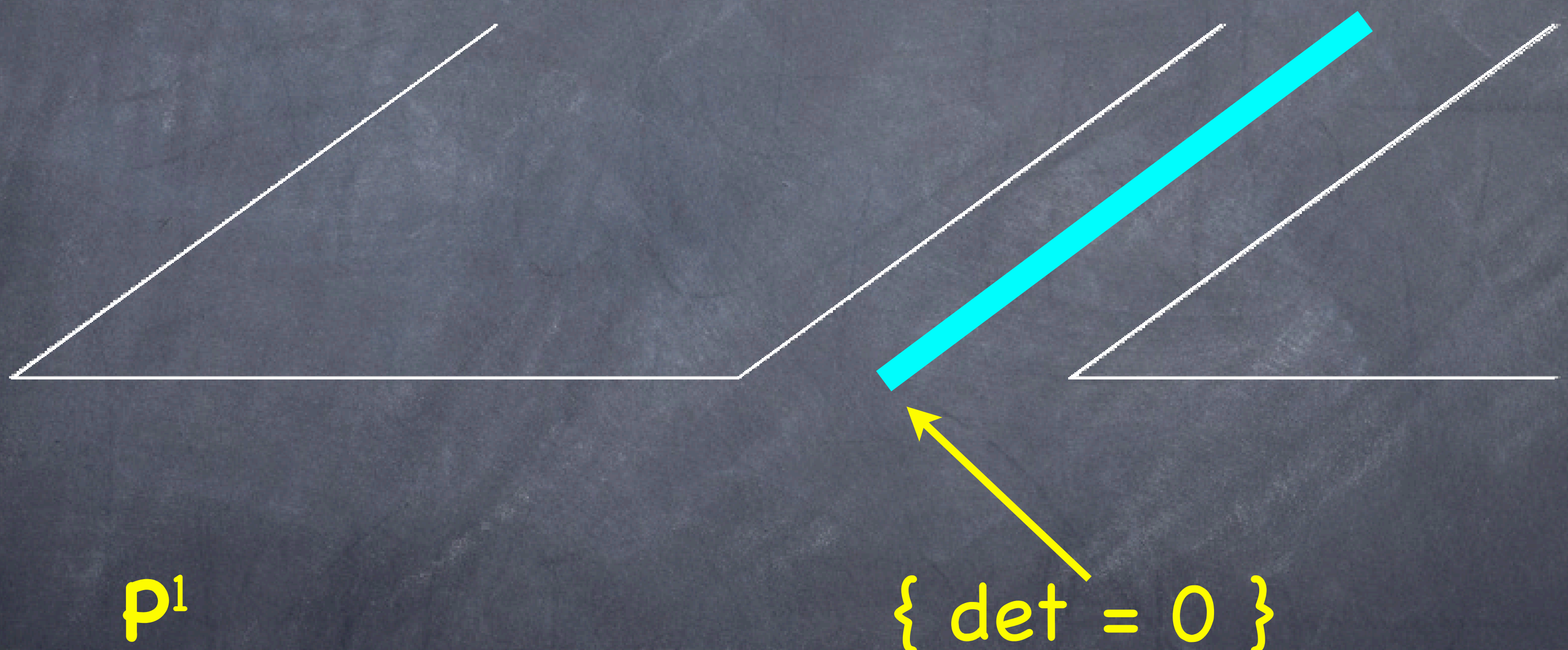
$V//_{-}\mathbf{C}^\times$:

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

* leaves just the p fields, of charge -2

* $\mathbf{Z}_2$ gerbe, hence double cover

$V//_{-}\mathbf{C}^{\times}$:



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{P}^1$....

$V//_{-}\mathbf{C}^{\times}$:

Double cover

$\mathbf{P}^1$

Berry phase

Result: branched double cover of $\mathbf{P}^1$

So far:

The GLSM realizes:

$//_{+}$:

$\mathbf{P}^3[2,2]$
$= T^2$

$\xleftrightarrow{\text{Kahler}}$

$//_{-}$:

branched double cover
of $\mathbf{P}^1$,
over deg 4 locus $= T^2$

where RHS realized at $//_{-}$ via
local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07;
A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

* novel realization of geometry
(via nonperturbative effects)

Next simplest example:

GLSM for $\mathbf{P}^5[2,2,2]$ = K3

$$V//\mathbf{C}^\times = \left(\mathbf{C}^6_{(1)} \oplus \mathbf{C}^3_{(-2)}\right)//\mathbf{C}^\times$$

$//_+$: LG on

Tot($O(-2)^3$ --> $\mathbf{P}^5$)

$\downarrow$ RG

NLSM on $\mathbf{P}^5[2,2,2]$

$//_-$: LG on

Tot($O(-1)^6$ --> $\mathbf{P}^2_{[2,2,2]}$)

$\downarrow$ RG

NLSM on branched 2-cover of $\mathbf{P}^2$, branched over deg 6 locus

$$\text{K3} \xleftrightarrow{\text{Kahler}} \text{K3}$$

(no surprise)

So far:

* easy low-dimensional examples of hpd

* RG endpt at $//_-$ is NLSM on geometry,
but **not** as the critical locus of a superpotential.

For physics, this is already neat, but there are much more interesting examples yet....

The next example in the pattern is more interesting.

GLSM for $\mathbf{CP}^7[2,2,2,2]$ = CY 3-fold

At $//_{-}$,

naively, same analysis says

get branched double cover of $\mathbf{CP}^3$,

branched over degree 8 locus.

-- another CY

(Clemens' octic double solid)

Here, **different** CY's;

not even birational

However, the analysis that worked well in lower dimensions, hits a snag here:

The branched double cover is singular,
but the GLSM is smooth at those singularities.

Hence, we're not precisely getting a branched double cover; instead, we're getting something slightly different.

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover, as hpd implies in this case.

Check that we are seeing K's noncomm' resolution:

Here, K's noncomm' res'n is defined by $(\mathbf{P}^3, B)$
where B is the sheaf of even parts of Clifford
algebras associated with the universal quadric over $\mathbf{P}^3$
defined by the GLSM superpotential.

B is analogous to the structure sheaf;
other sheaves are B-modules.

Physics?......

Physics picture of K's noncomm' space:

Matrix factorization for a quadratic superpotential:
even though the bulk theory is massive, one still has
D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$,
gives sheaves of Clifford algebras (determined by the
universal quadric / GLSM superpotential)
and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Summary so far:

This GLSM realizes:

$\mathbf{CP}^7[2,2,2,2] \xleftrightarrow{\text{Kahler}}$ nc res'n of branched double cover of $\mathbf{CP}^3$

where RHS realized at LG point via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Physical realization of a nc resolution

Geometry realized nonperturbatively

More examples:

CI of
n quadrics in $\mathbf{P}^{2n-1}$

$\xleftrightarrow{\text{Kahler}}$

(possible nc res'n of)
branched double
cover of $\mathbf{P}^{n-1}$,
branched over deg 2n
locus

Both sides CY

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$$\xleftrightarrow{\text{Kahler}}$$

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved	Negatively curved

r flows: $\dashrightarrow$ $\dashrightarrow$ $\dashrightarrow$

D-brane probes of nc resolutions

Let's now return to the branched double covers and nc resolutions thereof.

I'll outline next some work on D-brane probes of those nc resolutions.

(w/ N Addington, E Segal)

Idea: `D-brane probe' = roving skyscraper sheaf; by studying spaces of such, can sometimes gain insight into certain abstract CFT's.

Setup:

To study D-brane probes at the LG points,
we'll RG flow the GLSM a little bit,
to build an `intermediate' Landau-Ginzburg model.
(D-brane probes = certain matrix fact'ns in LG)

$\mathbf{P}^n[2,2,..,2]$ (k intersections) is hpd to

LG on $\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]}\right)$

with superpotential

$$W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$$

Our D-brane probes of this Landau-Ginzburg theory will consist of (sheafy) matrix factorizations:

$$\begin{tikzcd} \mathcal{E}_0 \arrow[d, bend right, "P"'] \\ \mathcal{E}_1 \arrow[u, bend right, "Q"'] \end{tikzcd}$$

where

$$P \circ Q, Q \circ P = W \, \mathrm{End}$$

up to a constant shift

(equivariant w.r.t. $\mathbf{C}^*{}_R$)

In a NLSM, a D-brane probe is a skyscraper sheaf.
Here in LG, idea is that we want MF's that RG flow to skyscraper sheaves.

That said, we want to probe nc res'ns (abstract CFT's), for which this description is a bit too simple.

First pass at a possible D-brane probe:
(wrong, but usefully wrong)

$$\begin{tikzcd} \mathcal{O}_x \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

where x is any point.

Since $W|_x$ is constant, $0 = W|_x$ up to a const shift, hence skyscraper sheaves define MF's.

This has the right `flavor' to be pointlike, but we're going to need a more systematic def'n....

When is a matrix factorization `pointlike'?

One necessary condition:
contractible off a pointlike locus.

Example: X = $\mathbf{C}^2$ $W = xy$

$$\begin{tikzcd} \mathcal{O} \arrow[d, bend right, "x"'] \\ \mathcal{O} \arrow[u, bend right, "y"'] \end{tikzcd}$$

is contractible on $\{y \neq 0\}$:

There exist maps s, t s.t. $1 = ys + tx$

namely $t = 0$, $s = y^{-1}$

Sim'ly, contractible on $\{x \neq 0\}$

hence support lies on $\{x = y = 0\}$

When is a matrix factorization `pointlike'?

Demanding contractible off a point,
gives set-theoretic pointlike support,
but to distinguish fat points, need more.

To do this, compute Ext groups.

Say a matrix factorization is `homologically pointlike'
if has same Ext groups as a skyscraper sheaf:

$$\dim \mathrm{Ext}^k_{\mathrm{MF}}(\mathcal{E}, \mathcal{E}) \; = \; \left(\begin{array}{c} n \\ k \end{array} \right)$$

We're interested in Landau-Ginzburg models on

$$\mathrm{Tot}\left(\mathcal{O}(-1/2)^{n+1} \longrightarrow \mathbf{P}^{k-1}_{[2,2,\cdots,2]}\right)$$

with superpotential $W = \sum_a p_a Q_a(\phi) = \sum_{i,j} \phi_i A^{ij}(p) \phi_j$

For these theories, it can be shown that the \`pointlike' matrix factorizations are of the form

$$\begin{tikzcd} \mathcal{O}_U \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

where U is an isotropic subspace of a single fiber.

Let's look at some examples, fiberwise, to understand what sorts of results these D-brane probes will give.

Example: Fiber $[\mathbf{C}^2/\mathbf{Z}_2]$, $\quad W|_F = xy$

Two distinct matrix factorizations:

$$\begin{tikzcd} \mathcal{O}_{\{y=0\}} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \sim \begin{tikzcd} \mathcal{O} \arrow[d, bend right, "x"'] \\ \mathcal{O}(1/2) \arrow[u, bend right, "y"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{O}_{\{x=0\}} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \sim \begin{tikzcd} \mathcal{O} \arrow[d, bend right, "y"'] \\ \mathcal{O}(1/2) \arrow[u, bend right, "x"'] \end{tikzcd}$$

D-brane probes see 2 pts over base => double cover

Example: Family $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}_\alpha$

$$W = x^2 - \alpha^2 y^2$$

Find branch locus:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha^2 \end{bmatrix} \qquad \det A = -\alpha^2$$

When $\alpha \neq 0$,

there are 2 distinct matrix factorizations:

$$(\mathcal{O}_{\{x=\alpha y\}} \rightleftharpoons 0), \quad (\mathcal{O}_{\{x=-\alpha y\}} \rightleftharpoons 0)$$

Over the branch locus $\{\alpha = 0\}$, there is only one.

=> **branched** double cover

Global issues:

Over each point of the base, we've picked an isotropic subspace U of the fibers, to define our ptlike MF's.

These choices can only be glued together up to an overall $\mathbf{C}^*$ automorphism, so globally there is a $\mathbf{C}^*$ gerbe.

Physically this ambiguity corresponds to gauge transformation of the B field; hence, characteristic class of the B field should match that of the $\mathbf{C}^*$ gerbe.

So far:

When the LG model flows in the IR to a smooth branched double cover,
D-brane probes see that branched double cover
(and even the cohomology class of the B field).

Case of an nc resolution:

Toy model: $$[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$$

$$W \;=\; ax^2 \;+\; bxy \;+\; cy^2$$

Branch locus:

$$A = \left[\begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right] \qquad \det A \;\propto b^2 - 4ac \;\equiv\; \Delta$$

Generically on $\mathbf{C}^3$, have 2 MF's, quasi-iso to

$$\begin{tikzcd} \mathcal{O}_F \arrow[d, bend right, "2ax+by+\sqrt{\Delta}y"'] \\ \mathcal{O}_F(1/2) \arrow[u, bend right, "2ax+by-\sqrt{\Delta}y"'] \end{tikzcd} \quad , \quad \begin{tikzcd} \mathcal{O}_F \arrow[d, bend right, "2ax+by-\sqrt{\Delta}y"'] \\ \mathcal{O}_F(1/2) \arrow[u, bend right, "2ax+by+\sqrt{\Delta}y"'] \end{tikzcd}$$

Gen'ly on branch locus, become a single MF,
but something special happens at $\{a=b=c=0\}$....

Case of an nc resolution, cont'd:

Toy model: $[\mathbf{C}^2/\mathbf{Z}_2]_{x,y} \times \mathbf{C}^3_{a,b,c}$

$$W \;=\; ax^2 \;+\; bxy \;+\; cy^2$$

At the point $\{a = b = c = 0\}$

there are 2 families of ptlike MF's:

$$\begin{tikzcd} \mathcal{O}_F \arrow[d, bend right, "0"'] \\ \mathcal{O}_F(1/2) \arrow[u, bend right, "\phi"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{O}_F \arrow[d, bend right, "\phi"'] \\ \mathcal{O}_F(1/2) \arrow[u, bend right, "0"'] \end{tikzcd}$$

where ϕ is any linear comb' of x, y (up to scale)

* 2 small resolutions (stability picks one)

I'm glossing over details,
but the take-away point is that for
nc resolutions
(naively, singular branched double covers),
D-brane probes see small resolutions.

Often these small resolutions will be non-Kahler,
and hence not Calabi-Yau.

(closed string geometry $\neq$ probe geometry;
also true in eg orbifolds)

So far:

* examples of hpd realizing singular non-birat'l pairs, in physics

* D-brane probes of nc resolutions appearing above

Next:

* predictions for analogues of GW inv'ts for those nc resolutions,
by applying GLSM localization techniques of Jockers et al

GW inv'ts of nc res'ns

Basic idea of Jockers, Morrison, Romo et al, 1208.6244:

Partition function of GLSM on S^2 can be computed exactly, for example:

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \left(\frac{\Gamma(\mathsf{q} - i\sigma - m/2)}{\Gamma(1 - \mathsf{q} + i\sigma - m/2)} \right)^8 \left(\frac{\Gamma(1 - 2\mathsf{q} + 2i\sigma + 2m/2)}{\Gamma(2\mathsf{q} - 2i\sigma + 2m/2)} \right)^4$$

(Benini, Cremonesi, 1206.2356; Doroud et al, 1206.2606)

After normalization, this becomes exp(-K):

$$\begin{aligned} \frac{Z}{\mathrm{stuff}} &= \exp(-K) \\ &= -\frac{i}{6}\kappa(t-\bar{t})^3 + \frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n \left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right) \\ &\quad - \frac{i}{(2\pi i)^2}\sum_n N_n \left(\mathrm{Li}_2(q^n) + \mathrm{Li}_2(\overline{q}^n)\right) n(t-\bar{t}) \end{aligned}$$

... and then read off the N_n's

Let's work through this in more detail.

For a U(1) gauge theory,

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \prod_i Z_{\Phi,i}$$

where

$$Z_\Phi = \frac{\Gamma(\mathrm{Q}/2 - Q(i\sigma + m/2))}{\Gamma(1 - \mathrm{Q}/2 + Q(i\sigma - m/2))}$$

Q = gauge U(1) charge

Q defines hol' Killing vector that combines with $U(1)_R$

$\mathcal{Q}$ vs Q :

To explain the difference, it's helpful to look at a NLSM lagrangian on S^2:

$$g_{i\bar{\jmath}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{\jmath}} - ig_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\gamma^m\mathcal{D}_m\psi^i + g_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} - F^i(\frac{1}{2}g_{i\bar{\jmath},\bar{k}}\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{k}} - W_i)$$

$$-\bar{F}^{\bar{\imath}}(\frac{1}{2}g_{j\bar{\imath},k}\psi^j\psi^k - \bar{W}_{\bar{\imath}}) - \frac{1}{2}W_{ij}\psi^i\psi^j - \frac{1}{2}\bar{W}_{\bar{\imath}\bar{\jmath}}\bar{\psi}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}} + \frac{1}{4}g_{i\bar{\jmath},k\bar{l}}\psi^i\psi^k\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{l}}$$

$$\underline{-\frac{1}{4r^2}g_{i\bar{\jmath}}X^iX^{\bar{\jmath}} + \frac{i}{4r^2}K_iX^i - \frac{i}{4r^2}K_{\bar{\imath}}X^{\bar{\imath}} - \frac{i}{2r}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}\nabla_jX^i\psi^j}$$

(B. Jia, 2013, to appear)

Specific to S^2

r = radius of S^2

X = holomorphic Killing vector (defines Q of previous slide)

Constraints: $2W = -iX^i\partial_iW$

so if $W \neq 0$ then $X \neq 0$ -- important for GLSM

As a warm-up,
let's outline the GW computation at $//_{+}$,
on $\mathbf{P}^7[2,2,2,2]$,
where the answer is known,
and then afterwards we'll repeat at $//_{-}$,
where the nc res'n lives.

For the GLSM for $\mathbf{P}^7[2,2,2,2]$:

$$Z = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \underbrace{\left(\frac{\Gamma(\mathsf{q} - i\sigma - m/2)}{\Gamma(1 - \mathsf{q} + i\sigma - m/2)} \right)^8}_{\substack{\Phi, Q = 1 \\ \mathsf{Q} = 2\mathsf{q}}} \underbrace{\left(\frac{\Gamma(1 - 2\mathsf{q} + 2i\sigma + 2m/2)}{\Gamma(2\mathsf{q} - 2i\sigma + 2m/2)} \right)^4}_{\substack{P, Q = -2 \\ \mathsf{Q} = 2 - 4\mathsf{q}}}$$

For $//_{+}$, r >> 0, so close contour on left.

Define $$f(\epsilon) = \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 2k - 2\epsilon)^4}{\Gamma(1 + k - \epsilon)^8} \right|^2$$

then

$$Z = \oint \frac{d\epsilon}{2\pi i} (z\overline{z})^{\mathsf{q} - \epsilon} \pi^4 \frac{(\sin 2\pi\epsilon)^4}{(\sin \pi\epsilon)^8} f(\epsilon)$$

$$= \frac{8}{3} (z\overline{z})^{\mathsf{q}} \left[-\ln(z\overline{z})^3 f(0) - 8\pi^2 f'(0) + 3\ln(z\overline{z})^2 f'(0) + \ln(z\overline{z}) \left(8\pi^2 f(0) - 3 f''(0) \right) + f^{(3)}(0) \right]$$

$\mathbf{P}^7$[2,2,2,2], cont'd

In principle,

$$
\begin{aligned}
Z \;\propto\; & \exp(-K) \\
= & -\frac{i}{6}\kappa(t-\bar{t})^3 + \frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n\left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right) \\
& - \frac{i}{(2\pi i)^2}\sum_n N_n\left(\mathrm{Li}_2(q^n) + \mathrm{Li}_2(\overline{q}^n)\right) n(t-\bar{t})
\end{aligned}
$$

We know $\kappa = 2^4 = 16$ and

$$
t \;=\; \frac{\ln z}{2\pi i} + (\text{terms invariant under } z \mapsto z e^{2\pi i})
$$

so we can solve for the normalization of Z, then plug in and compute the N_n's.

$\mathbf{P}^7[2,2,2,2]$, cont'd

Details:

$$Z = \frac{8}{3}(z\overline{z})^{q}\left[-\ln(z\overline{z})^3 f(0) \;-\; 8\pi^2 f'(0) \;+\; 3\ln(z\overline{z})^2 f'(0) \right.$$
$$\left. +\ \ln(z\overline{z})\left(8\pi^2 f(0) - 3f''(0)\right) \;+\; f^{(3)}(0)\right]$$

also $$\propto -\frac{i}{6}\kappa(t-\overline{t})^3 \;+\; \frac{\zeta(3)}{4\pi^3}\chi(X) \;+\; \frac{2i}{(2\pi i)^3}\sum_n N_n\left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right)$$
$$-\frac{i}{(2\pi i)^2}\sum_n N_n\left(\mathrm{Li}_2(q^n) \;+\; \mathrm{Li}_2(\overline{q}^n)\right) n(t-\overline{t})$$

Expect $$t-\overline{t} = \frac{\ln(z\overline{z})}{2\pi i} \;+\; \frac{\Delta(z)+\overline{\Delta}(\overline{z})}{2\pi i}$$ for some $\Delta(z)$

so we use the $\ln(z\, z^*)^3$ term to normalize.

$\mathbf{P}^7[2,2,2,2]$, cont'd

After normalization,

$$e^{-K} \;=\; -i\frac{16}{6}\left[\frac{\ln(z\overline{z})^3}{(2\pi i)^3} \;+\; \frac{8\pi^2}{(2\pi i)^3}\frac{f'(0)}{f(0)} \;-\; \frac{3}{2\pi i}\frac{\ln(z\overline{z})^2}{(2\pi i)^2}\frac{f'(0)}{f(0)}\right.$$
$$\left. -\;\frac{\ln(z\overline{z})}{2\pi i}\left(\frac{8\pi^2}{(2\pi i)^2} \;-\; \frac{3}{(2\pi i)^2}\frac{f''(0)}{f(0)}\right) \;-\; \frac{1}{(2\pi i)^3}\frac{f^{(3)}(0)}{f(0)}\right]$$

also $$= -\frac{i}{6}\kappa(t-\overline{t})^3 \;+\; \frac{\zeta(3)}{4\pi^3}\chi(X) \;+\; \frac{2i}{(2\pi i)^3}\sum_n N_n\left(\mathrm{Li}_3(q^n) + \mathrm{Li}_3(\overline{q}^n)\right)$$
$$-\;\frac{i}{(2\pi i)^2}\sum_n N_n\left(\mathrm{Li}_2(q^n) \;+\; \mathrm{Li}_2(\overline{q}^n)\right)n(t-\overline{t})$$

Expect $\quad t-\overline{t} = \dfrac{\ln(z\overline{z})}{2\pi i} \;+\; \dfrac{\Delta(z)+\overline{\Delta}(\overline{z})}{2\pi i}\quad$ for some $\Delta(z)$

so from ln(z z*)² term, $\quad \Delta + \overline{\Delta} \;=\; -\;\dfrac{\partial}{\partial\epsilon}\ln f(\epsilon)\Big|_{\epsilon=0}$

$\mathbf{P}^7[2,2,2,2]$, cont'd

So far,

$$q = \exp(2\pi i t) = z e^{2\pi i C}\left(1 + 64z + 7072z^2 + 991232z^3 + 158784976z^4 + \cdots\right)$$

Invert:

$$z \;=\; q e^{-2\pi i C} \;-\; 64 q^2 e^{-4\pi i C} \;+\; 1120 q^3 e^{-6\pi i C} \;-\; 38912 q^4 e^{-8\pi i C} \;+\; \cdots$$

Plug into remaining equations:

$$-\frac{i}{(2\pi i)^2}\sum_n n N_n \left(\mathrm{Li}_2(q^n) \;+\; \mathrm{Li}_2(\overline{q}^n)\right) = -i\frac{16}{6}\frac{1}{(2\pi i)^2}\left[3\left(\frac{\partial}{\partial\epsilon}\right)^2 \ln f(\epsilon)\right|_{\epsilon=0} \;-\; 8\pi^2$$

$$\frac{\zeta(3)}{4\pi^3}\chi(X) + \frac{2i}{(2\pi i)^3}\sum_n N_n\left(\mathrm{Li}_3(q^n) \;+\; \mathrm{Li}_3(\overline{q}^n)\right) = i\frac{16}{6}\frac{1}{(2\pi i)^3}\left(\frac{\partial}{\partial\epsilon}\right)^3 \ln f(\epsilon)\Big|_{\epsilon=0}$$

-- 2 equ'ns for the genus 0 GW inv'ts

Result for $\mathbf{P}^7[2,2,2,2]$:

n	N
1	512
2	9728
3	416256
4	25703936
5	1957983744
6	170535923200

matches Hosono et al, hep-th/9406055

Now, let's consider the opposite limit, $//_-$, where r << 0.

In order for the previous analysis to work, we needed

$$t \;=\; \frac{\ln z}{2\pi i} \;+\; (\text{terms invariant under } z \mapsto z e^{2\pi i})$$

-- characteristic of large-radius

-- don't typically expect to be true of LG models (so, computing Fan-Jarvis-Ruan using these methods will be more obscure),

but the present case is close enough to geometry that this should work, and indeed, one can extract integers.

Applying the same method, one finds

Compare GW inv'ts of smooth br' double cover

n	N	N
1	64	29504
2	1216	128834192
3	52032	1423720545880
4	3212992	23193056024793312

(Morrison, in "Mirror Symmetry I")

Interpretation?

We've found a set of integers, that play the same role as GW inv'ts, but for a nc res'n.

I don't know of a notion of GW theory for nc res'ns, but there's work on DT invt's

(see e.g. Szendroi, Nagao, Nakajima, Toda)

Perhaps some version of GW/DT can be used to define a set of integers that ought to be GW inv'ts?

Summary:

* physical realization of hpd

CI quadrics $\longleftrightarrow$ (nc res'n of) branched double cover

as phases of abelian GLSM, i.e. $V // \mathbf{C}^*$

* detour through physics of gerbes

* D-brane probes of nc res'ns

* Briefly: analogues of GW invt's for nc res'ns