

**PHYSICS 4455 — QUANTUM MECHANICS**  
**Problem Set 10 — due 12/2/2005, 5pm in Dr. Schmittmann's office.**

1. **Angular momentum and its commutators – part of every physicist's rate of passage!**

Confirm Liboff Eq. (9.8) and (9.13).

Do Liboff Problems 9.4 (takes just a few minutes), 9.7 (all of it!) and 9.11 (only parts a and c). Show every step explicitly.

2. **The isotropic harmonic oscillator in two dimensions.**

A non-relativistic, structureless particle of mass  $M$  is confined by a harmonic oscillator potential of the form

$$V(x,y) = \frac{1}{2}M\omega^2(x^2 + y^2)$$

The following sequence of steps takes you through the solution of the associated, two-dimensional Schrödinger equation. The notation follows the class notes.

(i) Write down the stationary Schrödinger equation for this problem, using two-dimensional polar coordinates  $\rho$  and  $\phi$ . Show that  $[\hat{H}, \hat{L}_z] = 0$  and reduce the eigenvalue problem to the radial differential equation for  $R_m(\rho)$ .

(ii) Examine this differential equation in the limit of  $\rho \rightarrow 0$  and show that  $R_m(\rho) \rightarrow \rho^{|m|}$  in this limit. Similarly, consider the limit  $\rho \rightarrow \infty$  and show that  $R_m(\rho) \rightarrow \exp[-M\omega\rho^2/(2\hbar)]$  in this limit.

(iii) Switch to dimensionless variables  $y \equiv \sqrt{M\omega/\hbar} \rho$  and  $\epsilon \equiv E/(\hbar\omega)$ . Assume that  $R_m(\rho)$  can be written in the form

$$R_m(\rho) = \rho^{|m|} e^{-y^2/2} U_m(y)$$

and show that  $U_m(y)$  obeys the equation

$$U_m'' + \left[ \frac{2|m|+1}{y} - 2y \right] U_m' + (2\epsilon - 2|m| - 2)U_m = 0$$

where  $U_m' = dU_m/dy$ .

(iv) Next, attempt to solve the differential equation for  $U_m$  via a series ansatz,

$$U_m(y) = \sum_{k=1} C_k y^k$$

and find the recursion equation satisfied by the coefficients  $C_k$ . How many coefficients need to be given in order to determine all others?

(v) Argue that the series must terminate at some finite  $k_o$  in order to lead to an acceptable solution in the  $y \rightarrow \infty$  limit. Show that the choice

$$\epsilon = k_o + |m| + 1$$

leads to the termination of the series after  $k_o$  terms.

(vi) Now, argue that  $k$  must be even, i.e.,  $k = 2l$ . In order to do this, show that odd  $k$  is incompatible with the expected  $\rho \rightarrow 0$  limit of  $R_m(\rho)$ . So, argue that you must end up with energy eigenvalues of the form

$$E = \hbar\omega(2l_o + |m| + 1) \quad , \quad \text{with } l_o = 0, 1, 2, \dots$$

Define  $n = 2l_o + |m|$  and show that, for a given  $n$ , the degeneracy of these energy eigenvalues is  $n + 1$ .

(vii) Write down all normalized eigenfunctions  $\Psi_{n,m}(\rho, \phi)$  associated with  $n = 0$  and  $n = 1$ . Compare these eigenfunctions to their counterparts in Cartesian coordinates,  $\psi_{n_x, n_y}(x, y)$ . Show that the  $n = 0$  eigenfunction is identical to the Cartesian eigenfunction with  $n_x = n_y = 0$ , while the two  $n = 1$  eigenfunctions are linear combinations of  $\psi_{1,0}(x, y)$  and  $\psi_{0,1}(x, y)$ .

**Comment:** The polynomials  $U_m(y)$  [or better:  $U_{l_0, m}(y)$ ] that you are constructing here are called “associated Laguerre polynomials”. Their properties have been extensively studied.

**You should review:**

Liboff, Chapter 7.1 – 7.9, 8.5 and 8.6, 9.1 – 9.3.