PHYSICS 4455 — QUANTUM MECHANICS
Problem Set 4 — Solutions.

1. **Becoming more familiar with operators.**
   Liboff Problem 3.3
   Nonlinear operators are: $\hat{P}$, $\hat{G}$ - why?

$$\hat{P}[a\varphi_1(x) + b\varphi_2(x)] = [a\varphi_1(x) + b\varphi_2(x)]^3 - 3[a\varphi_1(x) + b\varphi_2(x)]^2 - 4$$

which is not equal to :

$$a\hat{P}\varphi_1(x) + b\hat{P}\varphi_2(x) = a\{\varphi_1^3(x) - 3\varphi_1^2(x) - 4\} + b\{\varphi_2^3(x) - 3\varphi_2^2(x) - 4\}$$

$$\hat{G}[a\varphi_1(x) + b\varphi_2(x)] = 8$$

which is not equal to :

$$a\hat{G}\varphi_1(x) + b\hat{G}\varphi_2(x) = a \times 8 + b \times 8$$

All others are linear. I'm a bit lazy here: to get full credit for your homework, you should go through the operators one by one, and check the linearity condition. I'll skip that - if any of these others give you trouble, please ask about them during office hours.

Liboff Problem 3.16
We have

$$\hat{A}\varphi_n(x) = a_n\varphi_n(x)$$

and a function $f(x)$ with expansion

$$f(x) = \sum_{l=0}^{\infty} b_l x^l$$

We want to show that $\varphi_n(x)$ is an eigenfunction of $f(\hat{A}) \equiv \sum_{l=0}^{\infty} b_l \hat{A}^l$. To do so, note that

$$\hat{A}^3\varphi_n(x) = \hat{A}^2 a_n\varphi_n(x) = a_n\hat{A}^2\varphi_n(x) = a_n\hat{A}a_n\varphi_n(x) = \ldots = a_n^3\varphi_n(x)$$

etc., for any integer power of $\hat{A}$. So,

$$f(\hat{A})\varphi_n(x) = \sum_{l=0}^{\infty} b_l\hat{A}^l\varphi_n(x) = \sum_{l=0}^{\infty} b_l a_n^l\varphi_n(x) = f(a_n)\varphi_n(x).$$

2. **… and in particular, the displacement operator:**
   Liboff Problem 3.4
   The displacement operator is defined as

$$\hat{\mathcal{D}}f(x) \equiv f(x + \zeta)$$

This is of course a silly definition - what is ζ? For the purposes of this definition, it is simply a (given) constant. It would be much less obscure to define $\hat{\mathcal{D}}_\zeta$ as the displacement operator by an amount ζ, etc. To show that its eigenfunctions have the form $\varphi_\beta(x) = e^{\beta x}g(x)$ where $g(x)$ is a function satisfying $g(x) = g(x + \zeta)$, we must show that

$$\hat{\mathcal{D}}\varphi_\beta(x) = (\text{a number, say } \lambda_\beta) \times \varphi_\beta(x)$$

So, noting first that by definition, $\hat{\mathcal{D}}\varphi_\beta(x) = \varphi_\beta(x+\zeta)$, we have

$$\hat{\mathcal{D}}\varphi_\beta(x) = \varphi_\beta(x+\zeta) = e^{\beta(x+\zeta)}g(x+\zeta) = e^{\beta\zeta}e^{\beta x}g(x+\zeta) = e^{\beta\zeta}e^{\beta x}g(x) = e^{\beta\zeta}\varphi_\beta(x)$$

So, $\varphi_\beta(x)$ is an eigenfunction with eigenvalue $\lambda_\beta = e^{\beta\zeta}$.

Liboff Problem 3.17

Here, we want to show that

$$\exp\left[\frac{i\zeta\hat{p}}{\hbar}\right]$$

is a representation of the displacement operator, i.e.,

$$\exp\left[\frac{i\zeta\hat{p}}{\hbar}\right]f(x) = \hat{\mathcal{D}}f(x) = f(x+\zeta)$$

With $\hat{p} = -i\hbar(d/dx)$, we may write the Taylor series of the exponential as

$$\exp\left[\frac{i\zeta\hat{p}}{\hbar}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\zeta\hat{p}}{\hbar}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\zeta\frac{d}{dx}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n\left(\frac{d}{dx}\right)^n$$

Letting this act on $f(x)$, we find

$$\exp\left[\frac{i\zeta\hat{p}}{\hbar}\right]f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n\left(\frac{d}{dx}\right)^n f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n\frac{d^n}{dx^n}f(x)$$

On the other hand, the Taylor series of $f(x+\zeta)$, around x, is given by

$$f(x+\zeta) = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n\frac{d^n}{dx^n}f(x)$$

which is clearly the same result.

3. **Why is the Schrödinger equation – unlike the equations for classical waves – just first order in $\partial/\partial t$?**

Liboff Problem 3.20

Time-independent Schrödinger equation:

$$a\partial_t\psi(x,t) = \hat{H}\psi(x,t)$$

(a) Let's consider two independent subsystems (e.g., two non-interacting particles with coordinates x_1 and x_2, respectively). Subsystem 1 is described by a Hamiltonian $\hat{H}_1$ (acts on coordinate x_1) and a wave function $\psi_1(x_1,t)$; subsystem 2 has Hamiltonian $\hat{H}_2$ (acts on coordinate x_2) and stationary state $\psi_2(x_2,t)$. The Schrödinger equations for the two subsystems are:

$$a\partial_t\psi_i(x_i,t) = \hat{H}_i\psi_i(x_i,t) \quad \text{for } i = 1,2$$

If we consider the whole system of both particles, described by the Hamiltonian $\hat{H} = \hat{H}_1 + \hat{H}_2$ (recall: energies are additive), we can show that the associated Schrödinger equation is solved by the product, $\psi_1(x_1,t)\psi_2(x_2,t)$. To prove that statement, we need to show that

$$a\partial_t\psi_1(x_1,t)\psi_2(x_2,t) = \hat{H}\psi_1(x_1,t)\psi_2(x_2,t)$$

Let's work out the right hand side:

$$\begin{aligned}
\hat{H}\psi_1(x_1,t)\psi_2(x_2,t) &= \left(\hat{H}_1 + \hat{H}_2\right)\psi_1(x_1,t)\psi_2(x_2,t) \\
&= \left[\hat{H}_1\psi_1(x_1,t)\right]\psi_2(x_2,t) + \psi_1(x_1,t)\left[\hat{H}_2\psi_2(x_2,t)\right] \\
&= [a\partial_t\psi_1(x_1,t)]\psi_2(x_2,t) + \psi_1(x_1,t)[a\partial_t\psi_2(x_2,t)] \\
&= a\partial_t[\psi_1(x_1,t)\psi_2(x_2,t)]
\end{aligned}$$

which is the expected answer if $\psi_1(x_1,t)\psi_2(x_2,t)$ is a stationary state of $\hat{H}$.

(b) Now we show that this fails if we have second order time derivatives on the left hand side, i.e., even if

$$a^2\partial_t^2\psi_i(x_i,t) = \hat{H}_i\psi_i(x_i,t) \quad \text{for } i = 1,2$$

we find

$$a^2\partial_t^2[\psi_1(x_1,t)\psi_2(x_2,t)] \neq \hat{H}\psi_1(x_1,t)\psi_2(x_2,t)$$

Look at the right hand side:

$$\begin{aligned}
\hat{H}\psi_1(x_1,t)\psi_2(x_2,t) &= \left(\hat{H}_1 + \hat{H}_2\right)\psi_1(x_1,t)\psi_2(x_2,t) \\
&= \left[\hat{H}_1\psi_1(x_1,t)\right]\psi_2(x_2,t) + \psi_1(x_1,t)\left[\hat{H}_2\psi_2(x_2,t)\right] \\
&= [a^2\partial_t^2\psi_1(x_1,t)]\psi_2(x_2,t) + \psi_1(x_1,t)[a^2\partial_t^2\psi_2(x_2,t)]
\end{aligned}$$

Now look at the left hand side:

$$\begin{aligned}
&a^2\partial_t^2[\psi_1(x_1,t)\psi_2(x_2,t)] \\
&= a^2\partial_t\{[\partial_t\psi_1(x_1,t)]\psi_2(x_2,t) + \psi_1(x_1,t)[\partial_t\psi_2(x_2,t)]\} \\
&= a^2\{[\partial_t^2\psi_1(x_1,t)]\psi_2(x_2,t) + 2[\partial_t\psi_1(x_1,t)][\partial_t\psi_2(x_2,t)] + \psi_1(x_1,t)[\partial_t^2\psi_2(x_2,t)]
\end{aligned}$$

...and that does not agree with the right hand side: the term $2a^2[\partial_t\psi_1(x_1,t)][\partial_t\psi_2(x_2,t)]$ is missing there.

(c) So far, this only tells us that the product $\psi_1(x_1,t)\psi_2(x_2,t)$ is not a good wave function for (b). Maybe we can find a better wave function (different from the simple product form), if we work hard enough? As we shall see here, there is another reason why we would like to have a product wave function for our system, and this reason resides in the Born postulate, i.e., the connection between a wave function and the probability to find the system in a given state. Returning to the two noninteracting particles, we can ask for the probability density to find particle 1 at position x_1 *and* particle 2 at position x_2, at time t. Let's call that probability density $P_{12}(x_1,x_2;t)$. Since the two particles are noninteracting, their probabilities are independent, i.e., $P_{12}(x_1,x_2;t) = P_1(x_1;t)P_2(x_2;t)$, where $P_1(x_1;t)$ is the probability density to find particle 1 at position x_1 at time t (irrespective of what particle 2 is doing). Obviously, $P_2(x_2;t)$ is defined analogously for particle 2. If you want to preserve the Born postulate, namely, that $P_1(x_1;t) = |\psi_1(x_1,t)|^2$ and $P_2(x_2;t) = |\psi_2(x_2,t)|^2$, as well as the statistical properties of joint probabilities of independent events, the wave function of the two-particle system must be the product of the two single-particle wave function. This guarantees

$$P_{12}(x_1,x_2;t) = P_1(x_1;t)P_2(x_2;t)$$

because $P_{12}(x_1,x_2;t) = |\psi_1(x_1,t)\psi_2(x_2,t)|^2$ for the product wave function.