PHYSICS 4455 — QUANTUM MECHANICS
Problem Set 9 — due 11/17/2005, in class.

A few more items about harmonic oscillators.

1. Many of you wondered in class how the raising and lowering operator formalism connects back to the explicit wave functions in coordinate space. Here, we will see how this connection is established.

An important step in the operator algebra involved the (well-founded) assumption that there had to be a ground state, $|0\rangle$, with ground state energy $\epsilon_o = 1/2$ in dimensionless form. When the lowering operator $\hat{a}$ acts on this state, the result is 0: Start from the equation

$$\hat{a}|0\rangle = 0 \tag{1}$$

(i) Express $\hat{a}$ in terms of the operators $\hat{x}$ and $\hat{p}$ and write Eq. (1) in the position representation, as a differential equation for the ground state wave function $\varphi_o(x)$. Solve this differential equation and normalize its solution properly.

(ii) Next, we generate the first excited state, $\varphi_1(x)$, from the ground state by using the raising operator property

$$|1\rangle = \hat{a}^+|0\rangle \tag{2}$$

Write $\hat{a}^+$ in terms of the operators $\hat{x}$ and $\hat{p}$ and write Eq. (2) in the position representation. Find the explicit form of $\varphi_1(x)$. Is it already normalized?

(iii) Now, write a general expression for $\varphi_n(x)$, by translating the relation

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle$$

into position space. By comparing this general form with the explicit solution for $\varphi_n(x)$, show that the Hermite polynomials can be generated recursively from the relation

$$H_n(y) = e^{y^2/2}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}$$

2. The raising and lowering operator formalism is also very convenient when it comes to computing expectation values. Here is a simple example to demonstrate its power.

(i) Starting from the explicit form of the wave function $\varphi_3(x)$, compute the expectation values

$$\langle 3|\hat{x}|3\rangle = \int_{-\infty}^{+\infty} dx\varphi_3^*(x)x\varphi_3(x)$$

and

$$\langle 3|\hat{x}^2|3\rangle = \int_{-\infty}^{+\infty} dx\varphi_3^*(x)x^2\varphi_3(x)$$

by evaluating the integrals on the right hand side explicitly.

(ii) Now, compute the same expectation values by expressing $\hat{x}$ and $\hat{x}^2$ in terms of $\hat{a}$ and $\hat{a}^+$ and exploit the properties of the raising and lowering operators, as well as the orthonormality of the states $|n\rangle$. If all goes well, you will arrive at the same answer as under (i), but without doing a single integral!

(iii) *Extra credit.* Exploit the operator formalism to show that the average kinetic energy and the average potential energy are equal to each other, in eigenstate $|n\rangle$. That is the so-called "virial theorem"; see Liboff Problem 7.10 for parts of the solution.

You should review:

Liboff, Chapter 7.1 – 7.9, 8.5 and 8.6.