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# $\mathcal{N} = 1$ RG flows, product groups, and $a$ -maximization

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## Abstract

We explore new IR phenomena and dualities, arising for product groups, in the context of  $\mathcal{N} = 1$  supersymmetric gauge theories. The RG running of the multiple couplings can radically affect each other. For example, an otherwise IR interacting coupling can be driven to be instead IR free by an arbitrarily small, but nonzero, initial value of another coupling. Or an otherwise IR free coupling can be driven to be instead IR interacting by an arbitrarily small non-zero initial value of another coupling. We explore these and other phenomena in  $\mathcal{N} = 1$  examples, where exact results can be obtained using  $a$ -maximization. We also explore the various possible dual gauge theories, e.g., by dualizing one gauge group with the other treated as a weakly gauged flavor symmetry, along with previously proposed duals for the theories deformed by  $A_k$ -type Landau–Ginzburg superpotentials. We note that this latter duality, and all similar duality examples, always have nonempty superconformal windows within which both the electric and dual  $A_k$  superpotentials are relevant.

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## 1. Introduction

Asymptotically free gauge theories have various possible IR phases, one being the “non-Abelian Coulomb phase”, which is an interacting conformal field theory RG fixed point,

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where all beta functions vanish. A classic example is  $\mathcal{N} = 1$   $SU(N_c)$  SQCD with  $N_f$  massless flavors, which flows to a SCFT in the IR for  $N_f$  within the Seiberg superconformal window [1]  $\frac{3}{2}N_c < N_f < 3N_c$ . For  $N_f \leq \frac{3}{2}N_c$ , the theory is instead in a free magnetic  $SU(N_c - N_f)$  phase in the IR. (See, e.g., [2] for a review and references.) Our prejudice is that the interacting SCFT phase is rather generic for asymptotically free SUSY gauge theories with enough massless matter to avoid dynamical superpotentials, i.e. with massless matter representation  $R$  such that  $T(G) < T(R) < 3T(G)$ , with  $T(R)$  the quadratic Casimir of  $R$  and  $T(G)$  that of the adjoint. The theory at the origin is then either a non-trivial free field solution of 't Hooft anomaly matching (as in the free magnetic phase) or an interacting SCFT. Unfortunately, unless one has a conjectured dual description,<sup>1</sup> there is no simple test for directly determining if the IR phase is a SCFT or (fully or partially) IR free magnetic.

There is an essentially endless landscape of possible RG fixed point SCFTs to explore, coming from various gauge groups, including product groups, and matter representations. Here we will consider examples with product gauge groups, e.g., the theory

$$\begin{array}{llll}
 \text{gauge group:} & SU(N_c) \times SU(N'_c) & & \\
 \text{matter:} & X \oplus \tilde{X} & (\square, \bar{\square}) \oplus (\bar{\square}, \square), & \\
 & Q_f \oplus \tilde{Q}_{\tilde{f}} & (\square, \mathbf{1}) \oplus (\bar{\square}, \mathbf{1}) & (f, \tilde{f} = 1, \dots, N_f), \\
 & Q'_{f'} \oplus \tilde{Q}'_{\tilde{f}'} & (\mathbf{1}, \square) \oplus (\mathbf{1}, \bar{\square}) & (f', \tilde{f}' = 1, \dots, N'_f). \tag{1.1}
 \end{array}$$

We will be interested in when this theory flows to an (either fully or partially) interacting SCFT and when various dualities apply, e.g., dualizing one gauge group with the other treated as a spectator. We will also be interested in the superconformal window for a duality proposed in [5], for the above theory deformed by superpotential  $W_{A_{2k+1}} = \text{Tr}(X\tilde{X})^{k+1}$ .

With multiple couplings, e.g., the two gauge couplings of (1.1), the RG running of one coupling can radically affect that of the other, possibly driving it into another basin of attraction. For example, as depicted in Fig. 1, there can be saddle point IR fixed points (A) and (B) when one or the other coupling is tuned to precisely zero, but which are unstable to any perturbation in the other coupling: the generic RG flows then end up at point (C) in the IR, with both  $g_*$  and  $g'_*$  nonzero. Another possibility, shown in Fig. 2, is that  $g'$  is interacting in the IR only if  $g = 0$ , but that any arbitrarily small, nonzero,  $g$  would eventually overwhelm  $g'$ , and drive  $g'$  to be IR irrelevant,  $g' \rightarrow 0$  in the IR; generic RG flows then end up at point (A), with  $g'_* = 0$ . Fig. 3 depicts an opposite situation, where an otherwise IR free coupling  $g'$  is driven to be interacting in the IR by the coupling  $g$ . Fig. 4 depicts two separately IR free couplings, which can cure each other and lead to an interacting RG fixed point (this happens for, e.g., the gauge and Yukawa couplings of the  $\mathcal{N} = 4$  theory, when we break to  $\mathcal{N} = 1$  by taking them to be unequal).

The theory (1.1) realizes the RG flows depicted in Figs. 1 or 2, depending on the values of the parameters  $(N_c, N'_c, N_f, N'_f)$ . With an additional superpotential, as is present if we dualize one of the factors in (1.1), the phenomenon of Fig. 3 is also realized.

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<sup>1</sup> Even with a non-trivial free field solution of 't Hooft anomaly matching, e.g., as in the example of [3], there is the possibility that the matching is a fluke, and that the theory actually flows to an interacting SCFT after all, as was argued to be the case for another example in [4].

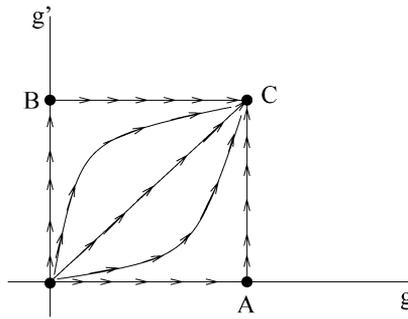


Fig. 1.  $A$  and  $B$  are saddlepoints.  $C$  is stable, and there both groups are interacting.

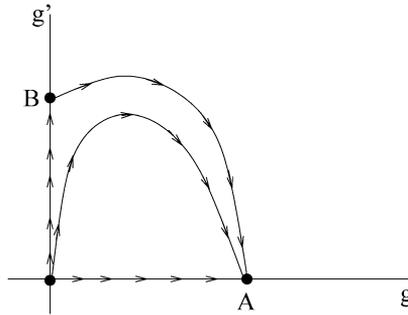


Fig. 2. The plop.  $B$  is a saddlepoint.  $A$  is stable, and there  $g'$  is driven to be IR free.

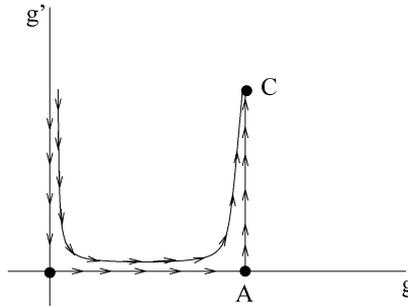


Fig. 3. The opposite of Fig. 2.  $g'$  is IR free for  $g = 0$  but  $g \neq 0$  drives  $g'$  IR interacting.

We will focus here on supersymmetric theories, such as (1.1) with  $\mathcal{N} = 1$  supersymmetry, where some exact results can be obtained. However, we expect the phenomena of Figs. 1 and 2 to occur even in nonsupersymmetric  $G \times G'$  gauge theories, with matter in mixed  $G \times G'$  representations, at least when the matter content is chosen such that each group is just barely asymptotically free. There can then be RG fixed points in the perturbative regime, as can be seen by considering the beta functions to two loops:

$$\beta_\alpha = \frac{\alpha^2}{2\pi}(-b_1 + b_2\alpha + c_2\alpha') + O(\alpha^4),$$

$$\beta_{\alpha'} = \frac{\alpha'^2}{2\pi}(-b'_1 + b'_2\alpha' + c'_2\alpha) + O(\alpha^4), \tag{1.2}$$

(writing  $\alpha = g^2/4\pi$  and  $\alpha' = g'^2/4\pi$ , and  $O(\alpha^4)$  refers to powers of either  $\alpha$  or  $\alpha'$ ), where the  $c_2$  and  $c'_2$  terms come from the matter in mixed representations; see, e.g., [6]. Choosing the matter content to be such that the groups are barely asymptotically free, i.e., such that  $b_1$  and  $b'_1$  are small positive numbers, it is then found that the two-loop coefficients ( $b_2, c_2, b'_2, c'_2$ ) in (1.2) are positive and not especially small (e.g., in large  $N_c$ ); this allows for RG fixed points to exist at relatively small values of the fixed point coupling, so that this argument for the RG fixed point's existence could be qualitatively reliable.

In particular, to two loops, we find zeros of the beta functions (1.2) at three points: point (A) at  $(\alpha_*, \alpha'_*)_A \approx (b_1/b_2, 0)$ , point (B) at  $(\alpha_*, \alpha'_*)_B \approx (0, b'_1/b'_2)$ , and point (C), at

$$\begin{pmatrix} \alpha_* \\ \alpha'_* \end{pmatrix}_C \approx \frac{1}{b_2b'_2 - c_2c'_2} \begin{pmatrix} b'_2 & -c_2 \\ -c'_2 & b_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b'_1 \end{pmatrix}. \tag{1.3}$$

For point (C) to actually exist, the values of  $\alpha_*$  and  $\alpha'_*$  in (1.3) must be positive. It is found that the determinant denominator in (1.3) is generally positive, so the positivity condition for RG fixed point (C) to exist is thus

$$b_1b'_2 > b'_1c_2 \quad \text{and} \quad b'_1b_2 > b_1c'_2 \quad \text{to have } g_* \neq 0 \text{ and } g'_* \neq 0. \tag{1.4}$$

These inequalities may or may not hold, depending on the choice of matter content. The intuition for these inequalities is that each gauge coupling makes the other less interacting in the IR (via the  $c_2$  or  $c'_2$  terms), so there can only be a RG fixed point (C), with both interacting, if the couplings flow in balance: if either flows too much faster than the other, it can drive the other to be IR free. For example, if the matter content is such that  $b_1c'_2 > b'_1b_2$ , then  $g' \rightarrow 0$  in the IR, as in Fig. 2, with the  $G$  dynamics overwhelming the  $G'$  dynamics in the IR. Likewise, if  $b'_1c_2 > b_1b'_2$ , then  $G'$  wins, and drives  $g \rightarrow 0$  in the IR. The inequality  $b_2b'_2 > c_2c'_2$  implies that both inequalities in (1.4) could not be reversed.

The criteria (1.4) for RG fixed point (C) to exist are equivalent to the condition that RG fixed points (A) and (B) be IR unstable to perturbations in the other coupling, as depicted in Fig. 5. For example, near (A), where  $\alpha' = 0$  and  $\alpha_* \approx b_1/b_2$ , (1.2) gives  $\beta_{\alpha'} = \alpha'^2(-b'_1 + c'_2b_1/b_2)/2\pi + O(\alpha'^3)$ . The second inequality in (1.4) is thus equivalent to having (A) be IR repulsive to  $\alpha'$  perturbations, as in Fig. 1. If (A) and (B) are both IR repulsive to

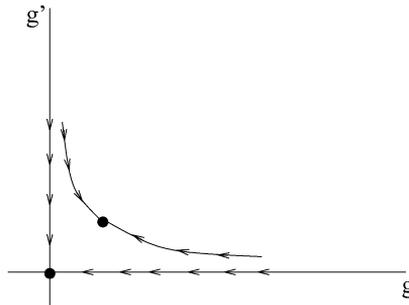


Fig. 4. Two separately irrelevant couplings combine to be interacting.  $\mathcal{N} = 4$  SYM is such an example.

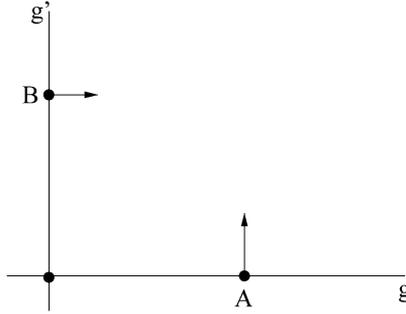


Fig. 5. If  $A$  and  $B$  are both IR unstable to perturbations, the theory flows to  $C$ , with both couplings interacting.

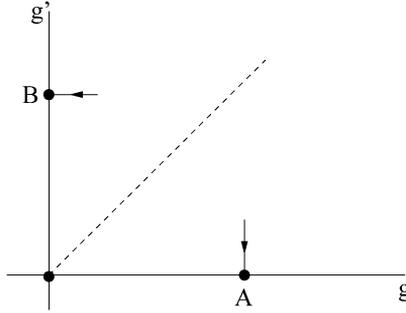


Fig. 6. We do not find examples of  $A$  and  $B$  both IR stable to perturbations. Would've required a separatrix between domains of attraction.

perturbations, generic couplings flow to having both interacting, and can end up at a fixed point ( $C$ ), as in Fig. 1. If either inequality of (1.4) is violated, then either ( $A$ ) or ( $B$ ) is IR attractive, and then RG fixed point ( $C$ ) does not exist (at least it does not exist within the basin of attraction of zero couplings). In that case, as depicted in Fig. 2, generic RG flows attract to the IR stable point ( $A$ ) or ( $B$ ). Because  $b_2 b'_2 > c_2 c'_2$ , both inequalities in (1.4) could not be reversed, i.e., we cannot have ( $A$ ) and ( $B$ ) both be IR attractive. As depicted in Fig. 6, such a hypothetical flow would have required an unstable separatrix ridge, depicted as a dashed line, dividing the RG flows into two different domains of attraction. In neither the perturbative analysis, nor the supersymmetric examples to follow, do we find examples of such flows.

We can go beyond the above perturbative analysis in  $\mathcal{N} = 1$  supersymmetric theories, where exact results can be obtained via the NSVZ [7] beta functions. For a general  $\mathcal{N} = 1$   $G \times G'$  gauge theory, with matter chiral superfield in representations  $\bigoplus_i (\mathbf{r}_i, \mathbf{r}'_i)$ , these are

$$\begin{aligned} \beta_g(g, g') &= -\frac{g^3 f}{16\pi^2} \left( 3T(G) - \sum_i T(\mathbf{r}_i) |\mathbf{r}'_i| (1 - \gamma_i(g, g')) \right) \\ &= -\frac{3g^3 f}{16\pi^2} \text{Tr} GGR, \end{aligned}$$

$$\begin{aligned}\beta_{g'}(g, g') &= -\frac{g'^3 f'}{16\pi^2} \left( 3T(G') - \sum_i T(r'_i) |r_i| (1 - \gamma_i(g, g')) \right) \\ &= -\frac{3g'^3 f'}{16\pi^2} \text{Tr } G' G' R.\end{aligned}\quad (1.5)$$

In the NSVZ scheme,  $f = (1 - \frac{g^2 T(G)}{8\pi^2})^{-1}$  and  $f' = (1 - \frac{g'^2 T(G')}{8\pi^2})^{-1}$ , while in other schemes these factors are replaced with other functions of the coupling [8], such that  $f = 1 + O(g^2)$ ; these scheme-dependent prefactors are unimportant for our discussion, except for the fact that they should be strictly positive in our range of coupling constants.

The last equality in each line of (1.5) involves  $\text{Tr } G G R$ , which is the coefficient of the  $U(1)_R$  ABJ triangle anomaly, with two external  $G$  gluons. This uses the fact that supersymmetry relates the dilatation current to a  $U(1)_R$  current, with the exact scaling dimensions of chiral fields related to their  $U(1)_R$  charges:

$$\Delta = \frac{3}{2}R, \quad \text{e.g.,} \quad \Delta(Q_i) \equiv 1 + \frac{1}{2}\gamma_i = \frac{3}{2}R(Q_i).\quad (1.6)$$

When the theory is conformally invariant, this  $U(1)_R$  is conserved, and part of the superconformal group  $SU(2, 2|1)$ . When the theory is not conformally invariant, e.g., along the RG flow from the UV to the IR, supersymmetry still relates the stress tensor to an R-current, whose charges run with the anomalous dimensions according to (1.6), and whose anomaly is related to the beta function according to (1.5). Among all possible R-symmetries, the superconformal R-symmetry is that which locally maximizes  $a_{\text{trial}}(R) \equiv 3 \text{Tr } R^3 - \text{Tr } R$  [9]. This method for determining the superconformal R-charges is referred to as “ $a$ -maximization”, because the value of  $a_{\text{trial}}$  at its unique local maximum equals the conformal anomaly coefficient  $a$  of the SCFT [10, 11] (we rescale  $a$  by a conventional factor of  $3/32$ ). An extension of  $a$ -maximization [12], further explored in [8, 13], has been proposed for determining the running R-charges, along the RG flow from the UV to the IR. See, e.g., [14–21] for further applications and extensions of  $a$ -maximization.

For our particular example (1.1), the exact beta functions (1.5) are

$$\begin{aligned}\beta_g(g, g') &= -\frac{3g^3 f}{16\pi^2} \text{Tr } SU(N_c)^2 R = -\frac{g^3 f}{16\pi^2} (b_1 + N_f \gamma_Q + N'_c \gamma_X), \\ \beta_{g'}(g, g') &= -\frac{3g'^3 f'}{16\pi^2} \text{Tr } SU(N'_c)^2 R = -\frac{g'^3 f'}{16\pi^2} (b'_1 + N'_f \gamma_{Q'} + N_c \gamma_X),\end{aligned}\quad (1.7)$$

where  $b_1 \equiv 3N_c - N_f - N'_c$  and  $b'_1 \equiv 3N'_c - N'_f - N_c$  are the one-loop beta functions. We will take both groups to be asymptotically free, i.e., take  $g = g' = 0$  to be UV attractive:

$$3N_c - N_f - N'_c > 0 \quad \text{and} \quad 3N'_c - N'_f - N_c > 0,\quad (1.8)$$

so that  $g = g' = 0$  is IR repulsive, as in Figs. 1 and 2. To have the theory flow to a SCFT in the IR, rather than dynamically generating a vev, from a dynamically generated superpotential or quantum moduli space constraint, we also require

$$N_f + N'_c > N_c \quad \text{and} \quad N'_f + N_c > N'_c \quad (\text{stability}).\quad (1.9)$$

Assuming that (1.8) and (1.9) hold, much as in the above perturbative analysis, we identify three possible RG fixed points:

$$(A) \quad g_* \neq 0, \quad g'_* = 0, \quad \text{where } \gamma_Q = \gamma_X = -\frac{b_1}{N_f + N'_c}, \text{ and } \gamma_{Q'} = 0. \quad (1.10)$$

$$(B) \quad g_* = 0, \quad g'_* \neq 0, \quad \text{where } \gamma_{Q'} = \gamma_X = -\frac{b'_1}{N'_f + N_c}, \text{ and } \gamma_Q = 0. \quad (1.11)$$

$$(C) \quad g_* \neq 0, \quad g'_* \neq 0, \quad \text{where} \\ b_1 + N_f \gamma_Q + N'_c \gamma_X = 0 = b'_1 + N'_f \gamma_{Q'} + N_c \gamma_X. \quad (1.12)$$

For point (A), we used the fact that there is an enhanced flavor symmetry which ensures that  $\gamma_X = \gamma_Q$  when  $g' = 0$ , and that  $Q'$  is a free field for  $g' = 0$ , so  $\gamma_{Q'} = 0$ . Analogous considerations apply for RG fixed point (B). Seiberg duality [1] shows that (A) and (B) are actually interacting SCFTs only if

$$N_f + N'_c > \frac{3}{2} N_c \quad \text{and} \quad N'_f + N_c > \frac{3}{2} N'_c, \quad (1.13)$$

respectively; otherwise (A) or (B) should be replaced with its free magnetic Seiberg dual.

Our interest here is in the possible RG fixed point (C). We will discuss when it exists as an interacting SCFT. We will find, for example, that (1.13) is modified, once the RG flow of both couplings is taken into account: the otherwise free magnetic dual can be driven interacting by the other gauge coupling, as depicted in Fig. 3.

Let us first discuss some simple necessary, though not sufficient, conditions for (C) to exist—at least within the domain of attraction of flows to the IR from the asymptotically free UV fixed point at zero couplings—by determining when the RG flow is as in Fig. 1, or as in Fig. 2, with one of the couplings driven IR free. (Our discussion here is essentially identical to one that already appeared in [22] for a chiral example similar to (1.1), having the field  $X$  but not  $\tilde{X}$ .) As in Fig. 5, (C) exists within the domain of attraction of the UV fixed point only if (A) and (B) are both IR unstable to perturbations in the other coupling. Using (1.7), (A) is IR stable to  $g'$  perturbations if  $\text{Tr} SU(N'_f)^2 R|_A < 0$ , i.e., we get  $\beta_{g'} \sim -g'^3 (b'_1 - N_c b_1 / (N_f + N'_c)) + O(g'^5)$ , with the second contribution from  $\gamma_X$  at (A), so  $g'$  is an IR irrelevant perturbation of (A) if  $b'_1 < N_c b_1 / (N_f + N'_c)$ , i.e.,

$$(A) \text{ is IR attractive, with } g' \rightarrow 0, \text{ if } (3N'_c - N'_f)(N'_c + N_f) - 3N_c^2 < 0. \quad (1.14)$$

Similarly,  $g$  will be an irrelevant perturbation of (B) if  $\text{Tr} SU(N_c)^2 R|_B < 0$ , which gives

$$(B) \text{ is IR attractive, with } g \rightarrow 0, \text{ if } (3N_c - N_f)(N_c + N'_f) - 3N_c'^2 < 0. \quad (1.15)$$

The two inequalities in (1.14) and (1.15) are mutually incompatible, so we do not find the situation of Fig. 6. The condition for RG fixed point (C) to exist (within the domain of attraction of the UV fixed point) is that neither (1.14) nor (1.15) holds, i.e., we have a flow as in Fig. 1 only if

$$(3N_c - N_f)(N_c + N'_f) - 3N_c'^2 > 0, \\ (3N'_c - N'_f)(N'_c + N_f) - 3N_c^2 > 0. \quad (1.16)$$

The inequalities (1.16) generally differ from the asymptotic freedom conditions (1.8) needed to have  $g = g' = 0$  not be IR attractive. For the special case  $N_c = N'_c$  and  $N_f = N'_f$ , (1.16) do reduce to the asymptotic freedom conditions (1.8).

When RG fixed point (C) does exist, the three independent anomalous dimensions,  $\gamma_Q$ ,  $\gamma_{Q'}$ , and  $\gamma_X$  are under-constrained by the two constraints of (1.12), so  $a$ -maximization [9] is required to determine the exact anomalous dimensions of chiral operators at (C). When the RG fixed point is not at sufficiently strong coupling for there to be accidental symmetries, the  $a$ -maximization result can be written as

$$\begin{aligned} \gamma_Q &= 1 - \sqrt{1 + \frac{\lambda_G}{2N_c}}, & \gamma_{Q'} &= 1 - \sqrt{1 + \frac{\lambda_{G'}}{2N'_c}}, \\ \gamma_X &= 1 - \sqrt{1 + \frac{\lambda_G}{2N_c} + \frac{\lambda_{G'}}{2N'_c}}, \end{aligned} \quad (1.17)$$

with  $\lambda_G$  and  $\lambda_{G'}$  determined by the two conditions in (1.12), for the two beta functions (1.7) to vanish. This way of writing the  $a$ -maximization result is motivated by the extension of  $a$ -maximization due to Kutasov [8,12,13], where the interaction constraints on the superconformal R-charges, e.g., that the ABJ anomalies should vanish, are imposed with Lagrange multipliers. The conjecture is that the Lagrange multipliers can be interpreted as the running coupling constants along the flow to the RG fixed point. In particular, the claim is that (1.17) gives the running anomalous dimensions along the entire RG flow, from  $g = g' = 0$  in the UV to the RG fixed point (C) in the IR, with  $\lambda_G = g^2|G|/2\pi^2$  and  $\lambda_{G'} = g'^2|G'|/2\pi^2$  the running couplings in some scheme.

This analysis needs to be supplemented when there are accidental symmetries [14], and we will find that many accidental symmetries do arise in these theories for general  $(N_c, N'_c, N_f, N'_f)$ .  $a$ -maximization with many accidental symmetries is best left to a computer (we used Mathematica), and then it is simpler to do the  $a$ -maximization at the RG fixed point, imposing the constraints at the outset rather than with Lagrange multipliers. We simplify the analysis by considering the limit of large numbers of flavors and colors, for arbitrary fixed values of the ratios, which for the example (1.1) are

$$x \equiv \frac{N_c}{N_f}, \quad x' \equiv \frac{N'_c}{N_f}, \quad n \equiv \frac{N'_f}{N_f}. \quad (1.18)$$

In this limit, the operator scaling dimensions then only depend on these ratios. Depending on  $(x, x', n)$ , a variety of accidental symmetries associated with gauge invariant operators hitting the unitarity bound are found to occur, and their effect on the  $a$ -maximization analysis [14] is accounted for in our numerical algorithm.

As a function of the parameters  $(x, x', n)$ , the theory either flows in the IR to a fully interacting RG fixed point, or can be partially or fully free. Our motivation for considering the examples (1.1) is that they have various possible dualities, and the  $a$ -maximization results can give insight into when they are applicable. For example, we could Seiberg dualize [1] one of the groups in (1.1), treating the other as a weakly gauged spectator. As we will discuss, there is a range of  $(x, x', n)$  for which this dual theory realizes the RG flow possibility of Fig. 3: an arbitrarily small nonzero coupling of the “spectator” group

can drive an otherwise free magnetic group to be interacting in the IR. This also occurs in an example discussed in [19], which appeared during the course of the present work.

Knowing the exact dimensions of chiral operators, we can classify the relevant superpotential deformations of  $(C)$ . In particular, we can now determine the “superconformal window” range of validity of a duality proposed in [5] for the theory (1.1) with added superpotential interaction  $W_{A_{2k+1}} = \text{Tr}(X\tilde{X})^{k+1}$ . The dual [5] has gauge group  $SU((k+1)(N_f + N'_f) - N_f - N'_c) \times SU((k+1)(N_f + N'_f) - N'_f - N_c)$  with similar matter content and additional gauge singlets (corresponding to the mesons), and a dual analog of the  $W_{A_{2k+1}}$  superpotential. The superconformal window, where both dual descriptions are useful, is the range of  $(N_c, N'_c, N_f, N'_f)$  within which both the electric  $W_{A_{2k+1}}$ , as well as its analog in the magnetic dual, are both relevant. The  $a$ -maximization results allow us to determine this subspace of  $(x, x', n)$  parameter space, as a function of  $k$ . For large  $k$ , we find that this subspace is necessarily close to the  $x \approx x'$  slice, i.e.,  $N_c \approx N'_c$ .

The outline of this paper is as follows. In Section 2 we briefly review  $a$ -maximization, and apply it to determine the superconformal R-charges for the  $SU(N_c) \times SU(N'_c)$  example (1.1). We find that there are accidental symmetries arising from gauge invariant operators hitting the unitarity bound  $\Delta \geq 1$ , and use the procedure of [14] to take these into account during  $a$ -maximization. We especially consider the parameter slice  $x = x'$  (i.e.,  $N_c = N'_c$ ) for large  $x$  (i.e.,  $N_c \gg N_f$ ), and general  $n$ . In this slice and limit,  $R(X) \rightarrow 0$ . We account for the many accidental symmetries, associated with generalized mesons hitting their unitarity bound, in this limit (and numerically check that no baryon operators hit the unitarity bound.) As we discuss, if we set  $n \equiv N'_f/N_f = 1$ , our results should—and indeed do—coincide with those of [14].

In Section 3 we consider the theory (1.1) deformed by the superpotential  $W_{A_{2k+1}} = \text{Tr}(X\tilde{X})^{k+1}$ , and the dual description of [5] of that theory. We use  $a$ -maximization to determine the exact chiral operator dimensions in the dual of [5]. Combining these results with those of Section 2, we can determine the superconformal window region of  $(x, x', n)$  parameter space, for any given value of  $k$ , within which the  $W_{A_{2k+1}}$  superpotential of both the electric theory (1.1) and its dual are both relevant. For large  $k$ , the superconformal window is necessarily close to the parameter slice  $x \approx x'$ . We check numerically that, for all  $k$ , there is always a nonempty superconformal window region of parameter space in which the duality of [5] is applicable.

In Section 4 we consider Seiberg dualizing [1] one of the groups in (1.1), treating the other gauge group as a spectator. We will discuss analogs  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{C}$  of the possible RG fixed points in Fig. 1, when they exist, and when they are IR stable to perturbations. We find that there is a range of the parameters (1.18)  $(x, x', n)$  where an otherwise IR free magnetic gauge group is driven to be interacting for any nonzero gauge coupling of the “spectator” group. This is the phenomenon depicted in Fig. 3. As seen from the exact beta functions (1.5), positive anomalous dimensions are needed to turn a 1-loop IR irrelevant coupling into an IR relevant one. The superpotential of the Seiberg dual theory plays a crucial role here, together with the spectator gauge coupling, to get the positive anomalous dimensions needed for the effect of Fig. 3. The condition that the RG fixed point  $(C)$  of the original electric theory (1.1) be interacting, rather than flowing to a free magnetic dual, is that the fully interacting RG fixed point  $(\tilde{C})$  exists in the dual theory; this issue is analyzed by the dual analog of Fig. 5. When  $(\tilde{C})$  does exist, we expect that it is equivalent to the RG

fixed point ( $C$ ) of the electric description. We verify that their superconformal R-charges and central charges indeed agree.

In Section 5 we briefly conclude, and present a topic for further research.

In Appendix A, we note that all of the many duality examples of [5] have a nonzero superconformal window. The theories in [5] with a single gauge group (either  $SU(N_c)$ ,  $SO(N_c)$ , or  $Sp(N_c)$ ) and matter in a two-index representation (e.g., adjoint, symmetric, or antisymmetric) are shown in the large  $N_c$  limit to all have the same superconformal R-charges, and superconformal window, as that of  $SU(N_c)$  with an adjoint; we can directly borrow the results obtained in [14], with the central charge differing from that of [14] by just a fixed overall multiplicative factor. We also note that all of the other examples in [5], involving product groups, all also have superconformal R-charges and superconformal window that reduce to those obtained in [14] for a 1d slice of their parameter space, when we take all of the group ranks equal and all numbers of flavors equal (e.g., taking  $x = x'$  and  $n = 1$  in (1.18)). This suffices to show that all of the duality examples of [5] indeed have a nonempty superconformal window.

## 2. $a$ -maximization analysis for the $SU(N_c) \times SU(N'_c)$ theory (1.1)

The superconformal  $U(1)_R$  symmetry is uniquely determined by the condition that it maximizes  $a_{\text{trial}}(R) = 3 \text{Tr} R^3 - \text{Tr} R$  among all possible R-symmetries [9]. The constraints on the superconformal R-symmetry, e.g., that it's ABJ anomaly free, can either be implemented at the outset, before maximizing  $a(R)$  w.r.t.  $R$ , or via Lagrange multipliers  $\lambda$  [12].  $a$ -maximization with the Lagrange multipliers yields simple general expressions for the R-charges of the fields  $R_i(\lambda)$ , with the conjectured interpretation of giving the running R-charges along the RG flow to the RG fixed point [8,12,13].

For example, for a general  $\mathcal{N} = 1$  supersymmetric  $G \times G'$  gauge theory, with zero superpotential, we determine the running R-charges by maximizing with respect to the  $R_i$

$$\begin{aligned} a(\lambda, R) &= 3 \text{Tr} R^3 - \text{Tr} R - \lambda_G \text{Tr} G^2 R - \lambda_{G'} \text{Tr} G'^2 R \\ &= 2|G| + 2|G'| - \lambda_G T(G) - \lambda_{G'} T(G') \\ &\quad + \sum_i |\mathbf{r}_i| |\mathbf{r}'_i| \left[ 3(R_i - 1)^2 - 1 - \lambda_G \frac{T(\mathbf{r}_i)}{|\mathbf{r}_i|} - \lambda_{G'} \frac{T(\mathbf{r}'_i)}{|\mathbf{r}'_i|} \right] (R_i - 1), \end{aligned} \quad (2.1)$$

holding fixed the Lagrange multipliers  $\lambda_G$  and  $\lambda_{G'}$ , which enforce the constraints that  $U(1)_R$  not have ABJ anomalies,  $\text{Tr} GGR = \text{Tr} G'G'R = 0$ . This yields

$$\begin{aligned} R_i(\lambda) &= 1 - \frac{1}{3} \sqrt{1 + \lambda_G \frac{T(\mathbf{r}_i)}{|\mathbf{r}_i|} + \lambda_{G'} \frac{T(\mathbf{r}'_i)}{|\mathbf{r}'_i|}}, \quad \text{i.e.,} \\ \gamma_i &= 1 - \sqrt{1 + \lambda_G \frac{T(\mathbf{r}_i)}{|\mathbf{r}_i|} + \lambda_{G'} \frac{T(\mathbf{r}'_i)}{|\mathbf{r}'_i|}} \end{aligned} \quad (2.2)$$

where we used  $\gamma_i = 3R_i - 2$  for the anomalous dimensions. The conjecture is that these expressions can be interpreted as giving the anomalous dimensions along the entire RG

flow, with  $\lambda_G = g^2|G|/2\pi^2$  and  $\lambda_{G'} = g'^2|G'|/2\pi^2$  in some scheme. For the example (1.1) this gives the result (1.17). As in [12], using (2.2) in (2.1) yields a monotonically decreasing  $a$ -function  $a(\lambda) = a(\lambda, R(\lambda))$  along the RG flow. The values of  $\lambda_G^*$  and  $\lambda_{G'}^*$  at the IR fixed point are the extremal values of  $a(\lambda)$ ; since  $a(\lambda)$ 's gradients are proportional to the exact beta functions [8,12,13], this is equivalent to the conditions that the anomalous dimensions (1.17) yield zeros of the beta functions (1.5).

Whenever a gauge invariant operator  $M$  hits or appears to violate the unitarity bound  $R(M) \geq 2/3$ ,  $M$  becomes a decoupled free field. This affects the  $a$ -maximization analysis by introducing an additive correction to the quantity  $a(R)$  to be maximized [14] (this can be derived from the presence of an accidental  $U(1)_M$  symmetry, acting only on  $M$  [17]):

$$a_{\text{trial}}(R) \rightarrow a_{\text{trial}}(R) + \frac{1}{9} \dim(M)(2 - 3R(M))^2(5 - 3R(M)). \tag{2.3}$$

This correction can also be included in the  $a$ -maximization analysis with Lagrange multipliers [13], but it becomes unwieldy to do so when there are many such contributions from operators that hit the unitarity bound, as is the case in our examples for general values of the numbers of flavors and colors. For this reason, we will here do the  $a$ -maximization analysis at the RG fixed point, numerically, with the constraints implemented at the outset rather than via Lagrange multipliers.

We consider the example (1.1) in the range of the parameters (1.18) where it is possible to have the RG fixed point like (C) in Fig. 1, with both groups interacting. For asymptotic freedom of  $g = g' = 0$  in the UV, and to avoid having it be attractive in the IR, we take

$$3x - x' - 1 > 0 \quad \text{and} \quad 3x' - x - n > 0. \tag{2.4}$$

We also impose the condition (1.9), which is

$$-n < x - x' < 1 \quad (\text{stability}), \tag{2.5}$$

to have the origin of the moduli space of vacua not be dynamically disallowed. Finally, to have the points (A) and (B) not be IR attractive, as in Fig. 2, we impose (1.16),

$$(3x - 1)(x + n) - 3x'^2 > 0 \quad \text{and} \quad (3x' - n)(x' + 1) - 3x^2 > 0. \tag{2.6}$$

If either inequality of (2.6) is not satisfied, one or the other group is driven IR free, to RG fixed point (A) or (B), with anomalous dimensions and R-charges given by (1.10) or (1.11). When both (2.6) are satisfied, RG flows generally end up with both couplings interacting, which can end up being a RG fixed point (C), where (1.12) is satisfied. As mentioned in the introduction, we do *not* impose the naive conditions (1.13) to avoid IR free magnetic dual groups: as we will see in Section 4, the conditions (1.13) are generally dramatically modified by the dynamics of the other gauge group.

As always, the conditions in (1.12) for the exact beta functions to vanish are equivalent to requiring that the superconformal  $U(1)_R$  have vanishing ABJ anomalies with respect to all of the interacting gauge groups. So at (C) we have the two anomaly free conditions

$$\begin{aligned} N_c + N'_c(R(X) - 1) + N_f(R(Q) - 1) &= 0, \\ N'_c + N_c(R(X) - 1) + N'_f(R(Q') - 1) &= 0 \end{aligned} \tag{2.7}$$

to have  $\text{Tr} SU(N_c)^2 U(1)_R = \text{Tr} SU(N'_c)^2 U(1)_R = 0$ . Enforcing (2.7) at the RG fixed point, we can solve for  $R(X)$  and  $R(Q')$  in terms of  $y \equiv R(Q)$

$$\begin{aligned} R(X) &= \frac{1-y}{x'} + 1 - \frac{x}{x'}, \\ R(Q') &= \frac{x}{nx'}(y-1) + \frac{x^2}{nx'} - \frac{x'}{n} + 1, \end{aligned} \quad (2.8)$$

where the parameters  $(x, x', n)$  of the theory are the ratios (1.18). We determine the superconformal R-charge  $y(x, x', n)$  by  $a$ -maximization (in the single variable  $y$ ).

Imposing (2.8), we compute  $a_{\text{trial}}(R) = 3 \text{Tr} R^3 - \text{Tr} R$  from the spectrum (1.1) to be

$$\begin{aligned} a^{(0)}/N_f^2 &= 2x^2 + 2x'^2 + 6x(y-1)^3 - 2x(y-1) \\ &+ 6nx' \left[ \frac{x}{nx'}(y-1) + \frac{x^2}{nx'} - \frac{x'}{n} \right]^3 - 2nx' \left[ \frac{x}{nx'}(y-1) + \frac{x^2}{nx'} - \frac{x'}{n} \right] \\ &+ 6xx' \left[ \frac{1-y}{x'} - \frac{x}{x'} \right]^3 - 2xx' \left[ \frac{1-y}{x'} - \frac{x}{x'} \right]. \end{aligned} \quad (2.9)$$

We then compute the superconformal R-charges by locally maximizing (2.9) w.r.t.  $y$ , for general fixed values of the parameters  $(x, x', n)$ ; we will denote the solution as  $y^{(0)}(x, x', n)$ . The superscript  $(0)$  is a reminder that these results are valid only in the range of  $(x, x', n)$  in which no gauge invariant operators have hit the unitarity bound; otherwise (2.9) will require corrections as in (2.3). Within this range of  $(x, x', n)$ , we can also use the Lagrange multiplier approach. Imposing (2.7) with Lagrange multipliers yields the simple expressions (1.17), which can be interpreted as the running R-charges along the RG flow, coinciding with the R-charges obtained above from  $y^{(0)}(x, x', n)$  at the RG fixed point.

The first gauge invariant, chiral, composite operators  $\mathcal{O}$  to hit the unitarity bound  $R(\mathcal{O}) \geq 2/3$  are the mesons  $M \equiv Q\tilde{Q}$  or  $M' \equiv Q'\tilde{Q}'$ .  $M$  hits the unitarity bound when  $y(x, x', n) \leq 1/3$ ; using (1.6), this happens when  $Q$  has the large, negative anomalous dimension,  $\gamma_Q(x, x', n) < -1$ , which can only happen if  $(x, x', n)$  are such that the RG fixed point values of the gauge couplings are large. The above result  $y^{(0)}(x, x', n)$  is valid within the range of  $(x, x', n)$  where neither  $M$  or  $M'$  have hit their unitarity bound, i.e., the range of  $(x, x', n)$  where  $y^{(0)}(x, x', n) > 1/3$  and where  $R(Q') \geq 1/3$ , with  $R(Q')$  computed from  $y^{(0)}(x, x', n)$  via (2.8). Outside of this range, the above  $a$ -maximization analysis has to be supplemented, as in (2.3), to account for the accidental symmetries associated with operators hitting the unitarity bound and becoming free fields.

For general  $(x, x', n)$  the gauge operators that will hit the unitarity bound are:

$$\begin{aligned} M_j &= Q(\tilde{X}X)^{j-1}\tilde{Q}, & M'_j &= Q'(\tilde{X}X)^{j-1}\tilde{Q}', \\ P_j &= Q(\tilde{X}X)^{j-1}\tilde{X}\tilde{Q}', & \tilde{P}_j &= \tilde{Q}X(\tilde{X}X)^{j-1}Q'. \end{aligned} \quad (2.10)$$

For every integer  $j \geq 1$ , there are  $N_f N'_f$  mesons  $P_j$  and  $\tilde{P}_j$ ,  $N_f^2$  mesons  $M_j$ , and  $N_f'^2$  mesons  $M'_j$ . We verified that it is self-consistent to assume that the baryon operators do not hit the unitarity bound; also, gauge invariants without fundamentals, such as  $\text{Tr}(X\tilde{X})^j$ ,

contribute negligibly in the large  $N$  limit. The quantity to maximize in general is then

$$\begin{aligned}
 a^{(p)}/N_f^2 &= \tilde{a}^{(0)}/N_f^2 + \frac{2n}{9} \sum_{j=1}^{p_P} [2 - 3R(P_j)]^2 [5 - 3R(P_j)] \\
 &+ \frac{1}{9} \sum_{j=1}^{p_M} [2 - 3R(M_j)]^2 [5 - 3R(M_j)] \\
 &+ \frac{n^2}{9} \sum_{j=1}^{p_{M'}} [2 - 3R(M'_j)]^2 [5 - 3R(M'_j)], \tag{2.11}
 \end{aligned}$$

where  $p$  denotes  $\{p_P, p_M, p_{M'}\}$ , with  $p_P = p_{\tilde{P}}$  the number of  $P$  (and also  $\tilde{P}$  type) mesons which have hit the unitarity bound. The quantities such as  $R(M_j)$  in (2.11) are given by, e.g.,  $R(M_j) = R[Q(X\tilde{X})^{j-1}\tilde{Q}] = 2y + 2(j-1)R(X)$ , with  $R(X)$  given by (2.8); so the corrections in (2.11) are complicated functions of the variable  $y$  that we're maximizing with respect to, along with the parameters  $(x, x', n)$ . Maximizing (2.11) yields  $y^{(p)}(x, x', n)$ , and  $y(x, x', n)$  is obtained by pasting these together, with the appropriate values of  $p$  depending on  $(x, x', n)$ , increasing, e.g.,  $p_M$  every time another value of  $j$  is obtained such that  $R(M_j)$  hits  $2/3$ . We numerically implemented this process to obtain  $y(x, x', n)$ , but it's difficult to produce an illuminating plot of a function of three variables.

Let us discuss some qualitative aspects of our results. From  $y(x, x', n)$  we can compute the anomalous dimensions  $\gamma_Q$ ,  $\gamma_X$ , and  $\gamma_{Q'}$ , using (1.6), and we find that all are negative within the range (2.6) where the RG fixed point ( $C$ ) can exist. This is to be expected, since our theory (1.1) has only gauge interactions, and no superpotential (gauge interactions yield negative anomalous dimension, and superpotentials yield positive contributions to the anomalous dimensions). When either inequality (2.6) is violated, the theory flows not to ( $C$ ), but rather to RG fixed points ( $A$ ) or ( $B$ ), as in Fig. 2 and the above  $a$ -maximization analysis, which assumed in (2.7) that both groups are interacting, is inapplicable. At the boundaries of (2.6), where either inequality is saturated, our  $a$ -maximization results properly reduce to (1.10) or (1.11).

It is interesting to note that there is a 1d slice of the  $(x, x', n)$  parameter space, given by  $x = x'$  and  $n = 1$ , for which the  $a$ -maximization analysis of this theory coincides with that of [14] for  $SU(N_c)$  SQCD with  $N_f$  fundamentals and an added adjoint. In this slice, for every contribution to the quantity  $a_{\text{trial}}$  to maximize in [14], we have here two analogous matter fields in the spectrum of our theory, with the same R-charges: twice as many gauge fields, twice as many fundamentals ( $Q$  and  $Q'$  and conjugate), the  $X$  and  $\tilde{X}$  fields contribute as two adjoints (using (2.8) for  $x = x'$  and  $n = 1$ ), and all the mesons (2.10) hitting the unitarity bound map to two copies of the mesons hitting the unitarity bound in [14]. Thus, for  $x = x'$  and  $n = 1$ , (2.11) is exactly twice the expression obtained in [14] for the theory considered there. Since  $a_{\text{trial}}$  is the same function of  $y$ , up to a factor of 2, it is maximized by the same solution  $y_{\text{KPS}}(x)$  obtained by the analysis of [14]. So the superconformal R-charges of our theory (1.1) satisfy  $y(x, x', n)|_{x=x', n=1} = y_{\text{KPS}}(x)$ .

For  $x \approx x'$  taken to be very large, i.e.,  $N_c \approx N'_c \gg N_f$ , the superconformal R-charge of the field  $X$  goes to zero,  $R(X) \rightarrow 0$ , for arbitrary fixed values of  $n \equiv N'_f/N_f$ , as seen from (2.8), and the fact that  $y$  remains finite in this limit. The asymptotic value  $y_{\text{as}}(n)$  in

this  $x = x' \rightarrow \infty$  limit is determined by our numerical  $a$ -maximization analysis, but it can also be approximated analytically. Because many mesons contribute to the sums (2.11), the sums can be approximated as integrals (following [14])

$$\frac{1}{9} \sum_{j=1}^p [2 - 3R_j]^2 [5 - 3R_j] \approx \frac{1}{27\beta} \int_0^{2-3\alpha} u^2 (3+u) du = \frac{1}{18\beta} (2-3\alpha)^3 \left(1 - \frac{1}{2}\alpha\right), \quad (2.12)$$

where  $\alpha$  and  $\beta$  are defined by  $R_j \equiv \alpha + (j-1)\beta$  (and  $u \equiv 2 - 3R_j$ ). The upper limit  $p$  in the sum is solved for by setting  $R_p = \alpha + (p-1)\beta$  equal to  $2/3$ . Applying (2.12) to the sums in (2.11),  $\beta = R(X\tilde{X}) = 2R(X)$  for all three, and for the first sum in (2.11)  $\alpha = R(Q\tilde{X}\tilde{Q}') = y + R(Q') + R(X)$ , while for the second and third  $\alpha = 2y$  and  $\alpha = 2R(Q')$  respectively; here,  $R(X)$  and  $R(Q')$  are to be written in terms of the variable  $y$  and the parameters  $(x, x', n)$  using (2.8).

Setting  $x = x'$  and taking both large, (2.11) then becomes

$$a/N_f^2 \simeq 6x \left[1 + \frac{1}{n^2}\right] (y-1)^3 - 20x(y-1) + \frac{x}{36} [2 - 6y]^3 + \frac{xn}{36} \left[\frac{6}{n}(1-y) - 4\right]^3 + \frac{xn}{36} \left(1 + \frac{1}{n}\right) \left[3(1-y) \left(1 + \frac{1}{n}\right) - 4\right]^3. \quad (2.13)$$

The first line of (2.13) is the large  $x = x'$  limit of (2.9), and the second line contains the meson sums of (2.11), evaluated using (2.12). Note that every term in (2.13) is linear<sup>2</sup> in  $x$  in this limit, so maximizing (2.13) w.r.t.  $y$  yields an asymptotic value,  $y_{\text{as}}(n)$ , that's independent of  $x$  in this limit of large  $x = x'$ . This asymptotic value depends on  $n \equiv N'_f/N_f$ , and the conditions (2.6) needed for neither gauge coupling to drive the other to be IR free here require  $n$  to lie in the range

$$3 > n > \frac{1}{3} \quad \text{for } x = x' \rightarrow \infty. \quad (2.14)$$

As expected from the discussion above, for  $n = 1$  (2.13) reduces to twice the expression obtained in the large  $x$  analysis of [14], and for  $n = 1$  our expression for  $y_{\text{as}}(n)$  coincides with the asymptotic large  $x$  value of  $y$  obtained there:  $y_{\text{as}}(n)|_{n=1} = (\sqrt{3} - 1)/3$ .

The asymptotic value  $y_{\text{as}}(n)$  will be used in the next section to find the minimal value of  $x \approx x'$  needed for the superpotential  $\Delta W_{A_{2k+1}} \equiv \text{Tr}(X\tilde{X})^{k+1}$  to be a relevant deformation of RG fixed point (C) in the limit of large  $k$ . This gives one side of the superconformal window for the duality of [5] (see Fig. 7). We have also checked, including away from the strict  $x = x'$  limit, that  $R(X)$  is nowhere negative, i.e., using (2.8) that the  $a$ -maximizing solution  $y(x, x', n)$  satisfies  $1 - y(x, x', n) + x' - x > 0$ . This is important for the self-consistency of our analysis since, if  $R(X)$  were negative, baryonic operators, formed by

<sup>2</sup> The fact that the expression in (2.13) grows for large  $x$  only linearly is a check of the conjectured  $a$ -theorem. Any greater exponent would've led to  $a$ -theorem violations, e.g., along a Higgs flat direction where  $X\tilde{X}$  gets an expectation value, higgsing each  $SU(N_c)$  gauge group factor to products of similar factors. This flat direction is analogous to that considered in a non-trivial check of the  $a$ -theorem in [14] (where it is also pointed out that the sub-leading constant term must be—and indeed is—negative for the  $a$ -theorem to hold for this Higgs RG flow).

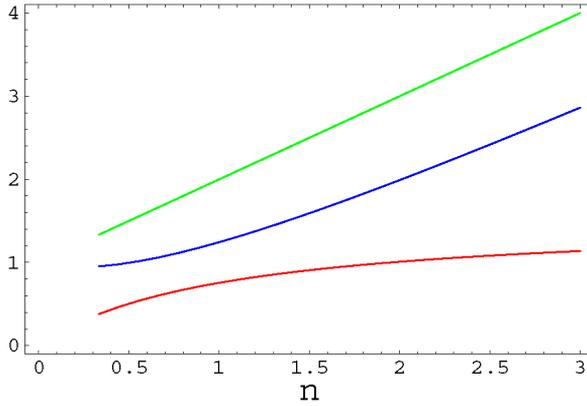


Fig. 7. The  $x/k$  conformal window: the upper line is the stability bound  $1 + n$ , the middle line is  $1 + n\tilde{y}_{\text{as}}(n)$  and the lower line is  $1 - y_{\text{as}}(n)$ .

dressing the quarks with many powers of  $X\tilde{X}$ , would hit the unitarity bound and lead to additional contributions analogous to (2.3).

### 3. The theory with $W_{A_{2k+1}} = \text{Tr}(X\tilde{X})^{k+1}$ and its dual

In [5] it was proposed that our theory (1.1), together with a superpotential  $W_{A_{2k+1}} = \text{Tr}(X\tilde{X})^{k+1}$  has a dual given by a similar theory:

gauge group:	$SU(\tilde{N}_c) \times SU(\tilde{N}'_c)$		
matter:	$Y \oplus \tilde{Y}$	$(\square, \bar{\square}) \oplus (\bar{\square}, \square)$ ,	
	$q_f \oplus \tilde{q}_{\tilde{f}}$	$(\square, \mathbf{1}) \oplus (\bar{\square}, \mathbf{1})$	$(f, \tilde{f} = 1, \dots, N'_f)$ ,
	$q'_{f'} \oplus \tilde{q}'_{\tilde{f}'}$	$(\mathbf{1}, \square) \oplus (\mathbf{1}, \bar{\square})$	$(f', \tilde{f}' = 1, \dots, N_f)$ ,

(3.1)

where  $\tilde{N}_c = (k + 1)(N_f + N'_f) - N_f - N'_c$  and  $\tilde{N}'_c = (k + 1)(N_f + N'_f) - N'_f - N_c$ . There are also singlets  $P_j$ , for  $j = 1, \dots, k$ , and  $M_j$  and  $M'_j$  for  $j = 1, \dots, k + 1$ , with superpotential

$$\begin{aligned}
 W = & \text{Tr}(Y\tilde{Y})^{k+1} + \sum_{j=1}^k [P_j q \tilde{Y} (Y\tilde{Y})^{k-j} \tilde{q}' + \tilde{P}_j \tilde{q}' (Y\tilde{Y})^{k-j} Y q] \\
 & + \sum_{j=1}^{k+1} [M_j q' (\tilde{Y}Y)^{k-j+1} \tilde{q}' + M'_j \tilde{q} (Y\tilde{Y})^{k-j+1} q].
 \end{aligned}$$
(3.2)

The first term is the dual analog of the  $W_{A_{2k+1}}$  superpotential, and the remaining terms are analogs of the  $Mq\tilde{q}$  superpotential in Seiberg duality [1].

The duality is useful within a superconformal window, which is the range of the parameters  $(x, x', n)$ , where the superpotential  $W_{A_{2k+1}}$  and its dual analog in (3.2) are both

relevant in controlling the IR dynamics, i.e., when the following conditions are satisfied:

$$(i) \quad R(X) = \frac{1 - y(x, x', n)}{x'} + 1 - \frac{x}{x'} < \frac{1}{k+1}, \quad (3.3)$$

$$(ii) \quad R(Y) = \frac{1 - \tilde{y}(\tilde{x}, \tilde{x}', \tilde{n})}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'} < \frac{1}{k+1}, \quad (3.4)$$

$$(iii) \quad (k+1)(N_f + N'_f) - N_f - N'_c > 0, \quad (k+1)(N_f + N'_f) - N'_f - N_c > 0. \quad (3.5)$$

If (i) is not satisfied,  $W_{\text{elec}} = \text{Tr}(X\tilde{X})^{k+1}$  is an irrelevant deformation of RG fixed point (C), and thus  $W_{\text{elec}} \rightarrow 0$  in the IR; this fact is obscured in the magnetic dual description. Likewise, if (ii) is not satisfied, one should use the magnetic description, with the coefficient of the  $\text{Tr}(Y\tilde{Y})^{k+1}$  superpotential term flowing to zero in the IR; the electric description then does not readily describe the true RG fixed point. Finally, condition (iii) is the “stability bound”, needed for the RG fixed point to exist (and for the dual groups (3.1) to have positive ranks): if (3.5) are not satisfied, the electric theory (1.1) with  $W_{A_{2k+1}}$  superpotential dynamically generates a superpotential, spoiling conformal invariance.

Using the results of the previous subsection, we can now determine the range of  $(x, x', n)$  in which condition (3.3) is satisfied, for  $\text{Tr}(X\tilde{X})^{k+1}$  to be relevant. The  $k = 0$  case is a mass term and (3.3) is then always satisfied (starting with  $W = 0$  at (C), all fields have  $R \leq 2/3$ ). For all  $k > 1$ , (3.3) is only satisfied in subspaces of the  $(x, x', n)$  parameter space for which the RG fixed point is at sufficiently strong enough coupling to give  $X$  a sufficiently negative anomalous dimension. The larger  $k$  is, the more strongly coupled the RG fixed point must be in order to have (3.3) be satisfied. For arbitrarily large  $k$ , there is a nonempty range of  $(x, x', n)$  in which (3.3) is satisfied: as we saw in the previous subsection,  $R(X) \rightarrow 0$  in parts of the parameter space. Let us consider, for example, the parameter slice  $x = x'$  and ask when (3.3) is satisfied for large values of  $k$ . Since satisfying (3.3) for large  $k$  requires large  $x$ , we can replace  $y(x, x', n)$  in (3.3) with the asymptotic value  $y_{\text{as}}(n)$  obtained by maximizing (2.13). Then the condition (3.3) for the superpotential to be relevant becomes

$$x > x_{\min}(n) \approx k(1 - y_{\text{as}}(n)) \quad \text{for } k \gg 1. \quad (3.6)$$

The above analysis of the electric theory gives one edge of the superconformal window of the parameters  $(x, x', n)$  for the duality of [5]. The other edge of the window is obtained by determining the range of these parameters in which (3.4) is satisfied, for the  $W_{A_{2k+1}}$  superpotential to be relevant in the magnetic theory. Again, we simplify the analysis by taking the numbers of flavors and colors in the electric theory to be large, so the same is true in the magnetic theory. The ratios on the magnetic side are defined to be  $\tilde{x} \equiv \tilde{N}_c/N'_f$ ,  $\tilde{x}' \equiv \tilde{N}'_c/N'_f$ , and  $\tilde{n} \equiv N_f/N'_f$ , which are related to the electric ones (1.18) as

$$\begin{aligned} \tilde{x} &= (k+1)(1+n^{-1}) - n^{-1} - x'n^{-1}, \\ \tilde{x}' &= (k+1)(1+n^{-1}) - 1 - xn^{-1}, \quad \tilde{n} = n^{-1}. \end{aligned} \quad (3.7)$$

In the magnetic theory (3.1), we assume that both magnetic gauge groups remain interacting. The superconformal R-charge is constrained by the magnetic analog of (2.7), that

it be anomaly free w.r.t. both gauge groups. As in (2.8), we can use this to solve for  $R(Y) = R(\tilde{Y})$  and  $R(q') = R(\tilde{q}') \equiv \tilde{y}'$  in terms of  $R(q) = R(\tilde{q}) \equiv \tilde{y}$ :

$$R(Y) = \frac{1 - \tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'}, \quad \tilde{y}' = \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y} - 1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} + 1. \quad (3.8)$$

The contribution to the magnetic  $\tilde{a}_{\text{trial}}$  from the fields in (3.1) is

$$\begin{aligned} \tilde{a}^{(0)}/N_f'^2 &= 2\tilde{x}^2 + 2\tilde{x}'^2 + 6\tilde{x}(\tilde{y} - 1)^3 - 2\tilde{x}(\tilde{y} - 1) \\ &+ 6\tilde{n}\tilde{x}' \left[ \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y} - 1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} \right]^3 - 2\tilde{n}\tilde{x}' \left[ \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y} - 1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} \right] \\ &+ 6\tilde{x}\tilde{x}' \left[ \frac{1 - \tilde{y}}{\tilde{x}'} - \frac{\tilde{x}}{\tilde{x}'} \right]^3 - 2\tilde{x}\tilde{x}' \left[ \frac{1 - \tilde{y}}{\tilde{x}'} - \frac{\tilde{x}}{\tilde{x}'} \right]. \end{aligned} \quad (3.9)$$

To this we must add the contributions from the singlets  $P_i, \tilde{P}_j, M_i, M'_j$ . Each of these fields couples only via a superpotential term in (3.2) and, initially taking that singlet's R-charge to be  $2/3$ , that superpotential term may be relevant or irrelevant in the IR. If it is relevant, then the singlet's R-charge is determined by the requirement that the superpotential term have  $R = 2$  total in the IR. If it is irrelevant, the singlet is a free field, with  $R = 2/3$ . If we assume that the last  $p_P$   $P_j$ 's (and  $\tilde{P}_j$ 's), the last  $p_M$   $M_j$ 's, and the last  $p_{M'}$   $M'_j$ 's are interacting, then the quantity to maximize is

$$\begin{aligned} \tilde{a}^{(p)}/N_f'^2 &= \tilde{a}^{(0)}/N_f'^2 + \frac{2\tilde{n}}{9} \sum_{j=1}^{p_P} (2 - 3\alpha_j^P)^2 (5 - 3\alpha_j^P) \\ &+ \frac{\tilde{n}^2}{9} \sum_{j=1}^{p_M} (2 - 3\alpha_j^M)^2 (5 - 3\alpha_j^M) + \frac{1}{9} \sum_{j=1}^{p_{M'}} (2 - 3\alpha_j^{M'})^2 (5 - 3\alpha_j^{M'}) \\ &+ \frac{4\tilde{n}}{9} (k - 2p_P) + \frac{2\tilde{n}^2}{9} (k + 1 - 2p_M) + \frac{2}{9} (k + 1 - 2p_{M'}), \end{aligned} \quad (3.10)$$

where we define

$$\begin{aligned} \alpha_j^P &\equiv \tilde{y} + \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y} - 1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} + 1 + (2j - 1) \left[ \frac{1 - \tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'} \right], \\ \alpha_j^M &\equiv 2 \left[ \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y} - 1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} + 1 \right] + 2(j - 1) \left[ \frac{1 - \tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'} \right], \\ \alpha_j^{M'} &\equiv 2\tilde{y} + 2(j - 1) \left[ \frac{1 - \tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'} \right]. \end{aligned} \quad (3.11)$$

The additional terms in (3.10) are the contributions from the singlets (see Section 6 of [17] for a detailed discussion of an analogous example). The full solution  $\tilde{y}(\tilde{x}, \tilde{x}', \tilde{n})$  is obtained by patching together the maximizing solutions of (3.10) with the appropriate values of  $p_M, p_{M'}$ , and  $p_P$ , depending on  $(\tilde{x}, \tilde{x}', \tilde{n})$ , given by the largest integer  $j$ 's such that the  $\alpha_j$  in (3.11) satisfy  $\alpha_j \leq 4/3$  (where the corresponding superpotential term becomes irrelevant).

For any given value of  $k$ , we can use the numerical  $a$ -maximization analysis to determine the range of  $(\tilde{x}, \tilde{x}', \tilde{n})$ , and thus the range of electric parameters  $(x, x', n)$ , in which

the condition (3.4) for  $\Delta W = \text{Tr}(Y\tilde{Y})^{k+1}$  to be relevant is satisfied. Using (3.7), we will express this in terms of the electric parameters  $(x, x', n)$ . To be concrete, let us consider the limit of large  $k$ . The condition (3.3) on the electric side gave the inequality (3.6), which shows that  $x \approx x'$  must get large, linearly in  $k$ , in the large  $k$  limit, while  $n$  is restricted to the range (2.14). Then (3.7) gives  $\tilde{x} \approx \tilde{x}' \approx k(1+n^{-1}) - xn^{-1}$ , and the condition (3.4) will require  $\tilde{x}$  to also be large. In this limit of large  $\tilde{x} \approx \tilde{x}'$ , (3.11) becomes

$$\begin{aligned} \tilde{a}/N_f'^2 &\simeq 6\tilde{x} \left[ 1 + \frac{1}{\tilde{n}^2} \right] (\tilde{y} - 1)^3 - 20\tilde{x}(\tilde{y} - 1) \\ &+ \frac{\tilde{x}\tilde{n}}{36} \left( 1 + \frac{1}{\tilde{n}} \right) \left[ 3(1 - \tilde{y}) \left( 1 + \frac{1}{\tilde{n}} \right) - 4 \right]^3 \\ &+ \frac{4\tilde{x}}{9} \left[ \frac{\tilde{n}\tilde{y}}{1 - \tilde{y}} - 1 \right] + \frac{\tilde{x}\tilde{n}}{36} \left[ \frac{6}{\tilde{n}}(1 - \tilde{y}) - 4 \right]^3 + \frac{2\tilde{n}\tilde{x}}{9} \left[ \frac{\tilde{n}}{1 - \tilde{y}} - 2 \right] \\ &+ \frac{\tilde{x}}{36} [2 - 6\tilde{y}]^3 + \frac{2\tilde{x}(2\tilde{y} - 1)}{9(1 - \tilde{y})} + \frac{2k}{9}(\tilde{n} + 1)^2. \end{aligned} \quad (3.12)$$

The first two terms are the large  $\tilde{x} \approx \tilde{x}'$  limit of (3.9), and the rest are the remaining terms in (3.10), with sums evaluated using (2.12) (modifying the lower limit of the integral (2.12) to be  $2 - 3R = -2$ , rather than 0, since  $\alpha_j = 4/3$  is the limit where the superpotential term becomes irrelevant). Maximizing (3.12) with respect to  $\tilde{y}$  gives  $\tilde{y}_{\text{as}}(\tilde{n})$ .

The condition (3.4) for  $\text{Tr}(Y\tilde{Y})^{k+1}$  to be relevant can then be written for  $k \gg 1$  as

$$\frac{(1 - \tilde{y}_{\text{as}})n}{k(n+1) - x} < \frac{1}{k}. \quad (3.13)$$

Rearranging and combining with (3.6), the electric and magnetic conditions (3.3) and (3.4) can be written together for  $k \gg 1$  as

$$1 - y_{\text{as}}(n) < \frac{x}{k} < 1 + n\tilde{y}_{\text{as}}(n) \quad \text{for } x \approx x' \text{ and } k \gg 1. \quad (3.14)$$

For the duality (3.1) of [5] to be useful, and the superconformal window be nonempty, the inequalities at the two ends of (3.14) had better be compatible with each other. This is verified to indeed be the case, as seen in the plots in Fig. 7, for the entire allowed range (2.14) of  $n$ . The vertical axis of Fig. 7 gives the allowed values of  $x/k$ , for a given value of  $n$ , and the superconformal window is the region between the lower two curves on Fig. 7. There is also the stability bound conditions (3.5), which in our  $k \gg 1$  limit, with  $x \approx x'$  scaling linearly in  $k$ , can both be written as  $x/k < 1 + n$ . In the plot of Fig. 7, the upper line is the stability bound, and the values of  $x/k$  in the superconformal window, between the lower two curves, is indeed always safely below the stability bound for the entire allowed range of  $n$ . All of these successes can be viewed as reassuring checks of the duality of [5].

For the case  $x = x', n = 1$ , the conformal window plotted in Fig. 7 coincides with that obtained in [14] for SQCD with an adjoint, for the reason discussed above.

#### 4. Dualizing one gauge group

With product gauge groups, such as (1.1), we can consider dualizing one of the gauge groups, treating the other gauge group as a spectator. The validity of doing this deserves scrutiny, because duality is only exact at the IR fixed point. Dualizing away from the extreme IR can be potentially justified if the dualized group’s dynamical scale is far above that of the other “spectator” group,  $\Lambda_d \gg \Lambda_s$  (and then holomorphic quantities can be analytically continued in  $\Lambda_d/\Lambda_s$ ) or if one group gets strong while the other gets weak in the IR (as in string theory examples, see, e.g., [23,24]).

Let us consider the  $SU(N_c) \times SU(N'_c)$  theory (1.1), and consider dualizing  $SU(N_c)$ , supposing that it is valid to treat  $SU(N'_c)$  as a weakly gauged flavor symmetry spectator. We will suppose that the original electric theory satisfies (2.6), so that both electric couplings RG flow to nonzero values. (If the second inequality (2.6) is violated,  $SU(N'_c)$  is IR free, and thus reasonably treated as a spectator. But if the first inequality in (2.6) is violated then  $SU(N_c)$  is actually IR free, and the validity of dualizing it with  $SU(N'_c)$  treated as a spectator is questionable.) The  $SU(N_c)$  group has  $N_f + N'_c$  flavors and its Seiberg [1] dual has  $SU(\tilde{N}_c)$  gauge group, with  $\tilde{N}_c \equiv N_f + N'_c - N_c$ , with  $N_f + N'_c$  flavors of dual quarks and  $(N_f + N'_c)^2$  singlet mesons. The stability condition (2.5) ensures that  $\tilde{N}_c > 0$ . Gauging  $SU(N'_c)_{\text{mag}}$ , with the subscript as a reminder that its spectrum now differs from that of (1.1), the dual is

gauge group:	$SU(\tilde{N}_c) \times SU(N'_c)_{\text{mag}}$	
matter:	$Y \oplus \tilde{Y}$	$(\square, \bar{\square}) \oplus (\bar{\square}, \square),$
	$q_f \oplus \tilde{q}_{\tilde{f}}$	$(\square, \mathbf{1}) \oplus (\bar{\square}, \mathbf{1}) \quad (f, \tilde{f} = 1, \dots, N_f),$
	$Q'_{f'} \oplus \tilde{Q}'_{\tilde{f}'}$	$(\mathbf{1}, \square) \oplus (\mathbf{1}, \bar{\square}) \quad (f', \tilde{f}' = 1, \dots, N'_{f'}),$
	$F'_{n'} \sim X \tilde{Q} \oplus \text{c.c.}$	$(\mathbf{1}, \square) \oplus (\mathbf{1}, \bar{\square}) \quad (n', \tilde{n}' = 1, \dots, N_f),$
	$M_{f, \tilde{g}} \sim Q \tilde{Q}$	$(\mathbf{1}, \mathbf{1}) \quad (f, \tilde{g} = 1, \dots, N_f),$
	$\Phi \sim X \tilde{X}$	$(\mathbf{1}, \text{Adj}) \oplus (\mathbf{1}, \mathbf{1}),$

with the superpotential of [1] yielding

$$W = Mq\tilde{q} + YF'\tilde{q} + \tilde{Y}q\tilde{F}' + \Phi Y\tilde{Y}. \tag{4.2}$$

The one loop beta function coefficients of the electric theory (1.1) were

$$b_1 = 3N_c - N'_c - N_f \quad \text{and} \quad b'_1 = 3N'_c - N_c - N'_f \tag{4.3}$$

(writing  $b_1 > 0$  if asymptotically free), and those of the dual (4.1) are

$$b_1^{\text{mag}} = 2N_f + 2N'_c - 3N_c \quad \text{and} \quad b'_1{}^{\text{mag}} = N'_c + N_c - 2N_f - N'_f. \tag{4.4}$$

Note that  $b_1^{\text{mag}}$  differs from  $b'_1$ , because the  $SU(N'_c)_{\text{mag}}$  fields in (4.1) differ from those of the original  $SU(N_c) \times SU(N'_c)$  theory (1.1); in fact,  $b'_1 - b_1^{\text{mag}} = 2(N_f + N'_c - N_c) = 2\tilde{N}_c > 0$ , so  $SU(N'_c)_{\text{mag}}$  is always less asymptotically free than the electric  $SU(N'_c)$  in the UV. Ignoring the  $SU(N'_c)_{\text{mag}}$  dynamics, we’d conclude that the magnetic  $SU(\tilde{N}_c)$  is IR free if  $N_f + N'_c < \frac{3}{2}N_c$ ; we will discuss here how the  $SU(N'_c)_{\text{mag}}$  dynamics can dramatically affect when the magnetic group is actually IR free.

The important quantities for the IR dynamics are the exact beta functions for the theory (4.1), which are

$$\begin{aligned} \beta_{g_{\text{mag}}} &= -\frac{3g_{\text{mag}}^3 f}{16\pi^2} \text{Tr} SU(\tilde{N}_c)^2 R, \\ \beta_{g'_{\text{mag}}} &= -\frac{3g_{\text{mag}}^3 f'}{16\pi^2} \text{Tr} SU(N'_c)^2_{\text{mag}} R, \end{aligned} \tag{4.5}$$

where again  $f$  and  $f'$  are unimportant, positive, scheme dependent factors. The beta functions (4.5) can be written in the usual NSVZ form using (1.6), which gives

$$\begin{aligned} 3 \text{Tr} SU(\tilde{N}_c)^2 R &= b_1^{\text{mag}} + N'_c \gamma_Y + N_f \gamma_q, \\ 3 \text{Tr} SU(N'_c)^2_{\text{mag}} R &= b_1^{\text{mag}} + \tilde{N}_c \gamma_Y + N'_f \gamma_{Q'} + N_f \gamma_{F'} + N'_c \gamma_{\Phi}. \end{aligned} \tag{4.6}$$

As with the electric theory, the dual (4.1) has three possible RG fixed points,

$$\begin{aligned} (\tilde{A}) \quad g_{\text{mag}*} &\neq 0, \quad g'_{\text{mag}*} = 0, \\ &\text{i.e., } SU(N'_c)_{\text{mag}} \text{ free and } \text{Tr} SU(\tilde{N}_c)^2 R|_{\tilde{A}} = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\tilde{B}) \quad g_{\text{mag}*} &= 0, \quad g'_{\text{mag}*} \neq 0, \\ &\text{i.e., } SU(\tilde{N}_c) \text{ free and } \text{Tr} SU(N'_c)^2_{\text{mag}} R|_{\tilde{B}} = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\tilde{C}) \quad g_{\text{mag}*} &\neq 0, \quad g'_{\text{mag}*} \neq 0 \\ &\text{so } \text{Tr} SU(\tilde{N}_c)^2 R|_{\tilde{C}} = \text{Tr} SU(N'_c)^2_{\text{mag}} R|_{\tilde{C}} = 0, \end{aligned} \tag{4.9}$$

which are depicted in Fig. 8. RG fixed point  $(\tilde{A})$  is simply the Seiberg dual description of RG fixed point  $(A)$  of the original electric theory (with  $SU(N'_c)$  part of the global flavor symmetry). We expect that RG fixed point  $(\tilde{C})$ , when it exists, is an equivalent, dual description of the SCFT at RG fixed point  $(C)$  of the original electric theory (1.1). The qualifier “when it exists” is because, as in the electric description, the RG flow may look like that of Fig. 2 rather than that of Fig. 1. In the electric description, the condition for the RG fixed point  $(C)$  to exist is (1.16). We will determine its analog in the magnetic theory (4.1), for  $(\tilde{C})$  to exist, by analyzing the IR stability of the RG fixed points  $(\tilde{A})$  and  $(\tilde{B})$  to

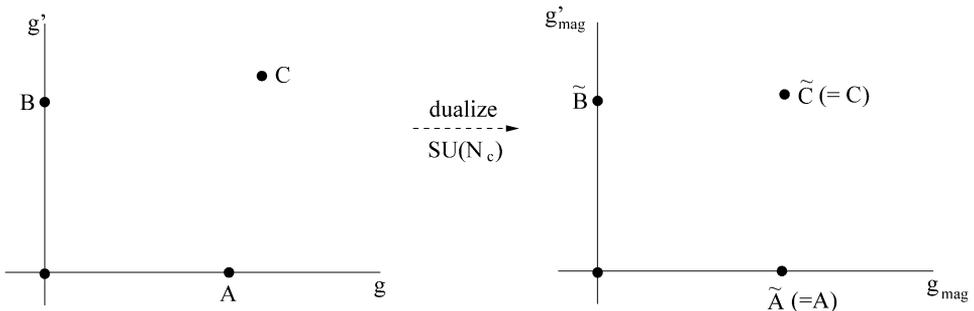


Fig. 8. The process of dualizing one group.

small nonzero perturbations in the couplings that are set to zero in (4.7) and (4.8), in analogy with Fig. 5. We will find that, for a particular range of flavors and colors, the theory (4.1) with superpotential (4.2) realizes the RG flow depicted in Fig. 3: even if the one-loop beta function might suggest that  $SU(\tilde{N}_c)$  is IR free, it can be driven to be interacting by the interactions of the other gauge group and the superpotential.

Finally, we note that RG fixed point ( $\tilde{B}$ ) is *not* the dual of RG fixed point ( $B$ ): as duality exchanges strong and weak coupling, the RG fixed point ( $\tilde{B}$ ), where the magnetic  $SU(\tilde{N}_c)$  is free, corresponds to strongly coupled electric  $SU(N_c)$ . In cases where RG fixed point ( $\tilde{B}$ ) is IR stable, our interpretation is that the electric side appears to flow to interacting RG fixed point ( $C$ ), but the magnetic dual reveals that the theory actually flows instead to having a free magnetic  $SU(\tilde{N}_c)$ , at the RG fixed point ( $\tilde{B}$ ).

#### 4.1. The RG fixed point ( $\tilde{A}$ ) and its IR stability to $g'_{\text{mag}}$ perturbations

RG fixed point ( $\tilde{A}$ ) is the Seiberg dual description of RG fixed point ( $A$ ) of the original electric theory. All of the superconformal R-charges at ( $\tilde{A}$ ) are immediately computable from the dual matter content (4.1) and superpotential, or from the Seiberg duality map [1] and the superconformal R-charges at RG fixed point ( $A$ ) in the electric description. Either way, the result is:  $R(Y) = R(q) = N_c/(N_f + N'_c)$ ,  $R(M) = R(F') = R(\Phi) = 2 - 2N_c/(N_f + N'_c)$ , and  $R(Q') = 2/3$ . Using (4.5), we see that  $g'_{\text{mag}}$  is an IR relevant perturbation of ( $\tilde{A}$ ) if  $\text{Tr } SU(N'_c)_{\text{mag}}^2 R|_{\tilde{A}}$  is positive, or an IR irrelevant perturbation if it is negative. This 't Hooft anomaly is easily directly computed, or we can use the fact that 't Hooft anomalies match in Seiberg duality [1] (since  $SU(N'_c)$  is a subgroup of the flavor group), so  $\text{Tr } SU(N'_c)_{\text{mag}}^2 R|_{\tilde{A}} = \text{Tr } SU(N'_c)^2 R|_A$ . The RG fixed point ( $\tilde{A}$ ) of the dual theory is thus IR stable if precisely the same inequality (1.14) found in the electric description holds. So our first necessary condition for RG fixed point ( $\tilde{C}$ ) to exist, at least within the domain of attraction of the UV fixed point at zero couplings, is that the opposite inequality of (1.14) should hold,

$$(3N'_c - N'_f)(N'_c + N_f) - 3N_c^2 > 0, \quad (4.10)$$

to have ( $\tilde{A}$ ) be IR repulsive. It is satisfying to see that the magnetic ( $\tilde{A}$ ) RG fixed point is IR repulsive precisely when the electric RG fixed point ( $A$ ) is. It is hard to imagine how it could have been otherwise, given that the RG fixed points ( $A$ ) and ( $\tilde{A}$ ) are identified.

#### 4.2. The RG fixed point ( $\tilde{B}$ ) and its IR stability to $g_{\text{mag}}$ perturbations

This case is considerably more difficult than that of the previous subsection, as  $a$ -maximization is needed to determine the superconformal R-charges at RG fixed point ( $\tilde{B}$ ). Notice that, with the  $SU(\tilde{N}_c)$  gauge coupling set to zero at ( $\tilde{B}$ ), the  $SU(N'_c)_{\text{mag}}$  spectrum in (4.1) is the same as that analyzed in [14]: SQCD with an additional adjoint. But here the  $a$ -maximization analysis is further complicated by the presence of the superpotential in (4.1), which couples some of the  $SU(N'_c)_{\text{mag}}$  fundamentals  $Y$  to the adjoint  $\Phi$ , and also to fundamentals  $F'$  and  $SU(N'_c)_{\text{mag}}$  singlets  $q$ . Rather than maximizing  $a_{\text{trial}}$  as a function

of one variable,  $y$ , depending on one parameter,  $x$ , as in [14], we will have here to maximize  $a_{\text{trial}}$  as function of two variables,  $R(Q') \equiv u$  and  $R(\Phi) \equiv v$ , depending on the three parameters  $(x, x', n)$  of (1.18).

Let us consider the constraints on the superconformal  $U(1)_R$  at  $(\tilde{B})$ . Having  $\beta_{g'_{\text{mag}}} = 0$  requires  $\text{Tr} SU(N'_c)_{\text{mag}}^2 R|_{\tilde{B}} = 0$  (4.8):

$$N'_c + \tilde{N}_c(R(Y) - 1) + N'_f(R(Q') - 1) + N_f(R(F') - 1) + N'_c(R(\Phi) - 1) = 0. \quad (4.11)$$

The superpotential terms (4.2) further impose

$$R(Y) + R(F') + R(q) = 2 \quad \text{and} \quad R(\Phi) + 2R(Y) = 2. \quad (4.12)$$

Note that the first term in the superpotential (4.2) is irrelevant for  $g_{\text{mag}} = 0$ , since none of its fields are charged under  $SU(N'_c)$ , so  $M$  is a free field, with  $R(M) = 2/3$ . The constraints (4.11) and (4.12) are three constraints on five R-charges; they can be solved for

$$\begin{aligned} R(Y) &= 1 - \frac{1}{2}v, & R(q) &= \frac{1}{2}(x + x')v + n(u - 1), \\ R(F') &= 1 + n(1 - u) + \frac{1}{2}(1 - x - x')v, \end{aligned} \quad (4.13)$$

with  $R(Q') \equiv u$ ,  $R(\Phi) \equiv v$ , and  $(x, x', n)$  defined as in (1.18).  $a$ -maximization w.r.t.  $u$  and  $v$  is needed to determine the values of  $u(x, x', n)$  and  $v(x, x', n)$ .

Once we have determined the superconformal R-charges at  $(\tilde{B})$ , we can determine whether or not  $(\tilde{B})$  is stable to nonzero  $g_{\text{mag}}$  perturbations. We see from  $\beta_{g_{\text{mag}}}$  in (4.5) that  $g_{\text{mag}}$  is a relevant perturbation of  $(\tilde{B})$  if  $\text{Tr} SU(\tilde{N}_c)_{\text{mag}}^2 R|_{\tilde{B}} > 0$ , i.e., if

$$\begin{aligned} 3\tilde{N}_c - N_f - N'_c + N_f\gamma_q + N'_c\gamma_Y &> 0, \\ \text{i.e., if } -N_c + N_f R(q) + N'_c R(Y) &> 0. \end{aligned} \quad (4.14)$$

This condition, together with (4.10), are the necessary conditions for RG fixed point  $(\tilde{C})$  to exist (at least within the domain of attraction of zero couplings). If the inequality in (4.14) is not satisfied, RG fixed point  $(\tilde{B})$  is IR attractive, and then we expect RG flows from generic values of the couplings to end up there in the IR. So if (4.14) is not satisfied, the original electric theory (1.1) flows to having a free magnetic  $SU(\tilde{N}_c)$  in the IR.

The condition (4.14), for  $SU(\tilde{N}_c)$  to not be free magnetic in the IR, is generally very different from the naive criterion,  $N_f + N'_c > \frac{3}{2}N_c$ , based on when  $SU(\tilde{N}_c)$  is asymptotically free in the UV. The difference is that (4.14) accounts for the  $SU(N'_c)_{\text{mag}}$  dynamics. If the numbers of flavors and colors are chosen such that the  $SU(N'_c)_{\text{mag}}$  matter spectrum is just barely asymptotically free (i.e.,  $b_1^{\text{mag}}$  in (4.4) is small and positive), then the RG fixed point coupling  $g'_{\text{mag}}$  at  $(\tilde{B})$  is small. In this case, the  $SU(N'_c)_{\text{mag}}$  dynamics does not much affect the running of the  $SU(\tilde{N}_c)$  coupling  $g_{\text{mag}}$ . In particular, when  $(\tilde{B})$  is at weak coupling, the  $a$ -maximization results properly give  $R(q) \approx 2/3$  and  $R(Y) \approx 2/3$ , since these fields are approximately free. We then find that (4.14) gives approximately the standard condition from Seiberg duality [1] for the magnetic dual  $SU(\tilde{N}_c)$  to be interacting

rather than IR free,  $N_f + N'_c > \frac{3}{2}N_c$ , which is the condition that  $g_{\text{mag}}$  be an IR relevant perturbation of free theory at  $g_{\text{mag}} = g'_{\text{mag}} = 0$ .

On the other hand, when the number of flavors and colors are such that  $SU(N'_c)_{\text{mag}}$  is very much asymptotically free, i.e.,  $b_1^{\text{mag}}$  in (4.4) is positive and large, the RG fixed point ( $\tilde{B}$ ) is at strong  $SU(N'_c)_{\text{mag}}$  coupling. In this case, the  $SU(N'_c)_{\text{mag}}$  dynamics can radically affect whether or not the  $SU(\tilde{N}_c)$  coupling  $g_{\text{mag}}$  is relevant. Indeed, depending on the values of  $(x, x', n)$ , this theory can realize the flow of Fig. 3: even if  $N_f + N'_c \leq \frac{3}{2}N_c$ , so  $SU(\tilde{N}_c)$  is IR free around  $g_{\text{mag}} = g'_{\text{mag}} = 0$ , the condition (4.14) for  $SU(\tilde{N}_c)$  to be an IR interacting deformation of ( $\tilde{B}$ ) can nevertheless be satisfied. In short, the  $SU(N'_c)_{\text{mag}}$  has driven an otherwise IR free  $SU(\tilde{N}_c)$  theory to instead be IR interacting. We will focus on this phenomenon of Fig. 3 in the rest of this subsection.

To have (4.14) be satisfied when  $b_1^{\text{mag}} < 0$  requires that  $q$  and/or  $Y$  have *positive* anomalous dimension, i.e., R-charge greater than  $2/3$ , at the RG fixed point ( $B$ ). Positive anomalous dimensions are possible provided that there is a  $W_{\text{tree}}$  superpotential, as in the case here (4.2): gauge interactions make negative contributions to the anomalous dimensions, and superpotential interactions make positive contributions. As we will now discuss, there is indeed a range of parameter space of flavors and colors for the magnetic theory (4.1) where the anomalous dimensions are sufficiently positive so as to have (4.14) satisfied, despite having  $b_1^{\text{mag}} < 0$ , realizing the effect of Fig. 3. (See also the example [19].)

Though the  $a$ -maximization analysis at RG fixed point ( $\tilde{B}$ ) is standard, it is computationally intensive. Using the spectrum (4.1) and (4.13), we compute the combination of 't Hooft anomalies  $a^{(0)} = 3 \text{Tr} R^3 - \text{Tr} R$ , as a function of the two variables,  $(u, v)$ , and the parameters  $(x, x', n)$ . Depending on  $(x, x', n)$ , we have to also add the additional contributions (2.3) for any gauge invariant operators with  $R \leq 2/3$ . (The operators hitting the unitarity bound are found to be  $Q' \Phi^{j-1} \tilde{Q}'$ ,  $Q' \Phi^{j-1} \tilde{F}'$ , and  $F' \Phi^{j-1} \tilde{F}'$  for values of  $j = 1, \dots$  increasing with  $x$  and  $x'$ , as the RG fixed point ( $\tilde{B}$ ) becomes more and more strongly coupled.) Again, we implemented this  $a$ -maximization analysis numerically, using Mathematica. While the numerics are similar in spirit to our previous cases, the fact that here we're maximizing a function of two, rather than one, variables, as a function of the three parameters, greatly prolongs the required computational timescale.

Let us focus on an interesting range of parameter space, where we take the parameters  $x$  and  $x'$  to be large. This is an interesting region of parameter space because then RG fixed point ( $\tilde{B}$ ) is at very strong  $SU(N'_c)_{\text{mag}}$  gauge coupling (as seen from the fact that the one-loop beta function is very large). For large  $x$  and  $x'$ , there are terms quartic in  $x$  in  $a^{(0)} = 3 \text{Tr} R^3 - \text{Tr} R$ , coming from the  $O(x)$  terms in  $R(q)$  and  $R(F')$  in (4.13). Note, however, that the  $O(x)$  terms in (4.13) all appear multiplied by  $v$ , so the leading terms for large  $x$  transform homogeneously, with degree one, under  $x \rightarrow \lambda x$  and  $v \rightarrow \lambda^{-1}v$ , e.g., the quartic term in  $a^{(0)}$  is  $\sim x^4 v^3$ . When we include the contributions from the meson hitting the unitarity bound, we find that they also have a leading large  $x$  term which is degree one under this scaling. Because of this homogeneity, when we maximize w.r.t.  $v$  for large  $x$ , we find  $v \sim 1/x$ , and the value of  $a$  at its maximum is linear in  $x$ , since it's degree one in the above rescaling. (The fact that the central charge of the SCFT grows linearly in  $x$  is a check of the  $a$ -theorem conjecture, since there is a higgsing RG flow analogous to that

of [14] which would violate the  $a$ -theorem with any higher degree.) To study the limit of large  $x$ , we can thus scale  $\lambda \rightarrow \infty$ , keeping only the terms of degree one in  $\lambda$ .

In this scaling limit,  $R(\Phi) \equiv v \rightarrow \lambda^{-1}v \rightarrow 0$  for  $\lambda \rightarrow \infty$ . Then (4.13) gives  $R(Y) \rightarrow 1$ , while  $R(Q') \equiv u$ ,  $R(q)$ , and  $R(F')$  asymptote to some finite values that are determined by  $a$ -maximization. The  $a$ -maximization answer for these quantities, in our limit of large  $x$  and  $x'$ , can also be obtained by borrowing the  $a$ -maximization results of [14]. The idea is that the  $SU(N'_c)_{\text{mag}}$  gauge coupling at  $(\tilde{B})$  is very strong for large  $x$  and  $x'$ , and the  $SU(N'_c)_{\text{mag}}$  matter content that it couples to coincides with that of [14], and adjoint and fundamentals. The theory at  $(\tilde{B})$  differs from that of [14] only because of the superpotential interactions (4.2). If not for the superpotential interactions, the  $a$ -maximization result of [14] would tell us that  $R(Y)$ ,  $R(Q')$  and  $R(F')$  all asymptote to  $(\sqrt{3} - 1)/3 \approx 0.244$  for  $x \rightarrow \infty$ . The superpotential nonnegligibly affects  $R(Y)$ : the  $\Phi Y \tilde{Y}$  term requires that  $R(Y) \rightarrow 1$ , since  $R(\Phi) \rightarrow 0$  for large  $x$ . But the superpotential negligibly affects  $R(Q')$  and  $R(F')$  in the strong  $SU(N'_c)_{\text{mag}}$  coupling, large  $x$  limit. For example, though there is a term  $Y F' \tilde{q}$  in the superpotential, its effect is to determine the R-charge of the otherwise free field  $q$ , leaving  $R(Y) \rightarrow 1$  and  $R(F') \rightarrow (\sqrt{3} - 1)/3 \approx 0.244$  unaffected. So in this large  $x$  limit we obtain (and the detailed  $a$ -maximization analysis bears this out)

$$R(Y) \rightarrow 1 \quad \text{and} \quad R(q) \rightarrow 1 - \left( \frac{\sqrt{3} - 1}{3} \right) \approx 0.756, \quad \text{for large } x \text{ and } x'. \quad (4.15)$$

The condition (4.14) for  $g_{\text{mag}}$  to be relevant at  $\tilde{B}$  is then (recalling (1.18))

$$N'_c - N_c + N_f(0.756) > 0, \quad \text{i.e.,} \quad x' - x + 0.756 > 0, \quad \text{for large } x \text{ and } x'. \quad (4.16)$$

This is very different from the condition that  $g_{\text{mag}}$  be asymptotically free for  $g'_{\text{mag}} = 0$ ,  $b_1^{\text{mag}} > 0$ , i.e.,  $1 + x' - \frac{3}{2}x > 0$ , and (4.16) can be satisfied even when  $b_1^{\text{mag}} < 0$ , i.e., we can have

$$1 + x' - \frac{3}{2}x < 0 \quad \text{but nevertheless} \quad x' - x + 0.756 > 0; \quad (4.17)$$

for example, we can take  $x \approx x' \rightarrow \infty$ . For values of  $(x, x', n)$  such that the inequalities in (4.17) both hold, the RG flow is as in Fig. 3: if  $SU(N'_c)_{\text{mag}}$ 's coupling were set to exactly zero, then  $SU(\tilde{N}_c)$  would be IR free, but any nonzero  $SU(N'_c)_{\text{mag}}$  coupling would eventually drive  $SU(\tilde{N}_c)$  to be instead interacting in the IR.

### 4.3. The RG fixed point $(\tilde{C})$

RG fixed point  $(\tilde{C})$  exists if  $(\tilde{A})$  and  $(\tilde{B})$  are both IR unstable to perturbations in the other coupling; we found this to be the case if (4.10) holds, and if, say for large  $x$  and  $x'$ , (4.16) holds, respectively. When  $(\tilde{C})$  exists, we expect that it's an equivalent, dual description of the RG fixed point  $(C)$  of the original electric theory. We will here check that the superconformal R-charges are compatible with this identification.

At  $(\tilde{C})$ , the six independent superconformal R-charges, on the six lines of (4.1), are subject to five constraints: for vanishing  $\beta_{g_{\text{mag}}}$  and  $\beta_{g'_{\text{mag}}}$ , we require (4.9),  $\text{Tr} SU(\tilde{N}_c)^2 R|_{\tilde{C}} =$

$\text{Tr} SU(N'_c)^2_{\text{mag}} R|_{\tilde{C}} = 0$ , along with three more constraints from requiring that the superpotential terms (4.2) all have total  $R(W) = 2$ . (All terms in (4.2) are relevant deformations of the  $W = 0$  theory when  $g_{\text{mag}*}$  and  $g'_{\text{mag}*}$  are both nonzero.) There is thus a one-variable family of R-charges, as for the electric RG fixed point ( $C$ ). These constraints are compatible with the duality map identification of the fields  $F'$ ,  $M$ , and  $\Phi$  in (4.1): the R-charges of the dual theory (4.1) can be related to those of the original electric theory (1.1), with  $R(Q) \equiv y$  as before, by

$$\begin{aligned} R(M) &= 2y, & R(F') &= R(X) + y, & R(\Phi) &= 2R(X), \\ R(q) &= 1 - y, & R(Y) &= 1 - R(X), \end{aligned} \tag{4.18}$$

with  $R(X)$  and  $R(Q')$  given by (2.8) in terms of the variable  $y$  and parameters  $(x, x', n)$ .

We compute the same function  $a_{\text{trial}}^{(0)} = 3 \text{Tr} R^3 - \text{Tr} R$  to maximize w.r.t.  $y$  as in the electric theory (2.9), as expected from the 't Hooft anomaly matching for the global flavor symmetries in Seiberg duality [1]. Compatible with our claim that the electric RG fixed point ( $C$ ) is equivalent to the dual one ( $\tilde{C}$ ), there is a one-to-one mapping of the operators that have hit the unitarity bound. Corresponding to the operators (2.10) we have

$$\begin{aligned} M_{j=1} &\leftrightarrow M, & M_{j>1} &\leftrightarrow \tilde{F} \Phi^{j-2} F, & M'_j &\leftrightarrow Q' \Phi^{j-1} \tilde{Q}', \\ P_j &\leftrightarrow F' \Phi^{j-1} \tilde{Q}', & \tilde{P}_j &\leftrightarrow \tilde{F}' \Phi^{j-1} Q'. \end{aligned} \tag{4.19}$$

So, even including the contributions of the operators hitting their unitarity bound, we find the same  $a_{\text{trial}}$  function of  $y$  and  $(x, x', n)$  to maximize w.r.t.  $y$ , and hence the same superconformal R-charges are given by (4.18) with  $y(x, x', n)$  the same superconformal R-charge as obtained by analyzing the electric theory (1.1).

### 5. Conclusions and comments

A general potential pitfall in applying  $a$ -maximization is that one must really have the full symmetry group under control, including all accidental symmetries, to obtain correct results. Overlooking some symmetries will lead to a value of the central charge  $a_{\text{SCFT}}$  that is too low. Seiberg duality [1] shows that there can be highly nonobvious accidental symmetries, such as those acting on the free magnetic  $SU(N_f - N_c)$  quarks and gluons when  $N_f < \frac{3}{2}N_c$ . More generally, without knowing the dual, we do not presently have a way to look for such accidental symmetries, which do not act on any of the “obvious” gauge invariant operators of the theory.

Ignoring the interplay of the two gauge couplings, the superconformal window of [1] for each gauge group in (1.1) separately is

$$\frac{3}{2}N_c < N_f + N'_c < 3N_c, \quad \frac{3}{2}N'_c < N'_f + N_c < 3N'_c. \tag{5.1}$$

These are the conditions for points (A) and (B) to be interacting SCFTs, respectively. The upper limits are needed for the electric coupling to not be driven to zero in the IR, and the lower limits are for the couplings of the dual [1] to not be driven to zero in the IR.

Accounting for the  $g$  and  $g'$  interplay, the conditions for point (C) to exist as a fully interacting SCFT differ from (5.1). The upper limits of (5.1) should be replaced with the conditions (1.16), for neither electric gauge coupling to be driven to zero in the IR. Similarly, the duality of Section 4 (assuming its validity) shows how the lower limits of (5.1) are modified, in order for neither magnetic gauge coupling to be driven to zero in the IR.

For example, taking  $x \equiv N_c/N_f$  and  $x' \equiv N'_c/N_f$  large, we found in Section 4 that  $(\tilde{B})$ , with  $g_{\text{mag}} \rightarrow 0$ , is IR attractive if

$$N'_c - N_c + (0.756)N_f < 0, \quad \text{i.e., if } x' - x + 0.756 < 0. \quad (5.2)$$

In this case, rather than flowing to the fully interacting RG fixed point (C), the theory flows to the free magnetic point  $(\tilde{B})$  in the IR, where the original electric  $SU(N_c)$  is very strongly coupled, but its  $SU(N_f + N'_c - N_c)$  magnetic dual is IR free. There is then a large, non-obvious, accidental symmetry of the original electric theory when (5.2) holds. Likewise, dualizing the  $SU(N'_c)$  factor of (1.1), we find for large  $x$  and  $x'$  that the apparent RG fixed point (C) of the electric theory instead flows to having a free magnetic  $SU(N'_f + N_c - N'_c)$  group when

$$N_c - N'_c + (0.756)N'_f < 0, \quad \text{i.e., if } x - x' + (0.756)n < 0. \quad (5.3)$$

So, for RG fixed point (C) to be fully interacting, rather than partially free magnetic, the lower limits in (5.1) are replaced, for large  $x$  and  $x'$ , with the conditions

$$-(0.756)n < x - x' < 0.756. \quad (5.4)$$

The range (5.4) is a subset of the stability range (2.5). Outside of the range (5.4), there are nonobvious accidental symmetries. Within the range (5.4), we have no evidence for nonobvious accidental symmetries. If there had been any such nonobvious accidental symmetries, our  $a$ -maximization analysis of Sections 2 and 3 would have to be appropriately modified. In particular, in our parameter slice of special interest in Sections 2 and 3,  $x = x'$ , i.e.,  $N_c = N'_c$ , the magnetic duals remain fully interacting.

As we noted, for  $x = x'$  and  $n = 1$ , the  $a$ -maximization analysis of our product group example (1.1) coincides with that in [14] for  $SU(N_c)$  with an adjoint and  $N_f$  fundamentals. In the analysis of [14] of that latter theory, it was assumed that the only accidental symmetries are the obvious ones, associated with gauge invariant operators hitting the unitarity bound. But, as in the example of [19], there is a possibility of a nonobvious accidental symmetry, associated with a free-magnetic gauge group in a deconfining dual. The idea of the deconfining dual [25] is that the dual (3.1) of our theory (1.1) would look quite a lot like  $SU(N'_c)$  SQCD with an adjoint if we chose the flavors and colors such that  $\tilde{N}_c = 1$  in (4.1). And a slight modification of the theory (1.1), with added fields and superpotential terms (designed to eliminate the analog of (4.2) in the dual), will lead to precisely SQCD with an adjoint and fundamentals, with no superpotential; see Table 8 of [26] for the needed field content. It would be interesting to carry out the  $a$ -maximization analysis of that theory, and

its duals, to determine whether or not any of the gauge groups of the deconfining duals can become IR free.

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**Appendix A. On the superconformal window of the other duals of [5]**

*A.1. Reviewing  $SU(N_c)$  SQCD, with  $N_f$  fundamental flavors, and an adjoint  $X$*

Let us briefly review the  $a$ -maximization analysis of Kutasov, Parnachev, and Sahakyan (KPS) [14] for this theory, with  $W_{\text{tree}} = 0$ . The anomaly free superconformal R-charges of the fields are  $R(Q) = R(\tilde{Q}) \equiv y$  for the fundamentals and  $R(X) = (1 - y)/x$  for the adjoint, where  $x \equiv N_c/N_f$ .  $a$ -maximization determines  $y(x)$  via maximizing

$$\begin{aligned}
 & a_{\text{KPS}}^{(p)}(y, x)/N_f^2 \\
 &= 2x^2 + x^2 \left( 3 \left( \frac{1-y}{x} - 1 \right)^3 - \left( \frac{1-y}{x} - 1 \right) \right) \\
 &\quad + 2x(3(y-1)^3 - (y-1)) \\
 &\quad + \sum_{j=0}^p \left( 2y + j \frac{1-y}{x} - \frac{2}{3} \right)^2 \left( 5 - 3 \left[ 2y + j \frac{1-y}{x} \right] \right), \tag{A.1}
 \end{aligned}$$

w.r.t.  $y$ ; this has solution  $y^{(p)}(x)$ . The sums account for the generalized mesons  $QX^j\tilde{Q}$  hitting their unitarity bound, with  $p$  the greatest integer such that  $R(QX^j\tilde{Q})$  would naively violated the unitarity bound. The solution  $y_{\text{KPS}}(x)$  is obtained by patching together the functions  $y^{(p)}(x)$ , with the appropriate value of  $p$  depending on  $x$ . The function  $y_{\text{KPS}}(x)$  is monotonically decreasing, with asymptotic value  $y(x \rightarrow \infty) \rightarrow y_{\text{as}} = (\sqrt{3} - 1)/3$ .  $R(X) = (1 - y)/x$  is also monotonically decreasing in  $x$  and, for  $x \rightarrow \infty$ ,  $R(X) \rightarrow (1 - y_{\text{as}})/x$ .

The superpotential  $W_{A_k} = \text{Tr } X^{k+1}$  is a relevant deformation of the  $W = 0$  SCFT if  $R(X^{k+1}) < 2$  (since  $\Delta(W) = \frac{3}{2}R(W)$ ), i.e., if  $R(X) = (1 - y)/x < 2/(k + 1)$ . Since  $R(X)$  monotonically decreases with  $x$ ,  $W_{A_k}$  can always be made relevant, by taking  $x$  sufficiently large,  $x > x_{A_k}^{\text{min}}$ , where  $x_{A_k}^{\text{min}}$  is determined by the condition that  $(1 - y(x_{A_k}^{\text{min}}))/x_{A_k}^{\text{min}} = 2/(k + 1)$ . Using the numerical solution for  $y(x)$ , the numerical values of  $x_{A_k}^{\text{min}}$  can be

obtained for arbitrary  $k$ . For large  $k$ ,  $x_{A_k}^{\min}$  is large, and then  $R(X) \approx (1 - y_{\text{as}})/x_{A_k}^{\min} = 2/(k+1)$  gives  $x_{A_k}^{\min} \rightarrow (\frac{4-\sqrt{3}}{6})k \approx 0.3780k$  [14].

The  $A_k$  theory has dual description [27–29] in terms of a magnetic  $SU(\tilde{N}_c)$  gauge theory, with  $\tilde{N}_c = kN_f - N_c$ . It has an adjoint field  $Y$ ,  $N_f$  fundamental flavors  $q, \tilde{q}$ , and  $N_f^2$  gauge singlet fields  $M_j$ , for  $j = 1, \dots, k$ . The superpotential is

$$W_{A_k} = \text{Tr} Y^{k+1} + \sum_{j=1}^k M_{k-j} q Y^{j-1} \tilde{q}. \quad (\text{A.2})$$

The analysis of the dual theory is similar to that of the electric theory, with  $x \rightarrow \tilde{x} = \tilde{N}_c/N_f = k - x$ , though the specifics are not identical, because of the effect of the additional gauge singlets  $M_j$  and superpotential terms in (A.2). We refer the reader to [14], for the details of the  $a$ -maximizing  $\tilde{y}(\tilde{x})$  in the magnetic theory. The qualitative result is that  $\tilde{y}(\tilde{x})$  drops to zero a little faster on the magnetic side than the electric  $y(x)$ , so the  $\text{Tr} Y^{k+1}$  term in (A.2) is relevant for  $\tilde{x} > \tilde{x}_{A_k}^{\min}$ , with  $\tilde{x}_{A_k}^{\min} < x_{A_k}^{\min}$ . In particular, for  $k \gg 1$ ,  $\tilde{x}_{A_k}^{\min} \approx 0.3578k$  [14].

The superconformal window, where both dual descriptions of the  $A_k(N_c, N_f)$  SCFTs are useful, is  $x_{A_k}^{\min} < x < k - \tilde{x}_{A_k}^{\min}$ ; for  $k \gg 1$ , it is  $0.3780k < x < 0.6422k$ . Within this range the electric and magnetic theories have the same central charge  $a$ , as guaranteed by 't Hooft anomaly matching. Outside of this range, there are accidental symmetries that are manifest in one of the dual descriptions but not in the other so, without accounting for these accidental symmetries, the central charge as computed by  $a$ -maximization for the electric and magnetic theories can appear to differ [14]. The larger of  $a_{\text{elec}}$  or  $a_{\text{mag}}$  is the correct one—it is larger because of maximizing  $a_{\text{trial}}$  over R-symmetries that can mix with the additional, accidental flavor symmetries.

## A.2. Some immediate generalizations, with other groups and matter content

Many analogs of the duality of [27–29] were soon given in [5,30,31], all for theories with  $W_{A_k}$  type LG superpotential. Without the LG superpotential, those theories are expected to flow to other SCFTs, which can now be analyzed via  $a$ -maximization. Doing so determines when the  $W_{A_k}$  superpotential is relevant. Doing a similar  $a$ -maximization analysis of the duals of [5,20,31] determines when the dual  $A_k$  LG superpotentials are relevant. Combining the two bounds, as in the analysis of [14], reviewed in the last section, determines the superconformal window for where the duals of [5,20,31] are useful. In particular, we can verify that the superconformal window is nonempty for all  $k$ .

As in the analysis of [14], it is most convenient to consider the limit  $N_c \gg 1$ ,  $N_f \gg 1$ , holding  $x \equiv N_c/N_f$  fixed. However, as we will now explain, the  $a$ -maximization analysis of all of the examples of [5,20,31] involving a single gauge group becomes simply identical to that of [14] in this limit, where we drop terms  $O(1/N_c)$  or  $O(1/N_f)$ .

The examples of [5,20,31] involving a single gauge group are:

Group	$X$	$Q$	# mesons $QX^jQ$	$a/a_{\text{KPS}}$
$SO(N_c)$	$\square\square$	$N_f \cdot \square$	$\frac{1}{2}N_f(N_f + 1)$	$\frac{1}{2}$
$Sp(N_c)$	$\square$	$2N_f \cdot \square$	$N_f(2N_f - 1)$	2
$SO(N_c)$	$\square$	$N_f \cdot \square$	$\frac{1}{2}N_f(N_f + (-1)^j)$	$\frac{1}{2}$
$Sp(N_c)$	$\square\square$	$2N_f \cdot \square$	$N_f(2N_f - (-1)^j)$	2
$SU(N_c)$	$\square \oplus \bar{\square}$	$N_f \cdot (\square \oplus \bar{\square})$	$N_f^2$ or $N_f(N_f - 1)$	1
$SU(N_c)$	$\square\square \oplus \bar{\square}\bar{\square}$	$N_f \cdot (\square \oplus \bar{\square})$	$N_f^2$ or $N_f(N_f + 1)$	1
$SU(N_c)$	$\square \oplus \bar{\square}$	$8 \cdot \square \oplus N_f \cdot (\square \oplus \bar{\square})$	$\sim N_f^2$	1

(A.3)

Our notation for  $Sp(N_c)$  is that  $SU(2) \cong Sp(1)$ .

Let us compare the theory on the first line of (A.3) with the  $SU(N_c)$  with adjoint theory analyzed in [14]. The anomaly free R-symmetry is constrained to satisfy  $2(N_c - 2) + 2N_f(R(Q) - 1) + 2(N_c + 2)(R(X) - 1) = 0$ . But in the  $N_c \gg 1$  and  $N_f \gg 1$  limit, holding fixed  $x \equiv N_c/N_f$ , this gives an identical relation,  $R(X) = (1 - y)/x$ , with  $x \equiv N_c/N_f$ , as in the case reviewed in the previous subsection. Computing the analog of (A.1) for the theory on the first line of (A.3), we find that every term is now simply half of that in (A.1), coming from the fact that the only difference (to leading order  $O(1/N_c)$  and  $O(1/N_f)$ ) is that there are half as many of each of the different fields. For example, the  $2x^2$  term in (A.1) is the contribution of the  $|SU(N_c)| \approx N_c^2$  gauge fields, which here becomes a similar contribution from the  $|SO(N_c)| = \frac{1}{2}N_c(N_c - 1) \approx \frac{1}{2}N_c^2$  gauge fields. Similarly, there are here half as many  $X$  fields ( $\frac{1}{2}N_c(N_c + 1) \approx \frac{1}{2}N_c^2$  here, vs.  $N_c^2 - 1 \approx N_c^2$  there), half as many  $Q$  fields ( $N_cN_f$  here, vs.  $2N_cN_f$  there) and half as many of the meson fields ( $\frac{1}{2}N_f(N_f + 1) \approx \frac{1}{2}N_f^2$  here, vs.  $N_f^2$  there). So the  $a$ -function to maximize for the theory in the first line of (A.3) is simply half  $a_{\text{KPS}}$  (A.1). Maximizing this obviously leads to the same solution for the superconformal R-charges as obtained in [14],  $y(x) = y_{\text{KPS}}(x)$ .

Likewise, all of the other theories in (A.3) similarly lead to the same results in the  $N_c \gg 1$ ,  $N_f \gg 1$  limit, for arbitrary  $x \equiv N_c/N_f$ . In this limit, the anomaly free condition on the superconformal R-symmetry gives  $R(X) = (1 - y)/x$ , with  $R(Q) \equiv y$ , for all of them. For all of these theories, the analog of every term in (A.1) becomes approximately simply the same as in (A.1), up to an overall factor, which is given in the last column of (A.3). For example, for the theory in the last line of (A.3), the generalized mesons  $QX^jQ$  which can hit the unitarity bound are given for even  $j$  by  $\tilde{Q}(X\bar{X})^rQ$  (which are  $N_f(N_f + 8) \approx N_f^2$  in number) and for odd  $j$  by  $Q(\bar{X}X)^2\tilde{X}Q$  (which are  $\frac{1}{2}(N_f + 8)(N_f + 9) \approx \frac{1}{2}N_f^2$  in number) and  $\tilde{Q}X(\bar{X}X)^r\tilde{Q}$  (which are  $\frac{1}{2}N_f(N_f - 1) \approx \frac{1}{2}N_f^2$  in number) so, whether  $j$  is even or odd, there are approximately the same number  $N_f^2$  of mesons, leading to the same contributions as in the last line of (A.1).

There are several interrelations among the theories (A.3) associated with giving  $X$  an expectation value (which we are free to do, since we are now discussing the theories with  $W_{\text{tree}} = 0$ ), and it can be verified that all of these higgsing RG flows satisfy  $a_{\text{IR}} < a_{\text{UV}}$ . These checks make use of the  $a/a_{\text{KPS}}$  factors in the last column of (A.3). For example, consider the  $SO(N_c)$  theory with adjoint  $X$ , on the third line of (A.3). Giving  $X$  an expect-

tation value, there is a RG flow connecting this  $SO(N_c)$  theory in the UV to an IR theory with gauge group  $U(\frac{1}{2}N_c)$ , adjoint matter  $X_{\text{low}}$ , and  $N_f$  fundamental flavors ( $\square \oplus \bar{\square}$ ). Using the last column of (A.3), the UV theory has  $a_{\text{UV}} \approx \frac{1}{2}N_f^2 \hat{a}_{\text{KPS}}(x)$ . The IR theory has  $a_{\text{IR}} \approx N_f^2 a_{\text{KPS}}(\frac{1}{2}x)$ , because the IR theory has  $N_c/2$  colors. The  $a$ -theorem conjecture thus requires  $\frac{1}{2}\hat{a}_{\text{KPS}}(x) > \hat{a}_{\text{KPS}}(\frac{1}{2}x)$ , which can be verified to be satisfied.

Since all of the theories in (A.3) have the same R-charge  $R(X)$ , given by  $R(X) = (1 - \gamma_{\text{KPS}}(x))/x$ , the minimal values  $x_{A_k}^{\text{min}}$  for the  $W_{A_k} = \text{Tr} X^{k+1}$  superpotential to be relevant is the same, for all of these theories,<sup>3</sup> as was obtained in [14] for the  $SU(N_c)$  with adjoint theory. E.g., for  $k \gg 1$ , all have  $x_{A_k}^{\text{min}} \rightarrow (\frac{4-\sqrt{3}}{6})k$ .

We can similarly analyze the magnetic duals of the above theories [5,30,31]. For example, the theory in the last line of (A.3), upon deforming by superpotential  $W_{A_k} = \text{Tr}(X\tilde{X})^{\frac{1}{2}(k+1)}$  (with  $k+1 = 0 \pmod{4}$  here), was argued to be dual to a similar theory, with gauge group  $SU(\tilde{N}_c)$ , with  $\tilde{N}_c \equiv k(N_f + 4) - N_c$ , along with some additional gauge singlets and superpotential terms. We can use  $a$ -maximization to analyze this dual  $SU(\tilde{N}_c)$  theory for  $W_{\text{tree}} = 0$ , and thereby determine when the various terms in the superpotential appearing in the duality of [5] are relevant. In particular, we can determine  $\tilde{x}_{A_k}^{\text{min}}$ , the lower bound on  $\tilde{x} \equiv \tilde{N}_c/N_f$  in order for the superpotential  $\tilde{W}_{A_k}$  to be relevant. As on the electric side, in the limit of large  $N_c$  and  $N_f$ , the  $a$ -maximization analysis becomes identical to that of [14] for the magnetic dual of the adjoint theory with superpotential  $W_{A_k}$ : the above  $\tilde{N}_c$  becomes  $\tilde{N}_c \approx kN_f - N_c$ , as in the adjoint theory, and every term in the  $a$ -maximization analysis here maps to a corresponding term there. In this limit, the values here of the  $\tilde{x}_{A_k}^{\text{min}}$  are the same as those obtained in [14] for the adjoint theory.

Thus, at least in the  $N_c \gg 1$  and  $N_f \gg 1$  limit, all of the above theories have exactly the same superconformal window as obtained in [14] for the  $SU(N_c)$  theory with adjoint.

We also note that all of the other theories in [5], with product gauge groups, also have the same superconformal window range of  $x$ , as long as we take all the groups to have the same (large) rank, and take all to have the same (large) number of fundamental flavors. This generalizes our observation of Section 2, that the  $SU(N_c) \times SU(N'_c)$  theory gives the superconformal window obtained in [14] for the slice of parameter space  $N_c = N'_c$  and  $N_f = N'_f$ .

For example, consider the  $SU(M) \times SO(N) \times SO(N')$  duality discussed in Section 11 of [5]. In the parameter slice,  $M = N = N' \equiv N_c$ , and  $m = n = n' \equiv N_f$ , taking  $N_c \gg 1$  and  $N_f \gg 1$  large, holding fixed the ratio  $x \equiv N_c/N_f$ , we find that every term in the quantity  $a_{\text{trial}}(y, x) = 3 \text{Tr} R^3 - \text{Tr} R$  equals twice a corresponding term in the corresponding function of [14],  $a_{\text{trial}}(y, x) = 2a_{\text{KPS}}(y, x)$  (even including the contributions of the gauge invariant operators that hit the unitarity bound). This is because there is a correspondence in this limit between every field, with the same R-charges and twice as many copies for the  $SU \times SO \times SO$  theory of [5] as compared with that of [14]. Since  $a_{\text{trial}} = 2a_{\text{KPS}}$ , it is maximized by the same function,  $y = \gamma_{\text{KPS}}(x)$ . The anomalous dimensions are thus the

<sup>3</sup> The theories in the third through sixth line of (A.3) must have  $k = \text{odd}$ , and that in the last line of (A.3) must have  $k+1 = 0 \pmod{4}$ .

same of those in [14] for this parameter slice. There is an analogous equality, up to the same factor of 2, between the function  $a_{\text{trial}}$  for the magnetic duals. It thus follows that the duality of [5] for this product group has a nonempty superconformal window, which reduces to the  $x$  interval of [14] in this 1d subspace of the full parameter space of flavors and colors. Likewise, all the dualities of [5] have a nonempty superconformal window, for any  $k$ , which reduces to the  $x$  interval of [14] in a 1d subspace of the full parameter space of flavors and colors.

## References

- [1] N. Seiberg, Electric–magnetic duality in supersymmetric non-Abelian gauge theories, Nucl. Phys. B 435 (1995) 129, hep-th/9411149.
- [2] K.A. Intriligator, N. Seiberg, Lectures on supersymmetric gauge theories and electric–magnetic duality, Nucl. Phys. B (Proc. Suppl.) 45 (1996) 1, hep-th/9509066.
- [3] K.A. Intriligator, N. Seiberg, S.H. Shenker, Proposal for a simple model of dynamical SUSY breaking, Phys. Lett. B 342 (1995) 152, hep-ph/9410203.
- [4] J.H. Brodie, P.L. Cho, K.A. Intriligator, Misleading anomaly matchings? Phys. Lett. B 429 (1998) 319, hep-th/9802092.
- [5] K.A. Intriligator, R.G. Leigh, M.J. Strassler, New examples of duality in chiral and nonchiral supersymmetric gauge theories, Nucl. Phys. B 456 (1995) 567, hep-th/9506148.
- [6] M.E. Machacek, M.T. Vaughn, Two loop renormalization group equations in a general quantum field theory, 1: Wave function renormalization, Nucl. Phys. B 222 (1983) 83.
- [7] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Exact Gell-Mann–Low function of supersymmetric Yang–Mills theories from instanton calculus, Nucl. Phys. B 229 (1983) 381.
- [8] D. Kutasov, A. Schwimmer, Lagrange multipliers and couplings in supersymmetric field theory, hep-th/0409029.
- [9] K. Intriligator, B. Wecht, The exact superconformal R-symmetry maximizes  $a$ , Nucl. Phys. B 667 (2003) 183, hep-th/0304128.
- [10] D. Anselmi, D.Z. Freedman, M.T. Grisaru, A.A. Johansen, Nonperturbative formulas for central functions of supersymmetric gauge theories, Nucl. Phys. B 526 (1998) 543, hep-th/9708042.
- [11] D. Anselmi, J. Erlich, D.Z. Freedman, A.A. Johansen, Positivity constraints on anomalies in supersymmetric gauge theories, Phys. Rev. D 57 (1998) 7570, hep-th/9711035.
- [12] D. Kutasov, New results on the ‘ $a$ -theorem’ in four-dimensional supersymmetric field theory, hep-th/0312098.
- [13] E. Barnes, K. Intriligator, B. Wecht, J. Wright, Evidence for the strongest version of the 4d  $a$ -theorem, via  $a$ -maximization along RG flows, hep-th/0408156.
- [14] D. Kutasov, A. Parnachev, D.A. Sahakyan, Central charges and  $U(1)$  R-symmetries in  $N = 1$  super-Yang–Mills, JHEP 0311 (2003) 013, hep-th/0308071.
- [15] K. Intriligator, B. Wecht, Baryon charges in 4D superconformal field theories and their AdS duals, Commun. Math. Phys. 245 (2004) 407, hep-th/0305046.
- [16] C.P. Herzog, J. Walcher, Dibaryons from exceptional collections, JHEP 0309 (2003) 060, hep-th/0306298.
- [17] K. Intriligator, B. Wecht, RG fixed points and flows in SQCD with adjoints, Nucl. Phys. B 677 (2004) 223, hep-th/0309201.
- [18] S. Franco, Y.H. He, C. Herzog, J. Walcher, Chaotic duality in string theory, hep-th/0402120.
- [19] C. Csaki, P. Meade, J. Terning, A mixed phase of SUSY gauge theories from  $a$ -maximization, JHEP 0404 (2004) 040, hep-th/0403062.
- [20] M. Bertolini, F. Bigazzi, A.L. Cotrone, New checks and subtleties for AdS/CFT and  $a$ -maximization, JHEP 0412 (2004) 024, hep-th/0411249.
- [21] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, J. Sparks, An infinite family of superconformal quiver gauge theories with Sasaki–Einstein duals, hep-th/0411264.

- [22] E. Poppitz, Y. Shadmi, S.P. Trivedi, Duality and exact results in product group theories, *Nucl. Phys. B* 480 (1996) 125, hep-th/9605113.
- [23] I.R. Klebanov, M.J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities, *JHEP* 0008 (2000) 052, hep-th/0007191.
- [24] F. Cachazo, B. Fiol, K.A. Intriligator, S. Katz, C. Vafa, A geometric unification of dualities, *Nucl. Phys. B* 628 (2002) 3, hep-th/0110028.
- [25] M. Berkooz, The dual of supersymmetric  $SU(2k)$  with an antisymmetric tensor and composite dualities, *Nucl. Phys. B* 452 (1995) 513, hep-th/9505067.
- [26] M.A. Luty, M. Schmaltz, J. Terning, A sequence of duals for  $Sp(2N)$  supersymmetric gauge theories with adjoint matter, *Phys. Rev. D* 54 (1996) 7815, hep-th/9603034.
- [27] D. Kutasov, A comment on duality in  $N = 1$  supersymmetric non-Abelian gauge theories, *Phys. Lett. B* 351 (1995) 230, hep-th/9503086.
- [28] D. Kutasov, A. Schwimmer, On duality in supersymmetric Yang–Mills theory, *Phys. Lett. B* 354 (1995) 315, hep-th/9505004.
- [29] D. Kutasov, A. Schwimmer, N. Seiberg, Chiral rings, singularity theory and electric–magnetic duality, *Nucl. Phys. B* 459 (1996) 455, hep-th/9510222.
- [30] K.A. Intriligator, New RG fixed points and duality in supersymmetric  $SP(N(c))$  and  $SO(N(c))$  gauge theories, *Nucl. Phys. B* 448 (1995) 187, hep-th/9505051.
- [31] R.G. Leigh, M.J. Strassler, Exactly marginal operators and duality in four-dimensional  $N = 1$  supersymmetric gauge theory, *Nucl. Phys. B* 447 (1995) 95, hep-th/9503121.