## Supplementary Information

# Photonic resource state generation from a minimal number of quantum emitters 

Bikun Li, Sophia E. Economou, ${ }^{\dagger}$ and Edwin Barnes ${ }^{\ddagger}$<br>Department of Physics, Virginia Tech, Blacksburg, Virginia 24061, USA

## Supplementary Note 1

This supplementary material contains several additional examples of generation protocols produced using our algorithm. We begin with a simple example that illustrates in detail how our algorithm works. We then provide solutions for more complicated examples of practical importance, including error correcting codes and repeater graph states of arbitrary size. These more complicated examples are solved numerically using MATLAB codes that are available on GitHub [1]. This code expresses the generation sequence in terms of an MPS [2] for bookkeeping purposes:

$$
\begin{equation*}
|\Psi\rangle=U_{p, \text { tot. }}\left\langle\psi_{f}\right|\left[\prod_{j=1}^{n_{p}}\left(\hat{M}_{j} U_{e, j} \hat{E}_{\eta_{j}}\right)\right] W_{0}\left|\psi_{0}\right\rangle \tag{1}
\end{equation*}
$$

in which the initial and final states of emitter qubits are simply product states of $|0\rangle:\left|\psi_{f}\right\rangle=\left|\psi_{0}\right\rangle=|0\rangle^{\otimes n_{e}}$. We denote $\eta_{j}$ as the emitter qubit that emits the $j$-th photon and $\mu_{j}$ as the emitter qubit that is measured after emitting the $j$-th photon. $\hat{E}_{\eta_{j}}$ is the emission tensor, $\hat{E}_{\eta_{j}}=|0\rangle_{j}|0\rangle_{\eta_{j}}\left\langle\left. 0\right|_{\eta_{j}}+\mid 1\right\rangle_{j}|1\rangle_{\eta_{j}}\left\langle\left. 1\right|_{\eta_{j}}\right.$, that describes emission of photon $j$ from emitter $\eta_{j}$, which can be represented as $\mathrm{CNOT}_{\eta_{j}, j} . U_{e, j}$ is the unitary operation obtained from the $j$-th photon absorption step, which transforms $g_{a}$ as explained in the main text (Sec.II.A). $\hat{M}_{j}$ is identity if no measurement happens ( $\mu_{j}$ is not assigned), otherwise, $\hat{M}_{j}=W_{j} H_{\mu_{j}} X_{\mu_{j}}^{s_{j}} \hat{\pi}_{\mu_{j}}$, with projection $\hat{\pi}_{\mu_{j}} \equiv \frac{1}{2}\left[\mathbb{1}+(-1)^{s_{j}} Z_{\mu_{j}}\right]$ and its random outcome $s_{j} \in\{0,1\}$. Here, $W_{j}$ is the unitary operation that is obtained from time-reversed measurement (Sec.IV.B), and $W_{0}$ is the unitary operation that disentangles all emitters at the final stage of the time-reversed sequence. Finally, $U_{p, \text { tot }}=\prod_{j}\left(X_{j}^{s_{j}} U_{p, j}\right)$ is the local Clifford operation that acts on photons with conditional $X_{j}^{s_{j}}$ flipping. The profile of the solution is stored as $\left\{U_{e, j}, U_{p, j}, \mu_{j}, \eta_{j}, W_{j}, W_{0}\right\}$. We note that such a solution is usually not unique due to there being multiple choices for how to choose the emitter gates and emitter sites in each photon absorption and time-reversed measurement.

As discussed in the main text, the height function plays a central role in determining the number of emitters and the operation sequence needed to generate a target photonic graph state. As shown in Eq. (1) of the main text, when the stabilizers $g_{m}$ are in the echelon gauge, the height function can be expressed as

$$
\begin{equation*}
h(x)=n-x-\#\left\{g_{m} \mid 1\left(g_{m}\right)>x\right\} \tag{2}
\end{equation*}
$$

where $l\left(g_{m}\right)$ is the index of the left-most (smallest index) site on which $g_{m}$ acts nontrivially. In the main text, we showed that the difference in the height function across adjacent sites determines whether we perform a photon absorption or a time-reversed measurement at each step of the algorithm. Therefore, we define

$$
\begin{equation*}
\delta h(x) \equiv h(x)-h(x-1)=\#\left\{g_{m} \mid 1\left(g_{m}\right)=x\right\}-1 \tag{3}
\end{equation*}
$$

from which it is apparent that this difference only depends on the number of stabilizers (in the echelon gauge) that have a left-ending on site $x$.

[^0]We begin by demonstrating our protocol solver algorithm in the case of the simple 4-photon graph state displayed in Supplementary Fig. 1(a). The stabilizers are given by

$$
\begin{align*}
& g_{1}=\sigma_{1}^{x} \sigma_{2}^{z} \sigma_{3}^{z}, \quad g_{2}=\sigma_{1}^{z} \sigma_{2}^{x} \sigma_{3}^{z} \sigma_{4}^{z},  \tag{4}\\
& g_{3}=\sigma_{1}^{z} \sigma_{2}^{z} \sigma_{3}^{x} \sigma_{4}^{z}, \quad g_{4}=\sigma_{2}^{z} \sigma_{3}^{z} \sigma_{4}^{x} .
\end{align*}
$$

We can switch to the echelon gauge by redefining $g_{3} \rightarrow g_{2} g_{3}$. We then calculate the height function using Supplementary Eq. (22, finding that the maximum is 2 . Therefore, at least $n_{e}=2$ emitter qubits are needed, and so we assemble a 6 -qubit lattice. We can depict the complete set of 6 stabilizers as a tableau, as shown in Supplementary Fig. 1(b).

In Supplementary Fig. 1(c), we first obtain inset (1) by transforming Supplementary Fig. 1(b) to the echelon gauge. The upper left sub-block of the tableau is exactly Supplementary Eq. (4) with $g_{3} \rightarrow g_{2} g_{3}$. Next we describe in detail how the generator set is updated from inset (1) to inset (17) step by step. The column label $j$ indicates which photon we are currently focusing on, and the labels (i),...,(iv) indicate the specific step of our algorithm. For each photon, we do the following steps:

- $j=4$ : (i) Obtain inset (1) by transforming to echelon gauge: $g_{3} \rightarrow g_{2} g_{3}$. (ii) Supplementary Eq. (3) gives $\delta h(4)=-1$. Perform a time-reversed measurement on emitter site 5 by applying a Hadamard $H_{5}$ followed by $\mathrm{CNOT}_{54}$, which yields inset (2). (iii) Let $g_{a}=g_{5}=\sigma_{4}^{x} \sigma_{5}^{x}$ in inset (2). One gets inset (3) by performing Hadamards on sites 4 and 5 . Then the 4 -th photon is absorbed into the emitter on site 5 by applying CNOT $_{54}$. Replace $g_{4} \rightarrow g_{4} g_{5}$ to eliminate the redundant $\sigma_{4}^{z}$, yielding inset (4). $\left(\mu_{4}=5, \eta_{4}=5, U_{p, 4}=H_{4}, U_{e, 4}=H_{5}\right.$, $W_{4}=H_{5}$.)
- $j=3$ : (i) Skip this step since inset (4) is already in echelon gauge. (ii) Supplementary Eq. (3) gives $\delta h(3)=-1$. Perform a time-reversed measurement on emitter site 6 by applying $H_{6}$ followed by CNOT 63 . Inset (6) is then obtained by redefining $g_{5} \leftrightarrow g_{6}$. (iii) Let $g_{a}=g_{5}=\sigma_{3}^{x} \sigma_{6}^{x}$ in inset (6). One gets inset (7) by applying Hadamards on sites 3 and 6 . Then the 3 rd photon is absorbed by applying CNOT $_{63}$. Replace $g_{3} \rightarrow g_{3} g_{5}$ to eliminate the redundant $\sigma_{3}^{z}$. Thus, inset (7) becomes (8). $\left(\mu_{j}=6, \eta_{3}=6, U_{p, 3}=H_{3}, U_{e, 3}=H_{6}\right.$, $W_{3}=H_{6}$.)
- $j=2$ : (i) Skip this step since inset (8) is already in echelon gauge. (ii) Skip this step since Supplementary Eq. (3) gives $\delta h(2)=1$. (iii) Choose $g_{a}=g_{4}=\sigma_{2}^{z} \sigma_{5}^{z} \sigma_{6}^{x}$ in inset (10). One gets inset (11) by applying $H_{6}$ followed by $\mathrm{CNOT}_{65}$ on the emitters, so that $g_{a} \rightarrow \sigma_{2}^{z} \sigma_{5}^{z}$. Then the 2 nd photon is absorbed into emitter 5 by applying $\mathrm{CNOT}_{52}$. Redefine $g_{k} \rightarrow g_{k} g_{4}$ for $k=1,3$ to eliminate the redundant $\sigma^{z}$ 's. Thus, inset (11) becomes (12). $\left(\eta_{2}=5, U_{p, 2}=\mathbb{1}, U_{e, 2}=H_{6} \mathrm{CNOT}_{65}, \hat{M}_{2}=\mathbb{1}\right.$.)
- $j=1$ : (i) Obtain inset (13) from (12) by transforming to echelon gauge: $\left(g_{3}, g_{4}, g_{5}, g_{6}\right) \rightarrow\left(g_{4}, g_{5}, g_{6}, g_{3}\right)$. (ii) Skip this step since Supplementary Eq. (3) gives $\delta h(1)=1$. (iii) Choose $g_{a}=g_{2}=\sigma_{1}^{z} \sigma_{5}^{x} \sigma_{6}^{z}$ in inset (14). One gets inset (15) by applying $H_{5}$ and then $\mathrm{CNOT}_{65}$ to transform $g_{a} \rightarrow \sigma_{1}^{z} \sigma_{5}^{z}$. Then the 1 st photon is absorbed into emitter site 5 by applying $\mathrm{CNOT}_{51}$. Thus, inset (15) becomes $(16) .\left(\eta_{1}=5, U_{p, 1}=\mathbb{1}\right.$, $U_{e, 1}=H_{5} \mathrm{CNOT}_{65}, \hat{M}_{1}=\mathbb{1}$.)
- (iv) Finally, to recover the state $|0\rangle^{\otimes n}$, one needs to disentangle the emitter qubits. This can be done with the following gate sequence: $H_{5} \mathrm{CNOT}_{56} H_{5}$. In the last step, we permute the $g_{m}$ to obtain inset (17). $\left(W_{0}=H_{6} \mathrm{CNOT}_{56} H_{5}.\right)$

Now that the algorithm is complete, we reverse all the operations to obtain the final generation sequence. This circuit is shown in Supplementary Fig. 2(a). It is worth noting that, in this example, the emission sequence can be further optimized by swapping the 1st and 3rd photons in the emission order, such that the maximum of $h(x)$ is reduced to 1 . Thus, only one emitter qubit is needed in this case, and the corresponding generation circuit is displayed in Supplementary Fig. 2(b).

## Supplementary Note 2

In this subsection, we demonstrate how to generate a useful quantum error correction code, with some continuous logical rotation. In particular, we present an emission sequence for the Shor code [3] with 9 photonic qubits, which is able to protect a qubit from single bit-flip and phase-flip errors. The stabilizer generators of this code are well known: $g_{j}=\sigma_{j}^{z} \sigma_{j+1}^{z}$ for $j=1,2,4,5,7,8$, and $g_{3}=\sigma_{1}^{x} \sigma_{2}^{x} \cdots \sigma_{6}^{x}, g_{6}=\sigma_{4}^{x} \sigma_{5}^{x} \cdots \sigma_{9}^{x}$ [4]. We can also define the logical operators $X_{L} \equiv \sigma_{1}^{z} \sigma_{2}^{z} \cdots \sigma_{9}^{z}, Z_{L} \equiv \sigma_{1}^{x} \sigma_{2}^{x} \cdots \sigma_{9}^{x}$ and $Y_{L}=i X_{L} Z_{L}$. Let the last stabilizer be $g_{9}= \pm X_{L}$, which
(a)

(b)


(ii)


(d)


- photon
- emitter

Supplementary Figure 1: Step-by-step illustration of the protocol solver. (a) A target graph state with 4 photons. (b) The set of generators $\mathcal{G}_{f}=\left\{g_{m}\right\}$ is depicted as a tableau in which each row corresponds to one generator. Different colors correspond to different Pauli operators. The first 4 columns correspond to photonic qubits, and the last 2 columns correspond to emitters. (c) Step by step demonstration of how to obtain the timereversed generation sequence, where $\mathcal{G}_{0}=\left\{\sigma_{i}^{z}\right\}$ is finally obtained. Explanations are in the main text. (d) Local Clifford equivalent graph state representations of tableaux in (c).


Supplementary Figure 2: Graph state generation circuits. In this figure, $p_{j}(j=1,2,3,4)$ labels different photonic qubits, and $e_{1}$ and $e_{2}$ are emitter qubits. At the end of each circuit, the photon qubits are in the target graph state displayed at the top right, while the emitter qubits are in state $|0\rangle$ after the measurements. (a) The emission circuit obtained from the steps in Supplementary Fig. 1. (b) A different generation circuit that produces the same target graph state as in (a). This circuit is obtained by swapping qubits $1 \leftrightarrow 3$, resulting in a circuit that requires only one emitter.
determines a pair of logical space basis states $| \pm\rangle_{L}$. For both choices, Supplementary Eq. (2) gives $h_{\max }=2$, and the emission circuit solutions for $| \pm\rangle_{L}$ are given in Supplementary Fig. 3, where $| \pm\rangle_{L}$ are separately given by $R=\mathbb{1}$ and $R=X_{e_{1}}$. Therefore, by replacing $R$ by a more general $x$-rotation, $e^{i \frac{\varphi}{2} X_{e_{1}}} \equiv \mathbb{1} \cos \frac{\varphi}{2}+i X_{e_{1}} \sin \frac{\varphi}{2}$, we can obtain a rotated logical qubit $|\varphi\rangle_{L}=e^{i \frac{\varphi}{2} X_{L}}|+\rangle_{L}$, with an arbitrary angle $\varphi$. That is, the circuit in Supplementary Fig. 3 allows us to transmit a rotated photonic logical qubit protected by the Shor code, with merely 2 emitter qubits, 1 two-qubit gate and 2 measurements, which is surprisingly simple.

## Supplementary Note 3

In this section, we generalize the repeater graph state example from the main text and present explicit generation circuits for repeater graphs with arbitrarily many photons. As shown in Supplementary Fig. 4, for a repeater graph state with $2 m$ photons, the maximum of the height function indicates that we need 2 emitters regardless how large $m$ is ( $m \geq 4$ ). The unitary operations $A, B$ and $C$ displayed in Supplementary Fig. 5(a) depend on $m$ :

$$
\begin{equation*}
A=X_{e_{2}}^{\lfloor m / 2\rfloor+1}, \quad B=X_{e_{1}}^{\lfloor m / 2\rfloor}, \quad C=P_{e_{2}}^{m}, \tag{5}
\end{equation*}
$$

where $X_{i}=\sigma_{i}^{x}$, and $P=\operatorname{diag}(1, i)$. Compared to the approach given in previous work [5], which requires $m-1$ two-qubit gates and $m$ measurements, the new solution in Supplementary Fig. 4 (a) uses $2 m-3$ two-qubit gates and


Supplementary Figure 3: Shor code example. The emission circuit that generates a logical state of the Shor code, controlled by a local operation $R$ (the yellow block). The inset displays the height function of $| \pm\rangle_{L}$, which has $h_{\max }=2$.
$m-1$ measurements, reducing the number of measurements needed to produce the state. We highlight that our method yields different solutions with flexible settings, so actually the solution from Ref. [5] can also be obtained from our algorithm.


Supplementary Figure 4: Example of a large RGS. (a) and (b) show the emission circuit and height function for the repeater graph state of $2 m$ photons displayed in (c). The boxed area of the circuit is repeated multiple times to generate photons $p_{4}, p_{5}, p_{6}, \cdots, p_{2 m-6}$.

## Supplementary Note 4

Finally, we consider a modified repeater graph state that includes some additional redundancy to further boost the likelihood of successful Bell measurements [5]. Supplementary Fig. 5(a) shows an example of such a repeater graph state with $6 m$ photons $(m>3)$. Note that compared to the state in Supplementary Fig. 4(c), this state contains twice as many external photon arms and is missing those internal edges that are not necessary for the functionality of this state as a repeater. Supplementary Fig. 5(b) shows that the height function is at most 2 for any $m>3$, i.e., only two emitter qubits are needed to generate the state in (a). We list all operations in the generation circuit in Supplementary Eq. (1). Denoting the $e_{1}$-th and the $e_{2}$-th qubits as the emitter qubits, where $e_{1}=6 m+1$ and


Supplementary Figure 5: Modified RGS example. (a) The graph for RGS, which has $6 m$ vertices. The labels represent the emission sequence. (b) The height function $h(x)$ for the target graph state in (a).
$e_{2}=6 m+2$, the circuit is given by:

$$
\begin{align*}
& U_{p, j}=\left\{\begin{array}{ll}
\mathbb{1}, & j=1(\bmod 3) \\
H_{j}, & \text { otherwise }
\end{array},\right.  \tag{6}\\
& U_{e, j}=\left\{\begin{array}{l}
\mathrm{CNOT}_{e_{2} e_{1}}, \quad j=3 \\
H_{e_{1}}, \quad j=3 k \text { with } 2 \leq k \leq 2 m \\
H_{e_{2}}, \quad j=6 m-3 \\
\mathbb{1}, \quad \text { otherwise }
\end{array}\right. \\
& W_{j}=\left\{\begin{array}{l}
\mathrm{CNOT}_{e_{1} e_{2}}, \quad j=3 k \text { with } 2 \leq k \leq 2 m-2, \\
\quad \text { and } k \neq m-1
\end{array}, .\right.
\end{align*}
$$

In the above circuit, there are $2 m-1$ measurements, and $2 m-1$ CNOT gates. Plugging these operations into Supplementary Eq. (1) gives the full sequence.

## Supplementary References

[1] Bikun Li. Photon-emission-circuit-solver: Emission circuit solver for photonics stabilizer state, (2021). (https://doi.org/10.5281/zenodo.5652105).
[2] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf. Sequential generation of entangled multiqubit states. Phys. Rev. Lett., 95:110503, (2005).
[3] Peter W. Shor. Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A, 52:R2493-R2496, (1995).
[4] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, (2010).
[5] Donovan Buterakos, Edwin Barnes, and Sophia E. Economou. Deterministic generation of all-photonic quantum repeaters from solid-state emitters. Phys. Rev. X, 7:041023, (2017).


[^0]:    *libk@vt.edu
    †economou@vt.edu
    ¥efbarnes@vt.edu

