Analytical approach to swift non-leaky entangling gates in superconducting qubits: Supplementary Information

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I. *CNOT* GATE FROM PARTIAL REVERSE-ENGINEERING

Consider two independent two-level systems with energies that differ by ϵ which are driven by the same external field with envelope Ω and frequency ω. We can express the total Hamiltonian for both systems as

$$H_{tot}^{lab} = \begin{pmatrix} -E/2 & \Omega(t)e^{i\omega t} & 0 & 0 \\ \Omega(t)e^{-i\omega t} & E/2 & 0 & 0 \\ 0 & 0 & -(E-\epsilon)/2 & \Omega(t)e^{i\omega t} \\ 0 & 0 & \Omega(t)e^{-i\omega t} & (E-\epsilon)/2 \end{pmatrix}, \tag{1}$$

where the first 2×2 block corresponds to system 1, the second to system 2. This Hamiltonian generates the evolution of the logical qubit subspace of two superconducting qubits in the case where only one qubit is driven. In particular, if we choose our basis states to be $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, then H_{tot}^{lab} applies when only the second qubit is driven. In this context, designing entangling two-qubit gates is tantamount to choosing $\Omega(t)$ in such a way that it performs a desired operation on system 1 while leaving system 2 alone. (Note that the results we obtain could also be applied in the context of single-qubit gates, in which case systems 1 and 2 correspond to separate qubits.) More precisely, we will find forms of $\Omega(t)$ such that $\Omega(0) = 0$ and $\Omega(\tau_p) = 0$, and where the evolution operator for system 2 at time $t = \tau_p$ coincides with the form it would have in the absence of any driving, namely

$$U_2^{lab}(\tau_p) = e^{i\frac{E-\epsilon}{2}\tau_p\sigma_z}, \tag{2}$$

where Δ is the detuning of the pulse relative to the resonant frequency of system 2:

$$\Delta \equiv \omega - E + \epsilon. \tag{3}$$

Note that our conventions are such that $\Delta = \epsilon$ when the pulse is resonant with system 1. The driving field should at the same time implement a desired operation on system 1; in the case where the pulse is resonant with system 1 ($E = \omega$), this is given by

$$U_1^{lab}(\tau_p) = e^{i\frac{E}{2}\tau_p\sigma_z}e^{-i\int_0^{\tau_p}dt\Omega(t)\sigma_x}. \tag{4}$$

In this case, we can only implement x rotations (on top of the system's free evolution), where the rotation angle is determined by the integral of Ω:

$$\phi = 2\int_0^{\tau_p} dt\Omega(t). \tag{5}$$

Thus when the pulse is resonant, we can fix the target operation on system 1 by choosing the area of the pulse appropriately.

It is more challenging to find conditions on Ω which will ensure that system 2 is unaffected by the pulse. To do this, we would like to borrow the formalism from Ref. [1]. In order to make use of this formalism, we first need to apply frame and basis transformations to bring the Hamiltonian into the form of Eq. (1):

$$U_2^{lab}(t) = e^{i\frac{\omega}{2}t\sigma_z}e^{-i\frac{\pi}{4}\sigma_y}U_2^{\chi}(t), \tag{6}$$

where U_2^{χ} is the evolution operator associated with the following Hamiltonian:

$$H_2^{\chi} = \begin{pmatrix} \Omega(t) & -\Delta/2 \\ -\Delta/2 & -\Omega(t) \end{pmatrix}. \tag{7}$$

H_2^{χ} has the form appropriate for a direct application of the methods derived in Ref. [1]. We modify slightly the formalism of that reference to render the initial conditions more natural for the present context, so that U_2^{χ} can be written as

$$U_2^{\chi} = \begin{pmatrix} \cos\chi e^{i\psi_-} & \sin\chi e^{-i\psi_+} \\ -\sin\chi e^{i\psi_+} & \cos\chi e^{-i\psi_-} \end{pmatrix}, \tag{8}$$

where

$$\psi_\pm = \int_0^t dt' \sqrt{\Delta^2/4 - \dot{\chi}^2(t')} \csc(2\chi(t')) \pm \frac{1}{2} \arcsin(2\dot{\chi}(t)/\Delta). \tag{9}$$

To obtain this evolution operator from the standard form given in Ref. [1], we multiplied the latter on the right by the constant matrix $e^{i\eta\frac{\pi}{4}\sigma_z}$. The advantage of using the form shown in Eq. (8) is that we may impose the initial condition $U_2^\chi(0) = e^{i\frac{\pi}{4}\sigma_y}$ through the following simple conditions on $\chi(t)$:

$$\chi(0) = \pi/4, \qquad \dot{\chi}(0) = 0. \tag{10}$$

The driving field for a chosen χ is determined from the formula

$$\Omega(t) = \frac{\ddot{\chi}}{2\sqrt{\Delta^2/4 - \dot{\chi}^2}} - \sqrt{\Delta^2/4 - \dot{\chi}^2} \cot(2\chi). \tag{11}$$

In order for the resulting control field to be physical, we must impose the constraint $|\dot{\chi}(t)| \leq |\Delta|/2$ for all times.

We are now ready to consider the condition for the pulse to not alter the evolution of system 2, Eq. (2). This condition requires

$$U_2^\chi(\tau_p) = e^{i\frac{\pi}{4}\sigma_y} e^{-i\frac{\Delta}{2}\tau_p\sigma_z} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\Delta\tau_p/2} & e^{i\Delta\tau_p/2} \\ -e^{-i\Delta\tau_p/2} & e^{i\Delta\tau_p/2} \end{pmatrix}. \tag{12}$$

Comparing this result to Eq. (8) reveals that we can satisfy the "invisibility" constraint if we impose the following set of conditions on χ at the final time τ_p:

$$\chi(\tau_p) = \pi/4, \qquad \dot{\chi}(\tau_p) = 0, \qquad \xi_0(\tau_p) \equiv \int_0^{\tau_p} dt \sqrt{\Delta^2/4 - \dot{\chi}^2} \csc(2\chi) = -\Delta\tau_p/2 + 2\pi n. \tag{13}$$

We thus have a total of six constraints that $\chi(t)$ must satisfy: $|\dot{\chi}(t)| \leq |\Delta|/2$, Eq. (10), and Eq. (13). If we multiply the right-hand side of Eq. (8) by $e^{i\frac{\pi}{2}\sigma_y}$, then we could alternatively impose the conditions

$$\chi(0) = \chi(\tau_p) = -\pi/4, \qquad \dot{\chi}(0) = \dot{\chi}(\tau_p) = 0, \qquad \xi_0(\tau_p) \equiv \int_0^{\tau_p} dt \sqrt{\Delta^2/4 - \dot{\chi}^2} \csc(2\chi) = \Delta\tau_p/2 + 2\pi n, \tag{14}$$

in addition to $|\dot{\chi}(t)| \leq |\Delta|/2$. This just shows that for any solution $\chi(t)$, we also have a solution $-\chi(t)$, which corresponds to flipping the sign of $\Omega(t)$ and equivalently that of the rotation angle ϕ.

In order to consider specific examples of χ which satisfy these constraints, we must first specify what sort of operation we wish to perform on system 1. For concreteness, we will take this operation to be a π rotation about x, requiring us to impose $\Delta = \epsilon$ and

$$\int_0^{\tau_p} dt\Omega(t) = \pi/2. \tag{15}$$

In the context where systems 1 and 2 correspond to two separate transitions in the logical subspace of two superconducting qubits, implementing a π rotation about x on system 1 while preserving the state of system 2 is tantamount to performing a two-qubit $CNOT$ gate. A simple ansatz for χ that can satisfy the constraints is

$$\chi(t) = A(t/\tau_p)^4(1 - t/\tau_p)^4 + \pi/4. \tag{16}$$

The constraints $\chi(0) = \pi/4 = \chi(\tau_p)$ and $\dot{\chi}(0) = 0 = \dot{\chi}(\tau_p)$ are automatically satisfied. We may then tune the parameters A and τ_p to satisfy the constraints $\int_0^{\tau_p} dt\Omega(t) = \pi/2$ and $\xi_0(\tau_p) = -\epsilon\tau_p/2 + 2\pi n$ while making sure that $|\dot{\chi}| \leq |\epsilon|/2$. We find that there is a solution with $n = 1$ and

$$A = 138.9, \qquad \tau_p = 5.87/|\epsilon|. \tag{17}$$

The resulting pulse is shown in Fig. 2 of the main text.

To test that this pulse performs as expected, we use it to implement a $CNOT$ gate in which system 1 is spanned by the states $|00\rangle$ and $|01\rangle$, while system 2 is spanned by $|10\rangle$ and $|11\rangle$. Thus our target transition will be $|00\rangle \Leftrightarrow |01\rangle$, and the harmful is $|10\rangle \Leftrightarrow |11\rangle$. We retain three qubit states and four cavity states in our numerical simulations, corresponding a total Hilbert space dimension of 48. The lower-energy states in the interacting spectrum are shown in Fig. S1. As discussed in the main text, the resulting CNOT gate has fidelity 99%.

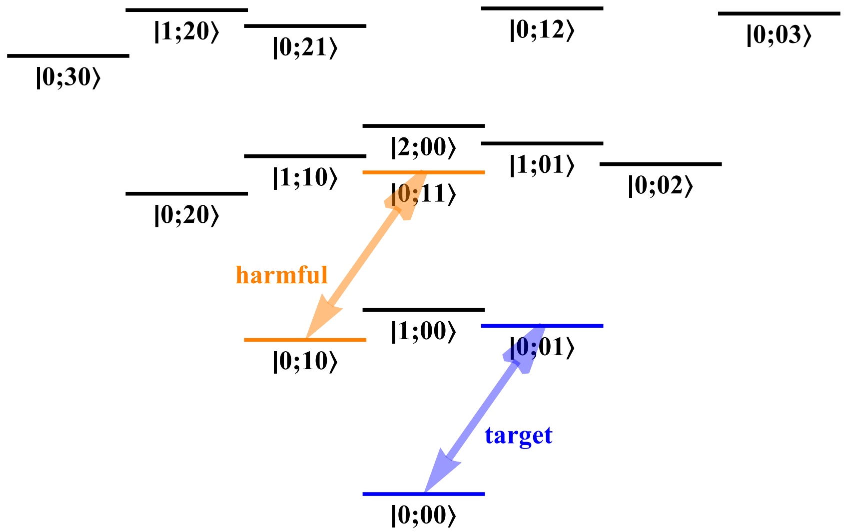

FIG. 1: (Color online) Level diagram for qubit-cavity system with ω_c=7.15 GHz, $\epsilon_{1,1}$=6.2 GHz, $\epsilon_{2,1}$=6.8 GHz, η=350 MHz, g=250 MHz. Three cavity and four qubit levels are kept in numerical simulations to yield converged results. The state labels indicate the largest component of each interacting state, with the first number corresponding to the cavity level. The target and harmful transitions chosen for the CNOT gate are indicated.

II. CZ GATE BASED ON SECH PULSES

We again consider two independent two-level systems as in the previous section, only this time the goal is to induce phases on each of them with a single pulse such that the difference of these phases is $\pm\pi$. In terms of the two-superconducting-qubit system, this amounts to performing a two-qubit CZ gate. A smooth, simple pulse shape which induces the desired phases in an analytically controllable fashion is a hyperbolic secant pulse: $\Omega(t) = \Omega_0 \mathrm{sech}(\sigma t)$. A hyperbolic secant pulse of area 4π acting on a two-level system induces the phase[2]

$$\phi = 2\arctan\left(\frac{4\Delta/\sigma}{(\Delta/\sigma)^2 - 3}\right), \tag{18}$$

where Δ is the detuning and σ is the bandwidth. This result can be derived by starting from the fact that the Schrödinger equation for a sech pulse can be solved exactly in terms of hypergeometric functions.[3,4] In particular, the amplitude to return to the ground state after the passage of the pulse is given by

$$c_g = {}_2F_1(a, -a, c^*, 1) = \frac{\Gamma(c^*)^2}{\Gamma(c^* - a)\Gamma(c^* + a)}, \tag{19}$$

where $c^* \equiv \frac{1}{2}(1 - \Delta/\sigma)$ and $a \equiv \Omega_0/\sigma$. In the case of a 4π pulse, $a = 2$, and the amplitude simplifies to

$$c_g = e^{-i\phi}, \tag{20}$$

with ϕ given in Eq. (18). Using this expression for ϕ, we obtain the difference

$$\phi_1 - \phi_2 = 2\arctan\left(\frac{4x_1}{x_1^2 - 3}\right) - 2\arctan\left(\frac{4x_2}{x_2^2 - 3}\right), \tag{21}$$

where ϕ_1 and ϕ_2 are the phases acquired by systems 1 and 2, respectively, and where we have defined $x_j \equiv \Delta_j/\sigma$. Making use of the simplification,

$$\begin{aligned} \tan\left(\frac{\phi_1 - \phi_2}{2}\right) &= \tan\left(\arctan\left(\frac{4x_1}{x_1^2 - 3}\right) - \arctan\left(\frac{4x_2}{x_2^2 - 3}\right)\right) \\ &= \frac{4(x_2 - x_1)(3 + x_1 x_2)}{9 + 16x_1 x_2 - 3x_1^2 - 3x_2^2 + x_1^2 x_2^2}, \end{aligned} \tag{22}$$

we see that the phase difference will be π when

$$9 + 16x_1x_2 - 3x_1^2 - 3x_2^2 + x_1^2x_2^2 = 0, \tag{23}$$

from which we obtain

$$x_2 = x_\pm(x_1) = \frac{-8x_1 \pm \sqrt{27 + 46x_1^2 + 3x_1^4}}{x_1^2 - 3}. \tag{24}$$

Using the fact that $\omega_p = \omega_1 + \Delta_1 = \omega_2 + \Delta_2$, we obtain for the bandwidth

$$\sigma = \frac{\omega_1 - \omega_2}{x_2 - x_1}. \tag{25}$$

Since we have two solutions for x_2 it is natural to define two functions for the bandwidth:

$$\sigma_\pm(x_1) = \frac{\omega_1 - \omega_2}{x_\pm(x_1) - x_1}. \tag{26}$$

Depending on the values of the parameters, the physical solution will change. Specifically, we have for $\omega_1 < \omega_2$

$$\sigma = \begin{cases} \sigma_+(x_1) & -\sqrt{3} < x_1 < \sqrt{3} \\ \sigma_\pm(x_1) & x_1 > \sqrt{3} \end{cases}. \tag{27}$$

Note that there are two solutions for $x_1 > \sqrt{3}$ and no solutions exist for $x_1 < -\sqrt{3}$. On the other hand, when $\omega_1 > \omega_2$,

$$\sigma = \begin{cases} \sigma_-(x_1) & -\sqrt{3} < x_1 < \sqrt{3} \\ \sigma_\pm(x_1) & x_1 < -\sqrt{3} \end{cases}, \tag{28}$$

and now there are two solutions for $x_1 < -\sqrt{3}$ and no solutions exist for $x_1 > \sqrt{3}$. We also define the maximum value of the bandwidth, which is given by

$$\sigma_{max} = |\omega_1 - \omega_2| \frac{2 + \sqrt{7}}{6}. \tag{29}$$

For $\omega_1 < \omega_2$ this occurs at $x_{1,max} = \sqrt{7} - 2$, while for $\omega_1 > \omega_2$ it occurs at $x_{1,max} = 2 - \sqrt{7}$.

Instead of viewing our prescription for the gate as being defined by fixing the bandwidth as a function of x_1, we may instead consider treating the detuning as a function of the bandwidth. We may assume without loss of generality that $\omega_1 > \omega_2$. In this case, we obtain the detuning Δ_1 as a function of bandwidth σ by solving the equation $\sigma_-(\Delta_1/\sigma) = \sigma$ for Δ_1. There are four solutions:

$$\Delta_1^{\pm\pm} = -\frac{1}{2}\left[\omega_1 - \omega_2 \pm \sqrt{(\omega_1 - \omega_2)^2 - 20\sigma^2 \pm 4\sigma\sqrt{16\sigma^2 + 3(\omega_1 - \omega_2)^2}}\right], \tag{30}$$

where the first $\pm$ in the exponent refers to the sign outside the square root and the latter to the sign inside the square root. It is clear from this expression that as $\sigma \to 0$, the solutions Δ_1^{-+} and Δ_1^{--} go to zero, while the solutions Δ_1^{+-} and Δ_1^{++} go to the value $\omega_2 - \omega_1$. In other words, the former two solutions correspond to the pulse becoming resonant with system 1 in the small bandwidth limit, while the latter two solutions correspond to the pulse becoming resonant with system 2 in this limit. If we wish, we can think of solutions Δ_1^{-+} and Δ_1^{--} as corresponding to the case where system 1 is the "target" transition, while solutions Δ_1^{+-} and Δ_1^{++} correspond to taking system 2 as the target. Also notice that the solutions Δ_1^{-+} and Δ_1^{++} exist for a broader range of σ (up to σ_{max}) because the quantity in the square root is more positive in these solutions. These two solutions therefore allow for faster pulses and are of primary interest. In summary, the solution Δ_1^{-+} allows for fast pulses and treats system 1 as the target transition in the slow pulse limit; the solution Δ_1^{++} allows for fast pulses and treats system 2 as the target transition in the slow pulse limit. In both cases, the pulse frequency is determined via

$$\omega_p = \omega_1 + \Delta_1^{\pm+}(\sigma). \tag{31}$$

When $\sigma = \sigma_{max}$, these two solutions take particularly simple forms:

$$\Delta_1^{\pm+}(\sigma_{max}) = (\omega_2 - \omega_1)/2 \quad \Rightarrow \quad \omega_p(\sigma_{max}) = (\omega_1 + \omega_2)/2. \tag{32}$$

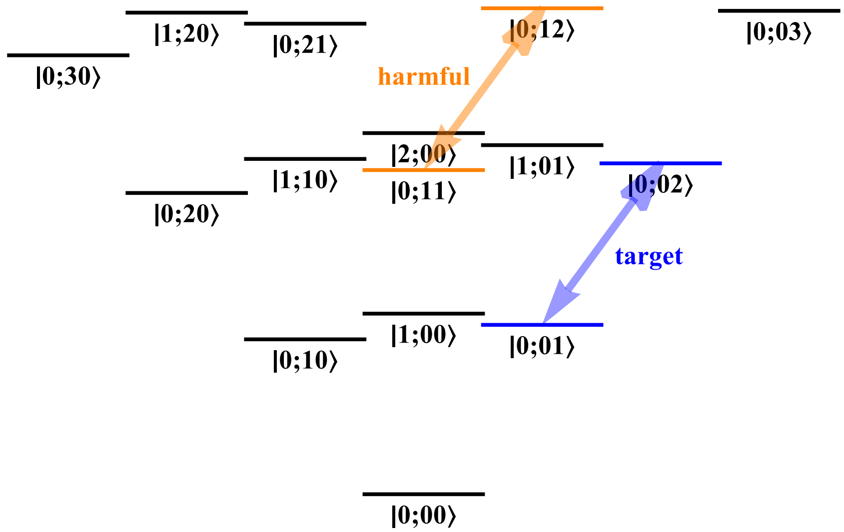

FIG. 2: (Color online) Level diagram for qubit-cavity system with ω_c=7.15 GHz, $\epsilon_{1,1}$=6.2 GHz, $\epsilon_{2,1}$=6.8 GHz, η=350 MHz, g=130 MHz. Three cavity and four qubit levels are kept in numerical simulations to yield converged results. The state labels indicate the largest component of each interacting state, with the first number corresponding to the cavity level. The target and harmful transitions chosen for the CZ gate are indicated.

A CZ gate can also be constructed from a 2π sech pulse. In this case, we set $a = 1$ in Eq. (19) and find that the phase acquired is[4]

$$\phi = 2\arctan(\sigma/\Delta). \tag{33}$$

Repeating the above analysis in this case leads to the following expression for the pulse frequency in terms of bandwidth:

$$\omega_p = \frac{\omega_1 + \omega_2}{2} \pm \frac{1}{2}\sqrt{(\omega_1 - \omega_2)^2 - 4\sigma^2}. \tag{34}$$

There is again a maximal value of the bandwidth, this time at $\sigma_{max} = |\omega_1 - \omega_2|/2$, at which value the pulse frequency is the average of the two transition frequencies:

$$\omega_p(\sigma_{max}) = (\omega_1 + \omega_2)/2. \tag{35}$$

To test this approach, we ran full simulations, keeping three cavity and four qubit levels. The interacting spectrum is shown in Fig. S2, where the chosen target and harmful transitions are indicated. As shown in Fig. 4 of the main text, the resulting CZ gate exhibits fidelities ranging from 99.7% to 99.99% depending on the chosen bandwidth and on whether one uses the 2π or 4π sech pulse.

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