5.8 A localized cylindrically symmetric current distribution is such that the ~~~~ current flows only in the azimuthal direction : the current density is a function only of r and θ (or ρ and z):

$$\bar{J} = \hat{\phi}\, J(r,\theta)$$

The distribution is 'hollow' in the sense that there is a current-free region near the origin, as well as outside.

a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L m_L\, r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L \mu_L\, r^{-L-1} P_L^1(\cos\theta)$$

outside the current distribution.

b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int d^3x\, r^{-L-1} P_L^1(\cos\theta)\, J(r,\theta)$$

$$\mu_L = -\frac{1}{L(L+1)}\int d^3x\, r^L P_L^1(\cos\theta)\, J(r,\theta)$$

We'll solve both parts simultaneously.

5.8, cont'd

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

Expand $\displaystyle \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \hat{\phi}' J(r',\theta')\,(4\pi)\sum_{\ell,m} \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)$$

Note that the direction of $\hat{\phi}'$ changes as one integrates over d^3x'.
Let's expand $\hat{\phi}'$ in terms of unprimed unit vectors.
To that end, recall that in Cartesian coordinates,

$$\hat{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\hat{\theta} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$$

$$\hat{\phi} = (-\sin\phi,\ \cos\phi,\ 0)$$

Expand

$$\begin{aligned}\hat{\phi}' &= (\hat{\phi}'\cdot\hat{r})\hat{r} + (\hat{\phi}'\cdot\hat{\theta})\hat{\theta} + (\hat{\phi}'\cdot\hat{\phi})\hat{\phi} \\ &= (-\sin\phi'\sin\theta\cos\phi + \cos\phi'\sin\theta\sin\phi)\,\hat{r} \\ &\quad + (-\sin\phi'\cos\theta\cos\phi + \cos\phi'\cos\theta\sin\phi)\,\hat{\theta} \\ &\quad + (\sin\phi'\sin\phi + \cos\phi'\cos\phi)\,\hat{\phi} \\ &= \sin(\phi-\phi')\sin\theta\,\hat{r} + \sin(\phi-\phi')\cos\theta\,\hat{\theta} + \cos(\phi-\phi')\,\hat{\phi}\end{aligned}$$

5.8, cont'd

Perform the ϕ' integral:

$$\int_0^{2\pi} d\phi' \, \hat{\phi}' e^{im\phi'}$$

$$= \int_0^{2\pi} d\phi' \left[\sin(\phi-\phi')(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) + \cos(\phi-\phi')\hat{\phi} \right] e^{im\phi'}$$

$$= \int_0^{2\pi} d\phi'' \left[-(\sin\phi'')(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) + (\cos\phi'')\hat{\phi} \right] e^{im\phi''} e^{im\phi}$$

for $\phi'' = \phi' - \phi$

Note

$$\int_0^{2\pi} d\phi\,(\sin\phi) e^{im\phi} = i \int_0^{2\pi} d\phi\,(\sin\phi)(\sin m\phi) = i\left(\tfrac{1}{2}\delta_{m,1} - \tfrac{1}{2}\delta_{m,-1}\right)(2\pi)$$

$$\int_0^{2\pi} d\phi\,(\cos\phi) e^{im\phi} = \int_0^{2\pi} d\phi\,(\cos\phi)(\cos m\phi) = \tfrac{1}{2}\left(\delta_{m,1} + \delta_{m,-1}\right)(2\pi)$$

Hence

$$\int_0^{2\pi} d\phi' \, \hat{\phi}' e^{im\phi'}$$

$$= -\tfrac{i}{2}\left(\delta_{m,1} - \delta_{m,-1}\right)(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) e^{im\phi} (2\pi)$$

$$+ \tfrac{1}{2}\left(\delta_{m,1} + \delta_{m,-1}\right)\hat{\phi}\, e^{im\phi} (2\pi)$$

Next, use

$$Y_{lm}(\theta,\phi) = \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} P_l^m(\cos\theta)\, e^{i\phi m}$$

$$Y_{l\,-1}(\theta,\phi) = -Y_{l1}^*(\theta,\phi) = -\left(\frac{2l+1}{4\pi} \frac{(l-1)!}{(l+1)!} \right)^{1/2} P_l^1(\cos\theta)\, e^{-i\phi}$$

5.8, cont'd

Next, evaluate

$$\sum_{m=-l}^{l} \int d^3x' \hat{\phi}' J(r', \theta') \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$= \sum_m \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^l}{r_>^{l+1}} J(r', \theta') P_l^m(\cos\theta') P_l^m(\cos\theta) e^{im\phi}$$

$$\hookrightarrow \int_0^{2\pi} d\phi' \hat{\phi}' e^{-im\phi'}$$

$$= \sum_m \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^l}{r_>^{l+1}} J(r', \theta') P_l^m(\cos\theta') P_l^m(\cos\theta)$$

$$\hookrightarrow \cancel{e^{im\phi}} \Big[(\sin\theta\, \hat{r} + \cos\theta\, \hat{\theta}) \cancel{e^{-im\phi}} \left(-\tfrac{i}{2}\right)(\delta_{-m,1} - \delta_{-m,-1}) + \tfrac{1}{2}(\delta_{-m,1} + \delta_{-m,-1}) \hat{\phi}\, \cancel{e^{-im\phi}} \Big] (2\pi)$$

$$= \frac{2l+1}{4\pi} \frac{(l-1)!}{(l+1)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^l}{r_>^{l+1}} J(r', \theta') P_l^1(\cos\theta') P_l^1(\cos\theta)$$

$$\hookrightarrow \Big[(\sin\theta\, \hat{r} + \cos\theta\, \hat{\theta}) \left(-\tfrac{i}{2}\right)(-) + \tfrac{1}{2}\hat{\phi} \Big] (2\pi)$$

$$+ \frac{2l+1}{4\pi} \frac{(l+1)!}{(l-1)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^l}{r_>^{l+1}} J(r', \theta') P_l^{-1}(\cos\theta') P_l^{-1}(\cos\theta)$$

$$\hookrightarrow \Big[(\sin\theta\, \hat{r} + \cos\theta\, \hat{\theta}) \left(-\tfrac{i}{2}\right)(+1) + \tfrac{1}{2}\hat{\phi} \Big] (2\pi)$$

Use $P_l^{-1}(x) = -\frac{(l-1)!}{(l+1)!} P_l^1(x)$

5.8, cont'd

$$\sum_{m=-\ell}^{\ell} \int d^3x' \, \phi' J(r', \theta') \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

$$= \frac{2\ell+1}{4\pi} \frac{(\ell-1)!}{(\ell+1)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^\ell}{r_>^{\ell+1}} J(r', \theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta) \cdot$$

$$\left[(\sin\theta \, \hat{r} + \cos\theta \, \hat{\theta})\left(-\frac{i}{2}\right)(-1+1) + \frac{1}{2}\hat{\phi}(1+1) \right](2\pi)$$

$$= \hat{\phi} (2\pi) \frac{2\ell+1}{4\pi} \frac{(\ell-1)!}{(\ell+1)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^\ell}{r_>^{\ell+1}} J(r', \theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta)$$

Thus, the only nonzero component of $\bar{A}$ is A_ϕ, and

$$A_\phi \hat{\phi} = \frac{\mu_0}{4\pi} \int d^3x' \, \phi' J(r', \theta') (4\pi) \sum_{\ell, m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

$$= \frac{\mu_0}{4\pi} \hat{\phi} \sum_{\ell=1}^{\infty} \frac{(\ell-1)!}{(\ell+1)!} \int_0^\infty r'^2 dr' \int_{-1}^{1} d(\cos\theta') \frac{r_<^\ell}{r_>^{\ell+1}} J(r', \theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta) (2\pi)$$

$$= \frac{\mu_0}{4\pi} \hat{\phi} \sum_{\ell=1}^{\infty} \frac{1}{\ell(\ell+1)} \int d^3x' \frac{r_<^\ell}{r_>^{\ell+1}} J(r', \theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta)$$

5.8, cont'd

In the interior, $r < r'$ for r' at which $J(r',\theta')$ has support

$\Rightarrow r_> = r', \; r_< = r$

$$\boxed{A_\phi = \frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty}\frac{1}{\ell(\ell+1)}\int d^3x' \frac{r^\ell}{r'^{\ell+1}} J(r',\theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta) = -\frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty} m_\ell r^\ell P_\ell^1(\cos\theta)}$$

where

$$\boxed{m_\ell = -\frac{1}{\ell(\ell+1)}\int d^3x' (r')^{-\ell-1} J(r',\theta') P_\ell^1(\cos\theta')}$$

Outside the current distribution, $r > r' \Rightarrow r_> = r, \; r_< = r'$

$$\boxed{A_\phi = \frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty}\frac{1}{\ell(\ell+1)}\int d^3x' \frac{r'^\ell}{r^{\ell+1}} J(r',\theta') P_\ell^1(\cos\theta') P_\ell^1(\cos\theta) = -\frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty} \mu_\ell r^{-\ell-1} P_\ell^1(\cos\theta)}$$

where

$$\boxed{\mu_\ell = -\frac{1}{\ell(\ell+1)}\int d^3x' (r')^{\ell} J(r',\theta') P_\ell^1(\cos\theta')}$$

(a, b) only

5.10 A circular current loop of radius a carrying a current I lies in the xy plane with its center at the origin.

a) Show that the only nonvanishing component of the vector potential is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, \cos kz \; I_1(k\rho_<) K_1(k\rho_>)$$

where $\rho_<, \rho_>$ are the smaller, larger of a, ρ.

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\bar{J}(\bar{x}')}{|\bar{x} - \bar{x}'|}$$

Here, $\bar{J}(\bar{x}') = I\,\delta(z')\,\delta(\rho' - a)\,\hat{\phi}'$

We need to relate $\hat{\phi}'$ to $(\hat{\rho}, \hat{\phi}, \hat{z})$.

First, relate to Cartesian: $\hat{\rho} = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi$
$\hat{\phi} = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi$

$$\hat{\phi}' = (\hat{\phi}'\cdot\hat{\rho})\hat{\rho} + (\hat{\phi}'\cdot\hat{\phi})\hat{\phi} + \underbrace{(\hat{\phi}'\cdot\hat{z})}_{=0}\hat{z}$$

$$= (-\sin\phi'\cos\phi + \cos\phi'\sin\phi)\hat{\rho} + (\sin\phi'\sin\phi + \cos\phi'\cos\phi)\hat{\phi}$$

$$= \sin(\phi - \phi')\hat{\rho} + \cos(\phi - \phi')\hat{\phi}$$

Then, use

$$\frac{1}{|\bar{x} - \bar{x}'|} = \frac{4}{\pi}\int_0^\infty dk \, \cos(k(z-z'))\left[\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^{\infty} \cos(m(\phi - \phi')) I_m(k\rho_<) K_m(k\rho_>)\right] \quad (3.149)$$

5.10 a), cont'd

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int \rho' d\rho' d\phi' dz' \; I\,\delta(z')\,\delta(\rho'-a)\left(\sin(\phi-\phi')\hat{\rho} + \cos(\phi-\phi')\hat{\phi}\right)$$

$$\frac{4}{\pi}\int_0^\infty dk \cos k(z-z')\left[\frac{1}{2} I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos m(\phi-\phi')\, I_m(k\rho_<)K_m(k\rho_>)\right]$$

$$= \frac{\mu_0 I a}{4\pi}\frac{4}{\pi}\int_0^{2\pi} d\phi' \int_0^\infty dk\,(\cos kz)\,\hat{\phi}\,\cos^2(\phi-\phi')\, I_1(k\rho_<)\,K_1(k\rho_>)$$

$$= \frac{\mu_0 I a}{\cancel{4}\pi}\,\frac{\cancel{4}}{\cancel{\pi}}\,\frac{\cancel{2}\cancel{\pi}}{\cancel{2}}\int_0^\infty dk\,(\cos kz)\, I_1(k\rho_<)\,K_1(k\rho_>)\;\hat{\phi}$$

$$\boxed{A_\phi = \frac{\mu_0 I a}{\pi}\int_0^\infty dk\,(\cos kz)\, I_1(k\rho_<)\,K_1(k\rho_>)}$$

5.10 cont'd

b) Show that an alternative expression for A_ϕ is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(ka)\, J_1(k\rho)$$

Here, we can largely repeat the analysis of (a), but using instead the expression

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho')\, e^{-k(z_>-z_<)} \quad \text{(from 3.16(b))}$$

$$= 2\int_0^\infty dk\, e^{-k(z_>-z_<)}\left[\frac{1}{2}J_0(k\rho)J_0(k\rho') + \sum_{m=1}^{\infty}(\cos m(\phi-\phi'))\, J_m(k\rho)\, J_m(k\rho')\right]$$

Plug in:

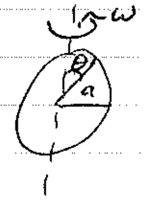
$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int \rho' d\rho' d\phi' dz'\; I\,\delta(z')\,\delta(\rho'-a)\left(\sin(\phi-\phi')\hat{\rho} + \cos(\phi-\phi')\hat{\phi}\right) \cdot$$
$$\cdot\; 2\int_0^\infty dk\, e^{-k(z_>-z_<)}\left[\frac{1}{2}J_0(k\rho)J_0(k\rho') + \sum_{m=1}^{\infty}(\cos m(\phi-\phi'))\, J_m(k\rho)\, J_m(k\rho')\right]$$

$$= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi' \cos^2(\phi-\phi')\,\hat{\phi}\int_0^\infty dk\, e^{-k|z|} J_1(k\rho)\, J_1(ka)\ (2)$$

$$= \frac{\mu_0 I a}{4\pi}\,\frac{2\pi}{2}\,\hat{\phi}\int_0^\infty dk\, e^{-k|z|} J_1(k\rho)\, J_1(ka)\ (2) = \boxed{\frac{\mu_0 I a}{2}\hat{\phi}\int_0^\infty dk\, e^{-k|z|} J_1(k\rho) J_1(ka)}$$

5.13 A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic field both inside and outside the sphere.



$$\bar{J} = \hat{\phi}\, \sigma \omega \sin\theta \, \delta(r' - a)$$

Note $\omega \sin\theta$ is the angular velocity of a point at θ.

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\bar{J}(\bar{x}')}{|\bar{x} - \bar{x}'|}$$

$$= \frac{\mu_0 \sigma \omega a^2}{4\pi} \int d\Omega' \, \frac{\hat{\phi}' \sin\theta'}{|\bar{x} - \bar{x}'|}$$

Need to convert $\hat{\phi}'$ to $(\hat{r}, \hat{\theta}, \hat{\phi})$:

In Cartesian basis,

$$\hat{r} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$\hat{\theta} = (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta)$$

$$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

Hence $\hat{\phi}' = (\hat{\phi}' \cdot \hat{r})\hat{r} + (\hat{\phi}' \cdot \hat{\theta})\hat{\theta} + (\hat{\phi}' \cdot \hat{\phi})\hat{\phi}$

$$= (-\sin\phi' \sin\theta \cos\phi + \cos\phi' \sin\theta \sin\phi)\,\hat{r}$$

$$+ (-\sin\phi' \cos\theta \cos\phi + \cos\phi' \cos\theta \sin\phi)\,\hat{\theta}$$

$$+ (\sin\phi' \sin\phi + \cos\phi' \cos\phi)\,\hat{\phi}$$

$$= \sin(\phi - \phi') \sin\theta \, \hat{r} + \sin(\phi - \phi') \cos\theta \, \hat{\theta} + \cos(\phi - \phi')\, \hat{\phi}$$

5.13, cont'd

Expand

$$\frac{1}{|\bar{x}-\bar{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \qquad (3.70)$$

Compute

$$\int_0^{2\pi} d\phi' \frac{\hat{\phi}'}{|\bar{x}-\bar{x}'|} = \int_0^{2\pi} d\phi' \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) \sin(\phi-\phi') + \hat{\phi}\cos(\phi-\phi') \Big]_2$$

$$\hookrightarrow 4\pi \sum_{m,l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

Note

$$\int_0^{2\pi} d\phi' \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) \sin(\phi-\phi') + \hat{\phi}\cos(\phi-\phi') \Big] e^{-im\phi'}$$

$\phi'' = \phi' - \phi$

$$= \int_0^{2\pi} d\phi'' \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})(\sin\phi'')(-) + \hat{\phi}\cos\phi'' \Big] e^{-im\phi''} e^{-im\phi}$$

$$= (\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})(-)\left(\frac{2\pi}{2}\right)(-i)(\delta_{m,1} - \delta_{m,-1})\, e^{-im\phi}$$

$$+ \hat{\phi}\left(\frac{2\pi}{2}\right)(\delta_{m,1} + \delta_{m,-1})\, e^{-im\phi}$$

5.13, cont'd

$$\int_0^{2\pi} d\phi' \frac{\hat{y}'}{|\bar{x}-\bar{x}'|} = \sum_{l,m} \frac{(l-m)!}{(l+m)!} \frac{r_<^l}{r_>^{l+1}} P_l^m(\cos\theta') P_l^m(\cos\theta)_2$$

$$\hookrightarrow \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})(i)(\pi)(\delta_{m,1} - \delta_{m,-1}) + \hat{\phi}(\pi)(\delta_{m,1} + \delta_{m,-1}) \Big]$$

$$= \sum_{l=1}^{\infty} \frac{(l-1)!}{(l+1)!} \frac{r_<^l}{r_>^{l+1}} P_l^1(\cos\theta') P_l^1(\cos\theta)_2$$

$$\hookrightarrow \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})(i\pi) + \pi\hat{\phi} \Big]$$

$$+ \sum_{l=1}^{\infty} \frac{(l+1)!}{(l-1)!} \frac{r_<^l}{r_>^{l+1}} P_l^{-1}(\cos\theta') P_l^{-1}(\cos\theta)_2$$

$$\hookrightarrow \Big[(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})(i\pi)(-) + \pi\hat{\phi} \Big]$$

$$= \sum_{l=1}^{\infty} \frac{(l-1)!}{(l+1)!} \frac{r_<^l}{r_>^{l+1}} P_l^1(\cos\theta') P_l^1(\cos\theta)\, \hat{\phi}(2\pi)$$

$$\int_{-1}^{1} d(\cos\theta') \int_0^{2\pi} d\phi' \frac{\hat{\phi}' \sin\theta'}{|\bar{x}-\bar{x}'|}$$

$$= \sum_{l=1}^{\infty} \frac{(l-1)!}{(l+1)!} \frac{r_<^l}{r_>^{l+1}} (2\pi)\hat{\phi}\, P_l^1(\cos\theta) \int_{-1}^{1} d(\cos\theta') \sin\theta' P_l^1(\cos\theta')$$

Now, $P_1^1(x) = -(1-x^2)^{1/2} \Rightarrow P_1^1(\cos\theta) = -\sin\theta$

5.13, cont'd

Use $\int_{-1}^{1} dx\, P_{\ell'}^{m}(x)\, P_{\ell}^{m}(x) = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell'\ell}$ (3.52)

$$\Rightarrow \int_{-1}^{1} d(\cos\theta') \sin\theta' P_\ell^1(\cos\theta') = -\int_{-1}^{1} d(\cos\theta') P_1^1(\cos\theta') P_\ell^1(\cos\theta')$$

$$= -\frac{2}{3}\frac{(2!)}{(1)} \delta_{\ell 1}$$

$$\Rightarrow \int d\Omega' \frac{\hat{\phi}' \sin\theta'}{|\bar{x}-\bar{x}'|}$$

$$= \frac{1}{\cancel{2!}} \frac{r_<}{r_>^2} (2\pi) \hat{\phi} P_1^1(\cos\theta) (-) \frac{2}{3} \cancel{2!}$$

$$= \frac{2}{3}(2\pi) \frac{r_<}{r_>^2} \hat{\phi} \sin\theta$$

$$\bar{A}(\bar{x}) = \frac{\mu_0 \sigma \omega a^2}{4\pi} \int d\Omega' \frac{\hat{\phi}' \sin\theta'}{|\bar{x}-\bar{x}'|}$$

$$= \frac{\mu_0 \sigma \omega a^2}{4\pi} \left(\frac{2}{3}\right)(2\pi) \frac{r_<}{r_>^2} \hat{\phi} \sin\theta$$

$$= \frac{\mu_0 \sigma \omega a^2}{3} \frac{r_<}{r_>^2} \hat{\phi} \sin\theta$$

5.13, cont'd

Inside the sphere:

$r_< = r, \quad r_> = a$

$$\boxed{\bar{A}(\bar{x}) = \frac{\mu_0 \sigma \omega a^2}{3} \frac{r}{a^2} \hat{\phi} \sin\theta = \frac{\mu_0 \sigma \omega r}{3} \hat{\phi} \sin\theta}$$

$$\bar{B}(\bar{x}) = \nabla \times \bar{A}$$

$$= \frac{\mu_0 \sigma \omega}{3} \nabla \times (r \sin\theta \, \hat{\phi})$$

$$= \frac{\mu_0 \sigma \omega}{3} \left[\frac{\hat{r}}{r \sin\theta} \frac{\partial}{\partial \theta} (r \sin^2\theta) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} (r^2 \sin\theta) \right]$$

$$= \frac{\mu_0 \sigma \omega}{3} \left[\frac{\hat{r}}{r \sin\theta} (2 r \sin\theta \cos\theta) - \frac{\hat{\theta}}{r} (2 r \sin\theta) \right]$$

$$= \frac{\mu_0 \sigma \omega}{3} \left[2 \hat{r} \cos\theta - 2 \sin\theta \, \hat{\theta} \right]$$

$$= \frac{2}{3} \mu_0 \sigma \omega (\hat{r} \cos\theta - \hat{\theta} \sin\theta)$$

5.13, cont'd

<u>Outside the sphere:</u>

$$r_< = a, \quad r_> = r$$

$$\boxed{\bar{A}(\bar{x}) = \frac{\mu_0 \sigma \omega a^2}{3} \frac{a}{r^2} \hat{\phi} \sin\theta}$$

$$\bar{B}(\bar{x}) = \nabla \times \bar{A}$$

$$= \frac{\mu_0 \sigma \omega a^3}{3} \nabla \times \left(\frac{\sin\theta}{r^2} \hat{\phi} \right)$$

$$= \frac{\mu_0 \sigma \omega a^3}{3} \left[\frac{\hat{r}}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\sin^2\theta}{r^2} \right) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} \left(\frac{\sin\theta}{r} \right) \right]$$

$$= \frac{\mu_0 \sigma \omega a^3}{3} \left[\frac{\hat{r}}{r \cancel{\sin\theta}} \frac{2 \cancel{\sin\theta} \cos\theta}{r^2} - \frac{\hat{\theta}}{r} \left(-\frac{1}{r^2} \right) \sin\theta \right]$$

$$= \frac{\mu_0 \sigma \omega a^3}{3} \left[\frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \right]$$

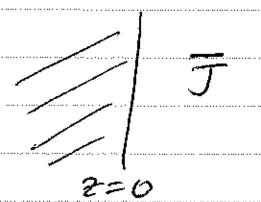
$$\boxed{= \frac{\mu_0 \sigma \omega}{3} \left(\frac{a}{r} \right)^3 \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)}$$

5.17 A current distribution $\bar{J}(\bar{x})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material of relative permeability μ_r and filling the half space $z<0$.

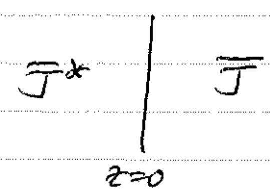
a) Show that for $z>0$ the magnetic field can be calculated by replacing the medium of permeability μ_r by an image current distribution $\bar{J}^*$ with component

$$\left(\frac{\mu_r-1}{\mu_r+1}\right) J_x(x,y,-z), \quad \left(\frac{\mu_r-1}{\mu_r+1}\right) J_y(x,y,-z), \quad -\left(\frac{\mu_r-1}{\mu_r+1}\right) J_z(x,y,-z)$$

Define $\sigma(\bar{x}) = (x, y, -z)$.



Using the method of images, the region $z>0$ sees



Hence

$$\bar{B}_> = \frac{\mu_0}{4\pi}\int d^3x' \, \bar{J}(\bar{x}') \times \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3} + \frac{\mu_0}{4\pi}\int d^3x' \, \bar{J}^*(\sigma\bar{x}') \times \frac{(\bar{x}-\sigma\bar{x}')}{|\bar{x}-\sigma\bar{x}'|^3}$$

The region $z<0$ sees only $\bar{J}$, but with modified magnitude:

$$\bar{B}_< = \frac{\mu\lambda}{4\pi}\int d^3x' \, \bar{J}(\bar{x}') \times \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}$$

5.17 a), cont'd

At the interface, require B_z, H_x, H_y continuous.

Note at $z=0$, $|\bar{x}-\bar{x}'| = |\bar{x} - \sigma\bar{x}'|$, so we only need match numerators.

B_z:

$$(J_x + J_x^*)(y-y') - (J_y + J_y^*)(x-x') = \mu_r \lambda \left(J_x (y-y') - J_y (x-x') \right)$$

H_x: $(\bar{B} = \mu \bar{H})$

$$J_y(-z') + J_y^*(+z') - (J_z + J_z^*)(y-y') = \lambda \left(J_y(-z') - J_z(y-y') \right)$$

H_y:

$$-J_x(-z') - J_x^*(+z') + (J_z + J_z^*)(x-x') = \lambda \left(-J_x(z-z') + J_z(x-x') \right)$$

This implies

$$J_x + J_x^* = \mu_r \lambda J_x, \quad J_y + J_y^* = \mu_r \lambda J_y$$

$$J_y - J_y^* = \lambda J_y, \quad J_z + J_z^* = \lambda J_z$$

$$J_x - J_x^* = \lambda J_x, \quad J_z + J_z^* = \lambda J_z$$

This implies

$$J_z^* = (\lambda - 1) J_z$$

$$J_y^* = (1-\lambda) J_y = (\mu_r \lambda - 1) J_y$$

$$J_x^* = (1-\lambda) J_x = (\mu_r \lambda - 1) J_x$$

Consistency requires $1-\lambda = \mu_r \lambda - 1 \Rightarrow \lambda = \frac{2}{1+\mu_r}$

5.17 a), cont'd

Plugging in d, we find

$$\boxed{\begin{aligned} J_x^*(x,y,z) &= \left(\frac{1+\mu_r-2}{1+\mu_r}\right) J_x(x,y,-z) = \left(\frac{\mu_r-1}{\mu_r+1}\right) J_x(x,y,-z) \\ J_y^*(x,y,z) &= \left(\frac{\mu_r-1}{\mu_r+1}\right) J_y(x,y,-z) \\ J_z^*(x,y,z) &= -\left(\frac{\mu_r-1}{\mu_r+1}\right) J_z(x,y,-z) \end{aligned}}$$

5.17, cont'd

b) Show that for $z<0$ the magnetic field appears to be due to a current distribution

$$\frac{2\mu_r}{\mu_r+1}\bar{J}$$

in a medium of unit relative permeability.

We recall our sol'n from part (a).
We assumed that in the region $z<0$,
one saw an effective current density $\lambda\bar{J}$,
or equivalently, in a medium of unit relative permeability, $\mu_r\lambda\bar{J}$.

Note $\mu_r\lambda\bar{J} = \boxed{\frac{2\mu_r}{1+\mu_r}\bar{J}}$

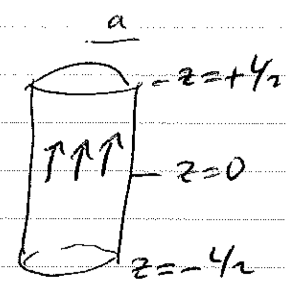
(a) only

5.19 A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis.

a) Determine the magnetic field $\bar{H}$ and $\bar{B}$ at all points on the axis of the cylinder, both inside and outside.

Since $\bar{J}=0$, we can use the magnetic scalar potential Φ_M.

$$\Phi_M(\bar{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\bar{M}(\bar{x}')}{|\bar{x}-\bar{x}'|}\,d^3x' + \frac{1}{4\pi}\int_{\partial V}\frac{\hat{n}'\cdot\bar{M}(\bar{x}')}{|\bar{x}-\bar{x}'|}\,da' \qquad (5.100)$$



Here, $\bar{M} = M_0\hat{z}$ & is constant inside the cylinder, hence
$$\nabla\cdot\bar{M}=0$$
Also, $\hat{n}\cdot\bar{M}$ is nonzero only along top & bottom.

$$\Rightarrow \Phi_M(\bar{x}) = \frac{1}{4\pi}\int_{top}\frac{M_0}{|\bar{x}-\bar{x}'|}\,da' - \frac{1}{4\pi}\int_{bottom}\frac{M_0}{|\bar{x}-\bar{x}'|}\,da'$$

$$|\bar{x}-\bar{x}'|^2 = (z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$$

However, we only need to determine $\bar{H} = -\nabla\Phi_M$ along the axis, & by symmetry, along the axis, only $H_z\neq 0$, so we only need the z dependence of Φ_M on axis.

In other words, even though we're computing a gradient, we know in advance that only $\frac{\partial}{\partial z}\Phi_M\neq 0$,

so we can safely evaluate at $\rho=0$.

5.19, cont'd

So long evaluating at $\rho = 0$:

$$\Phi_M(\vec{x}) = \underbrace{\frac{M_0}{4\pi}\int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \left[\left((z-L/2)^2 + \rho'^2 \right)^{-1/2} - \left((z+L/2)^2 + \rho'^2 \right)^{-1/2} \right]}$$

$$= \frac{M_0}{4\pi}(2\pi)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{-1}\left[\left((z-L/2)^2 + \rho'^2 \right)^{+1/2} - \left((z+L/2)^2 + \rho'^2 \right)^{+1/2} \right]_0^a$$

$$= \frac{M_0}{2}\left[\left((z-L/2)^2 + a^2 \right)^{1/2} - |z - L/2| \right.$$

$$\left. \underline{- \left((z+L/2)^2 + a^2 \right)^{1/2}} + |z + L/2| \right]$$

Compute $\vec{H}$:

Only H_z can be nonzero.

$$H_z = -\frac{\partial \Phi_M}{\partial z} = -\frac{M_0}{2}\left[\frac{1}{2}(2)\left(z-\frac{L}{2}\right)\left(\left(z-\frac{L}{2}\right)^2 + a^2 \right)^{-1/2} \right.$$

$$- \operatorname{sgn}(z - L/2)$$

$$- \frac{1}{2}(2)\left(z+\frac{L}{2}\right)\left(\left(z+\frac{L}{2}\right)^2 + a^2 \right)^{-1/2}$$

$$\left. + \operatorname{sgn}(z + L/2) \right]$$

$$= -\frac{M_0}{2}\left[\frac{z - L/2}{\left((z-L/2)^2 + a^2 \right)^{1/2}} - \frac{z + L/2}{\left((z+L/2)^2 + a^2 \right)^{1/2}} \right.$$

$$\left. + 2\Theta(L/2 - |z|) \right]$$

where $\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

This is valid both inside & outside.

S. (9a), cont'd

$\bar{B} = \mu_0 (\bar{H} + \bar{M})$

Here, $\bar{M} = M_0 \hat{k}\, \theta(\frac{L}{2} - |z|)$,

ie, $\bar{M}$ is nonzero only inside the magnet.

$$\Rightarrow \boxed{\begin{aligned} \bar{B} &= \mu_0 \hat{k} \Big\{ -\frac{M_0}{2}\left[\frac{z-\frac{L}{2}}{((z-L/2)^2+a^2)^{1/2}} - \frac{z+L/2}{((z+L/2)^2+a^2)^{1/2}}\right] \\ &\qquad \underline{-M_0\,\theta(L/2-|z|) \quad + M_0\,\theta(L/2-|z|)} \Big\} \\ &= -\frac{\mu_0 M_0}{2}\hat{k}\left[\frac{z-\frac{L}{2}}{((z-L/2)^2+a^2)^{1/2}} - \frac{z+L/2}{((z+L/2)^2+a^2)^{1/2}}\right] \end{aligned}}$$

Valid on axis both inside & outside.