5.20 a) Starting from the force equation

$$\bar{F} = \int d^3x\ \bar{J}(\bar{x}) \times \bar{B}(\bar{x})$$

and the fact that a magnetization $\bar{M}$ inside a volume V bounded by a surface ∂V is equivalent to a volume current density $\bar{J}_M = \nabla \times \bar{M}$ & a surface current density $\bar{M} \times \hat{n}$, show that in the absence of macroscopic conduction currents the total magnetic force on the body can be written

$$\bar{F} = -\int_V (\nabla \cdot \bar{M}) \bar{B}_e\, d^3x + \int_{\partial V} (\bar{M} \cdot \hat{n}) \bar{B}_e\, da$$

where $\bar{B}_e$ is the applied magnetic field (not including that of the body in question).

From the statement given, & taking into account both $\bar{J}_M$ & $\bar{M} \times \hat{n}$,

$$\bar{F} = + \int_V \bar{J}_M \times \bar{B}_e\, d^3x + \int_{\partial V} (\bar{M} \times \hat{n}) \times \bar{B}_e\, da$$

$$= \int_V (\nabla \times \bar{M}) \times \bar{B}_e\, d^3x + \int_{\partial V} (\bar{M} \times \hat{n}) \times \bar{B}_e\, da$$

Now

$$(\nabla \times \bar{M}) \times \bar{B}_e = -\bar{B}_e \times (\nabla \times \bar{M})$$

$$= \bar{M} \times \underbrace{(\nabla \times \bar{B}_e)}_{=0} + (\bar{B}_e \cdot \nabla) \bar{M} + (\bar{M} \cdot \nabla) \bar{B}_e - \nabla (\bar{M} \cdot \bar{B}_e)$$

$$(\bar{M} \times \hat{n}) \times \bar{B}_e = -\bar{B}_e \times (\bar{M} \times \hat{n})$$

$$= -(\bar{B}_e \cdot \hat{n}) \bar{M} + (\bar{B}_e \cdot \bar{M}) \hat{n}$$

5.20 a), cont'd

$$\bar{F} = \int_V (\nabla\times\bar{M})\times\bar{B}_e\, d^3x + \int_{\partial V} (\bar{M}\times\hat{n})\times\bar{B}_e\, da$$

$$= \int_V d^3x \left[(\bar{B}_e\cdot\nabla)\bar{M} + (\bar{M}\cdot\nabla)\bar{B}_e - \nabla(\bar{M}\cdot\bar{B}_e)\right]$$

$$+ \int_{\partial V} da \underline{\left[-(\bar{B}_e\cdot\hat{n})\bar{M} + (\bar{B}_e\cdot\bar{M})\hat{n}\right]}$$

However, $\int_V d^3x\ \nabla(\bar{M}\cdot\bar{B}_e) = \int_{\partial V} da\ (\bar{M}\cdot\bar{B}_e)\hat{n}$ (inside front cover)

so

$$\bar{F} = \int_V d^3x\left[(\bar{B}_e\cdot\nabla)\bar{M} + (\bar{M}\cdot\nabla)\bar{B}_e\right] - \int_{\partial V} da\ (\bar{B}_e\cdot\hat{n})\bar{M}$$

Another useful identity:

$$\int_V \Big((\bar{C}\cdot\nabla)\bar{D} + (\nabla\cdot\bar{C})\bar{D}\Big)\, d^3x = \int_{\partial V} da\ (\bar{C}\cdot\hat{n})\bar{D}$$

which can be checked by expanding in components.

$$\Rightarrow \int_V d^3x\,(\bar{B}_e\cdot\nabla)\bar{M} - \int_{\partial V} da\ (\bar{B}_e\cdot\hat{n})\bar{M} = -\int_V d^3x\ \underbrace{(\nabla\cdot\bar{B}_e)}_{=0}\bar{M}$$

$$\Rightarrow \boxed{\bar{F} = \boxed{\int_V d^3x\ (\bar{M}\cdot\nabla)\bar{B}_e} = -\int_V d^3x\ (\nabla\cdot\bar{M})\bar{B}_e + \int_{\partial V} da\ (\bar{M}\cdot\hat{n})\bar{B}_e}$$

using the second identity above.

5.20, cont'd

b) A sphere of radius R with uniform magnetization has its center at the origin of coordinates and its direction of magnetization making spherical angles θ_0, ϕ_0.
Suppose there is an external magnetic field with components

$$B_x = B_0(1+\beta y), \quad B_y = B_0(1+\beta x), \quad B_z = 0$$

Evaluate the components of the force acting on the sphere.

$$\bar{M} = M_0\left(\hat{\imath}\sin\theta_0\cos\phi_0 + \hat{\jmath}\sin\theta_0\sin\phi_0 + \hat{k}\cos\theta_0\right)$$

Since $\bar{M}$ is constant, $\nabla\cdot\bar{M} = 0$

$$\bar{F} = \int_{\partial V} da\ (\bar{M}\cdot\hat{n})\,\bar{B}_e$$

Here, $\hat{n} = \hat{r} = \hat{\imath}\sin\theta\cos\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta$

$$\bar{M}\cdot\hat{n} = M_0\left(\sin\theta\sin\theta_0\cos\phi\cos\phi_0 + \sin\theta\sin\theta_0\sin\phi\sin\phi_0 + \cos\theta\cos\theta_0\right)$$

$$= M_0\left(\sin\theta\sin\theta_0\cos(\phi-\phi_0) + \cos\theta\cos\theta_0\right)$$

$$F_x = \int_{\partial V} da\ (\bar{M}\cdot\hat{n})\,B_0(1+\beta y)$$

$$= R^2 M_0\int d\Omega\left[\sin\theta\sin\theta_0\cos(\phi-\phi_0) + \cos\theta\cos\theta_0\right]B_0\left(1+\beta R\sin\theta\sin\phi\right)$$

$$= R^2 M_0 B_0\int d\Omega\Big[\sin\theta\sin\theta_0\cos(\phi-\phi_0) + \cos\theta\cos\theta_0$$

$$+ \beta R\sin^2\theta\sin\theta_0\sin\phi\cos(\phi-\phi_0) + \beta R\sin\theta\cos\theta\cos\theta_0\sin\phi\Big]$$

5.20 b), cont'd

The ϕ integral removes terms proportional to $\cos(\phi-\phi_0)$ or $\sin\phi$.

$$F_x = R^2 M_0 B_0 \int d\Omega \left[\cos\theta \cos\theta_0 + \beta R \sin^2\theta \sin\theta_0 \sin\phi \cos(\phi-\phi_0) \right]$$

$$\int d\Omega \cos\theta \cos\theta_0 = 2\pi \cos\theta_0 \int_{-1}^{1} d(\cos\theta) \cos\theta$$

$$= 2\pi \cos\theta_0 \left. \frac{\cos^2\theta}{2} \right|_{-1}^{1} = 0$$

$$\int d\Omega \sin^2\theta \sin\theta_0 \sin\phi \cos(\phi-\phi_0)$$

$$= \sin\theta_0 \left[\int_{-1}^{1} d(\cos\theta) \left(1-\cos^2\theta\right) \right] \left[\int_0^{2\pi} d\phi \left(\tfrac{1}{2}\right) \left(\sin\phi_0 + \sin(2\phi-\phi_0) \right) \right]$$

$$= \sin\theta_0 \left[\cos\theta - \frac{\cos^3\theta}{3} \right]_{-1}^{1} \left(\tfrac{1}{2}\right)(2\pi) \sin\phi_0$$

$$= \pi \sin\theta_0 \sin\phi_0 \left[2 - \tfrac{2}{3} \right]$$

$$= \frac{4\pi}{3} \sin\theta_0 \sin\phi_0$$

$$F_x = R^2 M_0 B_0 (\beta R) \left(\frac{4\pi}{3} \right) \sin\theta_0 \sin\phi_0$$

$$= \left(\frac{4\pi}{3} R^3 \right) M_0 B_0 \beta \sin\theta_0 \sin\phi_0$$

5.20 b), cont'd

$$F_y = \int_{\partial V} da\, (\vec{M}\cdot\hat{n}) B_0 (1+\beta x)$$

$$= R^2 M_0 B_0 \int d\Omega \left[\sin\theta \sin\theta_0 \cos(\phi-\phi_0) + \cos\theta\cos\theta_0 \right] \cdot$$
$$\hookrightarrow \left[1 + \beta R \sin\theta \cos\phi \right]$$

$$= R^2 M_0 B_0 \int d\Omega \Big[\sin\theta \sin\theta_0 \cos(\phi-\phi_0) + \cos\theta\cos\theta_0$$
$$+ \beta R \sin^2\theta \sin\theta_0 \cos\phi \cos(\phi-\phi_0) + \beta R \sin\theta\cos\theta\cos\theta_0\cos\phi \Big]$$

The ϕ integral eliminates terms proportional to $\cos(\phi-\phi_0)$, $\cos\phi$.

$$= R^2 M_0 B_0 \int d\Omega \left[\cos\theta\cos\theta_0 + \beta R \sin^2\theta \sin\theta_0 \cos\phi\cos(\phi-\phi_0) \right]$$

$$\int d\Omega \cos\theta = 2\pi \int_{-1}^{1} d(\cos\theta)\cos\theta = 0 \text{ since } \overset{\text{integrand}}{\cancel{\text{odd}}}\ \text{odd}$$

$$\int d\Omega \sin^2\theta \cos\phi\cos(\phi-\phi_0)$$
$$= \left[\int_{-1}^{1} d(\cos\theta)\, (1-\cos^2\theta) \right] \left[\int_0^{2\pi} d\phi \left(\tfrac{1}{2}\right) \left(\cos\phi_0 + \cos(2\phi-\phi_0) \right) \right]$$
$$= \left[\cos\theta - \frac{\cos^3\theta}{3} \right]_{-1}^{1} \left[\frac{2\pi}{2}\cos\phi_0 \right]$$
$$= \left[2 - \frac{2}{3} \right] \left[\pi \cos\phi_0 \right]$$
$$= \frac{4\pi}{3}\cos\phi_0$$

$$\Rightarrow F_y = R^2 M_0 B_0 (\beta R) \sin\theta_0 \left(\frac{4\pi}{3} \right) \cos\phi_0$$

$$= \left(\frac{4\pi R^3}{3} \right) M_0 B_0 \beta \sin\theta_0 \cos\phi_0$$

5.20 b), cont'd

Since $B_z = 0$, $F_z = 0$.

Summarizing:

$$\boxed{F = \frac{4\pi R^3}{3} M_0 B_0 \beta \left(\hat{\imath} \sin\theta_0 \sin\phi_0 + \hat{\jmath} \sin\theta_0 \cos\phi_0\right)}$$

5.21 A magnetostatic field is due entirely to a localized distribution of permanent magnetization.

a) Show that

$$\int \bar{B}\cdot\bar{H}\,d^3x = 0$$

provided the integral is taken over all space.

Since there are no free current, $\bar{J}=0$ & $\nabla\times\bar{H}=0$, so we can write $\bar{H} = -\nabla\Phi_M$.

$$\int d^3x\,\bar{B}\cdot\bar{H} = -\int \bar{B}\cdot\nabla\Phi_M\,d^3x$$

$$= \int \Phi_M(\nabla\cdot\bar{B})\,d^3x - \int \nabla\cdot(\Phi_M\bar{B})\,d^3x$$

Use $\nabla\cdot\bar{B}=0$ & assume the fields are 'localized' & so fall off at infinity so that second term vanishes.

$$\Rightarrow \boxed{\int d^3x\,\bar{B}\cdot\bar{H} = 0}$$

Alternate solution:
Write $\bar{B} = \nabla\times\bar{A}$,

$$\int d^3x\,\bar{B}\cdot\bar{H} = \int d^3x\,\bar{H}\cdot(\nabla\times\bar{A}) = \int d^3x\,\bar{A}\cdot\underbrace{(\nabla\times\bar{H})}_{=0} + \underbrace{\int d^3x\,\nabla\cdot(\bar{A}\times\bar{H})}_{=0\text{ b/c 'localized'}}$$

5.21, cont'd

b) From the potential energy $U = -\bar{m}\cdot\bar{B}$ of a dipole in an external field, show that for a continuous distribution of permanent magnetization the magnetostatic energy can be written

$$W = \frac{\mu_0}{2}\int \bar{H}\cdot\bar{H}\, d^3x = -\frac{\mu_0}{2}\int \bar{M}\cdot\bar{H}\, d^3x$$

apart from an additive constant.

For a single dipole, $U = -\bar{m}\cdot\bar{B}$.
For a discrete collection of point dipoles,
the energy is obtained by successively bringing each in from infinity:

$$W = -\sum_{i<j}{}' \bar{m}_j\cdot\bar{B}_i$$

for $\bar{B}_i = \frac{\mu_0}{4\pi}\,\frac{3(\bar{m}_i\cdot\hat{x})\hat{x} - \bar{m}_i}{|\bar{x}|^3}$ the field due to i^{th} dipole

Note $\bar{m}_j\cdot\bar{B}_i = \bar{m}_i\cdot\bar{B}_j$,

hence $W = -\frac{1}{2}\sum_{i\neq j}{}' \bar{m}_j\cdot\bar{B}_i$

For a continuous distribution,

$$W = -\frac{1}{2}\int d^3x\, \bar{M}\cdot\bar{B}$$

(Note this does include self-interactions, but we can treat those as an additive constant.)

5.21(b), cont'd

Write $\overline{B} = \mu_0 (\overline{H} + \overline{M})$,

$$\boxed{W = -\frac{\mu_0}{2}\int d^3x\, \overline{M}\cdot(\overline{H}+\overline{M}) = W_0 - \frac{\mu_0}{2}\int d^3x\, \overline{M}\cdot\overline{H}}$$

for $W_0 = -\frac{\mu_0}{2}\int d^3x\, \overline{M}\cdot\overline{M}$,

which we treat as a contribution to additive constant

Use $\overline{M} = \overline{B}/\mu_0 - \overline{H}$:

$$\boxed{W = W_0 - \frac{\mu_0}{2}\int d^3x\, \left(\frac{\overline{B}}{\mu_0} - \overline{H}\right)\cdot\overline{H} = W_0 + \frac{\mu_0}{2}\int d^3x\, \overline{H}\cdot\overline{H}}$$

using the result from part (a) that $\int d^3x\, \overline{B}\cdot\overline{H} = 0$.

5.26 A two-wire transmission line consists of a pair of nonpermeable parallel wires of radii a, b separated by a distance $d \geq a+b$.
A current flows down one wire and back the other.
It is uniformly distributed over the cross-section of each wire.
Show that the self inductance per unit length is

$$L = \frac{\mu_0}{4\pi}\left[1 + 2\ln\left(\frac{d^2}{ab}\right)\right]$$

$$L = \frac{\mu_0}{4\pi I^2}\int d^3x\, d^3x' \, \frac{\bar{J}(\bar{x})\cdot\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

$$= \frac{1}{I^2}\int d^3x \quad \bar{J}(\bar{x})\cdot\bar{A}(\bar{x})$$

Let's compute $\bar{A}$ for each wire separately.

For a single wire of radius a,
from Ampère's law,

$$\bar{B} = \frac{\mu_0 I}{2\pi a}\frac{\rho_<}{\rho_>}\hat{\varphi} \qquad \begin{matrix}\rho_< = \min(a,\rho)\\ \rho_> = \max(a,\rho)\end{matrix}$$

Since $\bar{B} = \nabla\times\bar{A}$,

$$B_\phi = -\frac{\partial A_z}{\partial\rho}$$

$$\Rightarrow A_z = -\int B_\phi(\rho)\,d\rho$$

$$= \begin{cases} -\dfrac{\mu_0 I}{4\pi a^2}\rho^2 & \rho < a \\[2ex] -\dfrac{\mu_0 I}{4\pi}\left(\ln\left(\dfrac{\rho}{a}\right)^2 + 1\right) & \rho > a \end{cases}$$

where the constants were chosen to make A_z continuous at $\rho = a$.

S.26, cont'd

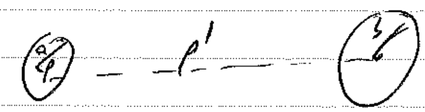
Let's now compute the contribution to the self-inductance per length from the wire of radius a.

$$L_a \equiv \frac{1}{I^2}\int d^2x\ \bar{J}_a(\bar{x})\cdot\bar{A}(\bar{x})$$ = inductance per length, (integrated over cross section)

$\bar{J}_a$ = current density in the wire of radius a
$= \frac{I}{\pi a^2}\hat{h}$

A = total vector potential inside the wire of radius a

$$= \underbrace{-\frac{\mu_0 I}{4\pi}\left(\frac{\rho}{a}\right)^2}_{\text{contribution from (a)}} + \underbrace{\frac{\mu_0 I}{4\pi}\left(\ln\left(\frac{\rho'}{d}\right)^2 + 1\right)}_{\text{contribution from (b)}}$$



$$(\rho')^2 = \rho^2 + d^2 - 2\rho d\cos\phi$$

$$L_a = \frac{1}{I^2}\int_0^{2\pi} d\phi\int_0^a \rho\, d\rho\left(\frac{\mu_0 I}{4\pi}\right)\left(-\left(\frac{\rho}{a}\right)^2 + 1 + \ln\left(\frac{\rho'}{d}\right)^2\right)\left(\frac{I}{\pi a^2}\right)$$

$$= \frac{\mu_0}{4\pi^2 a^2}\left[-\frac{2\pi}{a^2}\frac{1}{4}a^4 + 2\pi\frac{a^2}{2} + \int_0^{2\pi} d\phi\int_0^a \rho\, d\rho\ln\left(\frac{\rho'}{d}\right)^2\right]$$

$$= \frac{\mu_0}{4\pi a^2}\left[-\frac{a^2}{2} + a^2 + \frac{1}{\pi}\int_0^{2\pi} d\phi\int_0^a \rho\, d\rho\ln\left(\frac{\rho'}{d}\right)^2\right]$$

5.26, cont'd

Next, we need to evaluate the integral of the log.

It can be shown that the log is a harmonic function:

$$\nabla^2 \ln\left(\frac{\rho'}{b}\right)^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\ln\left(\frac{\rho'}{b}\right)^2\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\ln\left(\frac{\rho'}{b}\right)^2$$

$$= 0$$

The average value of a harmonic function around a circle equals its value at the center, which in complex analysis is a result of Cauchy's integral formula.

$$\Rightarrow \int_0^{2\pi} d\phi \ln\left(\frac{\rho'}{b}\right)^2 = 2\pi \ln\left(\frac{d}{b}\right)^2$$

$$\Rightarrow L_a = \frac{\mu_0}{4\pi a^2}\left[\frac{a^2}{2} + \frac{2\pi}{\pi}\int_0^a \rho\, d\rho \ln\left(\frac{d}{b}\right)^2\right]$$

$$= \frac{\mu_0}{4\pi a^2}\left[\frac{a^2}{2} + \frac{2\pi}{\pi}\frac{a^2}{2}\ln\left(\frac{d}{b}\right)^2\right]$$

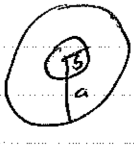
$$= \frac{\mu_0}{4\pi}\left[\frac{1}{2} + \ln\left(\frac{d}{b}\right)^2\right]$$

Similarly,

$$L_b = \frac{\mu_0}{4\pi}\left[\frac{1}{2} + \ln\left(\frac{d}{a}\right)^2\right]$$

$$\boxed{L = L_a + L_b = \frac{\mu_0}{4\pi}\left[1 + \ln\frac{d^4}{a^2 b^2}\right]}$$

5.27 A circuit consists of a long thin conducting shell of radius a and a parallel return wire of radius b on axis inside. If the current is assumed distributed uniformly throughout the cross-section of the wire, calculate the self-inductance per unit length. What is the self-conductance if the inner conductor is a thin hollow tube?



From Ampère's law,

$$\bar{B} = \begin{cases} \dfrac{\mu I}{2\pi b}\dfrac{\rho}{b}\,\hat{\phi} & 0<\rho<b \\ \dfrac{\mu_0 I}{2\pi\rho}\,\hat{\phi} & b<\rho<a \\ 0 & \rho>a \end{cases}$$

Energy in the magnetic field per unit length

$$= \frac{1}{2} L I^2 \qquad (5.152)$$

$$= \frac{1}{2}\int d^2x\ \bar{B}\cdot\bar{H} \qquad (5.148), \text{ modified to give energy / length}$$

$$= \frac{1}{2\mu}\int_0^{2\pi} d\phi \int_0^b \rho\, d\rho\ \bar{B}^2 + \frac{1}{2\mu_0}\int_0^{2\pi} d\phi \int_b^a \rho\, d\rho\ \bar{B}^2$$

$$= \frac{1}{2\mu}(2\pi)\int_0^b \rho\, d\rho \left(\frac{\mu I}{2\pi b^2}\right)^2 \rho^2 + \frac{1}{2\mu_0}(2\pi)\int_b^a \rho\, d\rho \left(\frac{\mu_0 I}{2\pi\rho}\right)^2$$

$$= \frac{\pi}{\mu}\frac{\mu^2 I^2}{(2\pi)^2 b^4}\frac{\rho^4}{4}\bigg|_0^b + \frac{\pi}{\mu_0}\frac{\mu_0^2 I^2}{(2\pi)^2}\ln\rho\bigg|_b^a$$

$$= \frac{\mu I^2}{4\pi}\frac{1}{4} + \frac{\mu_0 I^2}{4\pi}\ln(a/b)$$

$$\Rightarrow \boxed{L = \frac{1}{2\pi}\left[\frac{\mu}{4} + \mu_0 \ln\left(\frac{a}{b}\right)\right]} = \text{self-inductance / length}$$

~~If the inner conductor were a~~

5.27, cont'd

If the inner conductor were a thin hollow tube,
then $\bar{B}=0$ for $\rho<b$, so the first term would vanish,

$$\boxed{L = \frac{1}{2\pi}\mu_0 \ln(a/b)}$$