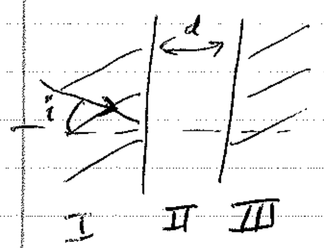
(a) only. 1

7.3 Two plane semi-infinite slabs of the same uniform isotropic nonpermeable lossless dielectric with index of refraction n are parallel and separated by an air gap ($n=1$) of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i. For linear polarization both parallel to and perpendicular to the plane of incidence,

a) calculate the ratio of power transmitted into the second slab to the incident power & the ratio of reflected to incident power.



We write

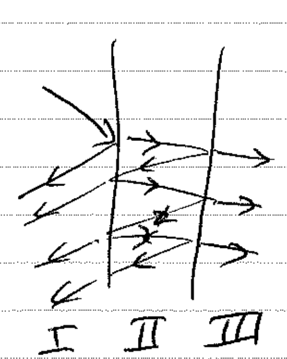
$$\bar{E}_{I} = \bar{\varepsilon}_0 E_0 e^{i(\frac{n}{c}\omega\hat{k}\cdot\bar{x}-\omega t)} + \bar{\varepsilon}_1 E_1 e^{i(\frac{n}{c}\omega_1\hat{k}_1\cdot\bar{x}-\omega_1 t)}$$

$$\bar{E}_{II} = \bar{\varepsilon}_2 E_2 e^{i(\frac{1}{c}\omega_2\hat{k}_2\cdot\bar{x}-\omega_2 t)} + \bar{\varepsilon}_3 E_3 e^{i(\frac{1}{c}\omega_3\hat{k}_3\cdot\bar{x}-\omega_3 t)}$$

$$E_{III} = \bar{\varepsilon}_4 E_4 e^{i(\frac{n}{c}\omega_4\hat{k}_4\cdot(\bar{x}-\bar{d})-\omega_4 t)}$$

Here $\hat{k} = \frac{\bar{k}}{|\bar{k}|}$, not necessarily $\hat{z}$.

$\bar{\varepsilon}_0 E_0$ -- corresponds to the original incoming wave. The other terms are all sums of contributions from all possible reflections.



7.3 a) cont'd

Since the boundary conditions must hold at all times & all positions on the boundary,

$$e^{i(\frac{n}{c}\omega\hat{k}\cdot\bar{x}-\omega t)}\Big|_{z=0} = e^{i(\frac{n}{c}\omega_1\hat{k}_1\cdot\bar{x}-\omega_1 t)}\Big|_{z=0}$$

$$= e^{i(\frac{1}{c}\omega_2\hat{k}_2\cdot\bar{x}-\omega_2 t)}\Big|_{z=0} = e^{i(\frac{1}{c}\omega_3\hat{k}_3\cdot\bar{x}-\omega_3 t)}\Big|_{z=0}$$

& $$e^{i(\frac{1}{c}\omega_2\hat{k}_2\cdot\bar{x}-\omega_2 t)}\Big|_{z=d} = e^{i(\frac{1}{c}\omega_3\hat{k}_3\cdot\bar{x}-\omega_3 t)}\Big|_{z=d}$$

$$= e^{i(\frac{n}{c}\omega_4\hat{k}_4(\bar{x}-\bar{d})-\omega_4 t)}\Big|_{z=d}$$

Since these conditions must hold for all t,
the coefficients of t must match separately,

$$\Rightarrow \omega = \omega_1 = \omega_2 = \omega_3 = \omega_4$$

That leaves

$$n\hat{k}\cdot\bar{x}\Big|_{z=0} = n\hat{k}_1\cdot\bar{x}\Big|_{z=0} = \hat{k}_2\cdot\bar{x}\Big|_{z=0} = \hat{k}_3\cdot\bar{x}\Big|_{z=0}$$

$$\hat{k}_2\cdot\bar{x}\Big|_{z=d} = \hat{k}_3\cdot\bar{x}\Big|_{z=d} = n\hat{k}_4\cdot(\bar{x}-\bar{d})\Big|_{z=d}$$

Assume the plane of incidence is aligned with the xz plane,
so that none of these vectors has any y dependence.

Taking x components,

$$n\sin\theta_i = n\sin\theta_1 = \sin\theta_2 = \sin\theta_3$$

$$\sin\theta_2 = \sin\theta_3 = n\sin\theta_4$$

$$\Rightarrow \theta_i = \theta_1,\quad \theta_2 = \theta_3,\quad n\sin\theta_i = \sin\theta_2 = n\sin\theta_4$$

7.3 a), cont'd

We can now rewrite the eqn's for the fields:

$$\bar{E}_{I} = \bar{\varepsilon}_0 E_0 e^{i\left(\frac{n}{c}\omega(x\sin\theta_i + z\cos\theta_i) - \omega t\right)} + \bar{\varepsilon}_1 E_1 e^{i\left(\frac{n}{c}\omega(x\sin\theta_i - z\cos\theta_i) - \omega t\right)}$$

$$\bar{E}_{II} = \bar{\varepsilon}_2 E_2 e^{i\left(\frac{1}{c}\omega(xn\sin\theta_i + z\sqrt{1-n^2\sin^2\theta_i}) - \omega t\right)} + \bar{\varepsilon}_3 E_3 e^{i\left(\frac{1}{c}\omega(xn\sin\theta_i - z\sqrt{1-n^2\sin^2\theta_i}) - \omega t\right)}$$

$$\bar{E}_{III} = \bar{\varepsilon}_4 E_4 e^{i\left(\frac{n}{c}\omega(x\sin\theta_i + (z-d)\cos\theta_i) - \omega t\right)}$$

For simplicity, let us restrict to $x=0$ and $t=0$.

$$\bar{E}_{I} = \bar{\varepsilon}_0 E_0 e^{i\frac{n}{c}\omega z\cos\theta_i} + \bar{\varepsilon}_1 E_1 e^{-i\frac{n}{c}\omega z\cos\theta_i}$$

$$\bar{E}_{II} = \bar{\varepsilon}_2 E_2 e^{i\frac{1}{c}\omega z\cos\theta_2} + \bar{\varepsilon}_3 E_3 e^{-i\frac{1}{c}\omega z\cos\theta_2}$$

$$E_{III} = \bar{\varepsilon}_4 E_4 e^{i\frac{n}{c}\omega(z-d)\cos\theta_i}$$

where $\cos\theta_2 = \sqrt{1-n^2\sin^2\theta_i}$

7.3 a), cont'd

Polarization perpendicular to plane of incidence:

$\bar{\varepsilon}_0 = \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = \hat{y} = \hat{\jmath}$

$\bar{B} = \frac{n}{c}\hat{k}\times\bar{E}$

$$\bar{B}_I = \frac{n}{c}(\sin\theta_i\,\hat{z} - \cos\theta_i\,\hat{x})E_0\,e^{i\frac{n}{c}\omega z\cos\theta_i} + \frac{n}{c}(\sin\theta_i\,\hat{z} + \cos\theta_i\,\hat{x})E_1\,e^{-i\frac{n}{c}\omega z\cos\theta_i}$$

$$\bar{B}_{II} = \frac{1}{c}(n\sin\theta_i\,\hat{z} - \cos\theta_2\,\hat{x})E_2\,e^{i\frac{\omega}{c}z\cos\theta_2} + \frac{1}{c}(n\sin\theta_i\,\hat{z} + \cos\theta_2\,\hat{x})E_3\,e^{-i\frac{\omega}{c}z\cos\theta_2}$$

$$\bar{B}_{III} = \frac{n}{c}(\sin\theta_i\,\hat{z} - \cos\theta_i\,\hat{x})E_4\,e^{i\frac{n}{c}\omega(z-d)\cos\theta_i}$$

Boundary conditions: (analogues of (7.37))

~~At the boundary~~

① $\varepsilon_1\bar{E}_I\cdot\hat{z}\big|_{z=0} = \varepsilon_0\bar{E}_{II}\cdot\hat{z}\big|_{z=0}$, ② $\bar{E}_I\times\hat{z}\big|_{z=0} = \bar{E}_{II}\times\hat{z}\big|_{z=0}$

③ $\bar{B}_I\cdot\hat{z}\big|_{z=0} = \bar{B}_{II}\cdot\hat{z}\big|_{z=0}$, ④ $\bar{B}_I\times\hat{z}\big|_{z=0} = \bar{B}_{II}\times\hat{z}\big|_{z=0}$

& similarly at $z=d$.

Since the materials are nonmagnetic, we assume $\mu=\mu_0$ everywhere, hence $\varepsilon_1/\varepsilon_0 = n^2$.

7.3 a), cont'd

Polarization $\perp$ plane of incidence, cont'd

at $z=0$,

① $\Rightarrow$ $0=0$

② $\Rightarrow$ $E_0+E_1 = E_2+E_3$

③ $\Rightarrow$ $n \sin\theta_i\,(E_0+E_1) = n\sin\theta_i\,(E_2+E_3)$, same as above

④ $\Rightarrow$ $n\cos\theta_i\,(-E_0+E_1) = \cos\theta_2\,(-E_2+E_3)$

at $z=d$,

① $\Rightarrow$ $0=0$

② $\Rightarrow$ $E_2\, e^{i\frac{\omega}{c}d\cos\theta_2} + E_3\, e^{-i\frac{\omega}{c}d\cos\theta_2} = E_4$

③ $\Rightarrow$ $n\sin\theta_i\left(E_2\, e^{i\frac{\omega}{c}d\cos\theta_2} + E_3\, e^{-i\frac{\omega}{c}d\cos\theta_2}\right) = n\sin\theta_i\, E_4$, same as above

④ $\Rightarrow$ $\cos\theta_2\left(-E_2\, e^{i\frac{\omega}{c}d\cos\theta_2} + E_3\, e^{-i\frac{\omega}{c}d\cos\theta_2}\right) = -n\cos\theta_i\, E_4$

After a lot of algebra, one can show

$$\frac{E_1}{E_0} = \frac{(1-b^2)\, i\sin\alpha}{2b\cos\alpha - (1+b^2)\, i\sin\alpha} \qquad \alpha = \frac{\omega}{c}d\left(1-n^2\sin^2\theta_i\right)^{1/2}$$

$$\frac{E_2}{E_0} = \frac{b(1+b)(\cos\alpha - i\sin\alpha)}{2b\cos\alpha - i(1+b^2)\sin\alpha} \qquad b = \frac{n\cos\theta_i}{\sqrt{1-n^2\sin^2\theta_i}}$$

$$\frac{E_3}{E_0} = \frac{b(1-b)(\cos\alpha + i\sin\alpha)}{2b\cos\alpha - i(1+b^2)\sin\alpha}$$

$$\frac{E_4}{E_0} = \frac{2b}{2b\cos\alpha - i(1+b^2)\sin\alpha}$$

7.3 a), cont'd

Polarization $\perp$ plane of incidence, cont'd

Below the critical angle of total internal reflection:

~~α,β~~ α, b are real-valued

Reflected power $R = \left|\frac{E_1}{E_0}\right|^2 = \dfrac{(1-b^2)^2 \sin^2\alpha}{4b^2\cos^2\alpha + (1+b^2)^2\sin^2\alpha}$

Transmitted power $T = \left|\frac{E_4}{E_0}\right|^2 = \dfrac{4b^2}{4b^2\cos^2\alpha + (1+b^2)^2\sin^2\alpha}$

Above the critical angle, α, b become pure imaginary.

Write $\alpha = i\lambda = i\,\frac{\omega}{c} d \sqrt{n^2\sin^2\theta_i - 1}$

$b = i\gamma = -i\,\dfrac{n\cos\theta_i}{\sqrt{n^2\sin^2\theta_i - 1}}$

Reflected power $R = \left|\frac{E_1}{E_0}\right|^2 = \dfrac{4\gamma^2}{4\gamma^2\cosh^2\lambda + (1-\gamma^2)^2\sinh^2\lambda}$

7.3 d, cont'd

Polarization ∥ plane of incidence: Here, specify $\bar{B} \parallel \hat{y}$.

$$\bar{E}_I = (-\cos\theta_i\,\hat{x} + \sin\theta_i\,\hat{z})E_0\,e^{i\frac{n}{c}\omega z\cos\theta_i}$$

$$+ (\cos\theta_i\,\hat{x} + \sin\theta_i\,\hat{z})E_1\,e^{-i\frac{n}{c}\omega z\cos\theta_i}$$

$$\bar{E}_{II} = (-\cos\theta_2\,\hat{x} + n\sin\theta_i\,\hat{z})E_2\,e^{i\frac{\omega z}{c}\cos\theta_2}$$

$$+ (\cos\theta_2\,\hat{x} + n\sin\theta_i\,\hat{z})E_3\,e^{-i\frac{\omega}{c}z\cos\theta_2}$$

$$\bar{E}_{III} = (-\cos\theta_i\,\hat{x} + \sin\theta_i\,\hat{z})E_4\,e^{i\frac{n}{c}\omega\cos\theta_i(z-d)}$$

$$\bar{B}_I = -\hat{y}\left(\frac{n}{c}\right)\left(E_0\,e^{i\frac{n}{c}\omega z\cos\theta_i} + E_1\,e^{-i\frac{n}{c}\omega z\cos\theta_i}\right)$$

$$\bar{B}_{II} = -\hat{y}\left(\frac{1}{c}\right)\left(E_2\,e^{i\frac{\omega}{c}z\cos\theta_2} + E_3\,e^{-i\frac{\omega}{c}z\cos\theta_2}\right)$$

$$\bar{B}_{III} = -\hat{y}\left(\frac{n}{c}\right)E_4\,e^{i\frac{n}{c}\omega(z-d)\cos\theta_i}$$

Applying the same boundary conditions as before, we find

$$E_2 + E_3 = n(E_0 + E_1)$$

$$n(E_2 - E_3) = \frac{n\cos\theta_i}{\cos\theta_2}(E_0 - E_1)$$

$$nE_4 = E_2\,e^{i\frac{\omega}{c}d\cos\theta_2} + E_3\,e^{-i\frac{\omega}{c}d\cos\theta_2}$$

$$n\cos\theta_i\,E_4 = n\cos\theta_2\left(E_2\,e^{i\frac{\omega}{c}d\cos\theta_2} - E_3\,e^{-i\frac{\omega}{c}d\cos\theta_2}\right)$$

7.3 a), cont'd

After some algebra, one can show

$$\frac{E_1}{E_0} = \frac{i(n^4-b^2)\sin\alpha}{2n^2 b\cos\alpha - i(n^4+b^2)\sin\alpha} \qquad b = \frac{n\cos\theta_i}{\sqrt{1-n^2\sin^2\theta_i}}$$

$$\frac{E_2}{E_0} = \frac{bn(n^2+b)(\cos\alpha - i\sin\alpha)}{2n^2 b\cos\alpha - i(n^4+b^2)\sin\alpha} \qquad \alpha = \frac{\omega}{c} d\sqrt{1-n^2\sin^2\theta_i}$$

$$\frac{E_3}{E_0} = \frac{bn(n^2-b)(\cos\alpha + i\sin\alpha)}{2n^2 b\cos\alpha - i(n^4+b^2)\sin\alpha}$$

$$\frac{E_4}{E_0} = \frac{2n^2 b}{2n^2 b\cos\alpha - i(n^4+b^2)\sin\alpha}$$

Below the critical angle,

$$\text{Reflected power} = R = \left|\frac{E_1}{E_0}\right|^2 = \frac{(n^4-b^2)^2\sin^2\alpha}{4n^4b^2\cos^2\alpha + (n^4+b^2)^2\sin^2\alpha}$$

$$\text{Transmitted power } T = \left|\frac{E_4}{E_0}\right|^2 = \frac{4n^4b^2}{4n^4b^2\cos^2\alpha + (n^4+b^2)^2\sin^2\alpha}$$

Above the critical angle,

define $a = \frac{\omega}{c} d\sqrt{n^2\sin^2\theta_i - 1}$, $\quad \beta = \frac{n\cos\theta_i}{\sqrt{n^2\sin^2\theta_i - 1}}$

$$\text{Reflected power } R = \left|\frac{E_1}{E_0}\right|^2 = \frac{(n^4+\beta^2)^2\sinh^2 a}{4n^4\beta^2\cosh^2 a + (n^4-\beta^2)^2\sinh^2 a}$$

$$\text{Transmitted power } T = \left|\frac{E_4}{E_0}\right|^2 = \frac{4n^4\beta^2}{4n^4\beta^2\cosh^2 a + (n^4-\beta^2)^2\sinh^2 a}$$

7.12 The time dependence of electrical disturbances in good conductors is governed by the frequency-dependent conductivity

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \qquad (7.58)$$

Consider longitudinal electric fields in a conductor, using Ohm's law, the continuity equation, and the differential form of Coulomb's law.

a) Show that the time-Fourier-Transformed charge density satisfies the equation

$$[\sigma(\omega) - i\omega\varepsilon_0]\,\rho(\bar{x},\omega) = 0.$$

Continuity eq'n: $\nabla\cdot\bar{J} + \frac{\partial\rho}{\partial t} = 0$

Ohm's law: $\bar{J} = \sigma\bar{E}$

$$\Rightarrow \sigma\underbrace{\nabla\cdot\bar{E}}_{\rho/\varepsilon_0 \text{ from Gauss's law}} + \frac{\partial\rho}{\partial t} = 0$$

$$\Rightarrow \frac{\sigma}{\varepsilon_0}\rho + \frac{\partial\rho}{\partial t} = 0$$

Assume $\rho(\bar{x},t) = \frac{1}{\sqrt{2\pi}}\int d\omega\, \rho(\bar{x},\omega)\, e^{-i\omega t}$

$$\Rightarrow \frac{\partial\rho}{\partial t} = \frac{1}{\sqrt{2\pi}}\int d\omega\,(-i\omega)\,\rho(\bar{x},\omega)\,e^{-i\omega t}$$

So $\frac{\sigma}{\varepsilon_0}\rho(\bar{x},t) + \frac{\partial\rho}{\partial t} = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\left(\frac{\sigma}{\varepsilon_0}\rho(\bar{x},\omega) - i\omega\rho(\bar{x},\omega)\right)$

$$\Rightarrow \boxed{(\sigma - i\omega\varepsilon_0)\,\rho(\bar{x},\omega) = 0}$$

7.12 cont'd

b) Using the representation $\sigma(\omega) = \sigma_0 (1 - i\omega\tau)^{-1}$,
where $\sigma_0 = \epsilon_0 \omega_p^2 \tau$ and τ is a damping time,
show that in the approximation $\omega_p \tau \gg 1$ any initial disturbance
will oscillate with the plasma frequency and decay in
amplitude with a decay constant $\lambda = 1/2\tau$.

Note that if you use $\sigma(\omega) = \sigma(0) = \sigma_0$ in (a),
you will find no oscillations & extremely rapid damping.

From (a), $\sigma(\omega) = i\epsilon_0 \omega$.

$$\Rightarrow \frac{\sigma_0}{1 - i\omega\tau} = i\epsilon_0 \omega \quad \Rightarrow \quad \frac{\omega_p^2 \tau}{1 - i\omega\tau} = i\omega$$

$$\Rightarrow i\omega(1 - i\omega\tau) = \omega_p^2 \tau$$

$$i\omega + \omega^2\tau - \omega_p^2 \tau = 0$$

$$\Rightarrow \omega = \frac{-i \pm \left(-1 - 4(\tau)(-\omega_p^2\tau)\right)^{1/2}}{2\tau}$$

$$= \frac{-i \pm \left(4\tau^2\omega_p^2 - 1\right)^{1/2}}{2\tau}$$

If $\omega_p \tau \gg 1$,

$$\omega \simeq \frac{-i \pm 2\omega_p\tau}{2\tau} = \pm\omega_p - \frac{i}{2\tau}$$

$\Rightarrow$ $\boxed{\text{oscillation at plasma frequency } \omega_p, \text{ with decay constant } \frac{1}{2\tau}}$

7.16 Plane waves propagate in a homogeneous nonpermeable but anisotropic dielectric. The dielectric is characterized by a tensor ε_{ij}, but if coordinate axes are chosen as the principal axes, the components of displacement along these axes are related to the electric field component by $D_i = \varepsilon_i E_i$, where ε_i are the eigenvalues of the matrix ε_{ij}.

a) Show that plane waves with frequency ω and wavevector $\bar{k}$ must satisfy

$$\bar{k}\times(\bar{k}\times\bar{E}) + \mu_0\omega^2\bar{D} = 0.$$

This is a general result that does not depend on the form of the dielectric tensor.

Recall Maxwell's equations for harmonic fields include

$$i\bar{k}\times\bar{H} + i\omega\bar{D} = 0, \qquad i\bar{k}\times\bar{E} - i\omega\bar{B} = 0$$

$$\Rightarrow i\bar{k}\times(i\bar{k}\times\bar{E}) - i\omega(i\bar{k}\times\bar{B}) = 0$$

Use $\bar{B} = \mu_0\bar{H} \Rightarrow i\bar{k}\times(i\bar{k}\times\bar{E}) + \mu_0\omega^2(\bar{k}\times\bar{H}) = 0$

$$\Rightarrow -\bar{k}\times(\bar{k}\times\bar{E}) - \mu_0\omega^2\bar{D} = 0$$

7.16, cont'd

b) Show that for a given wave vector $\bar{k} = k\hat{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_i \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component of $\hat{n}$ along the i^{th} principal axis.

Write $\bar{k} = k\hat{n}$

$$\bar{k} \times (\bar{k} \times \bar{E}) = (\bar{k} \cdot \bar{E})\bar{k} - (k^2)\bar{E}$$

$$= k^2 \left((\hat{n} \cdot \bar{E})\hat{n} - \bar{E} \right)$$

Hence $\bar{k} \times (\bar{k} \times \bar{E}) + \mu_0 \omega^2 \bar{D} = 0$

$$\Rightarrow (\hat{n} \cdot \bar{E})\hat{n} - \bar{E} + \mu_0 v^2 \bar{D} = 0$$

Work with the principal axes, and define

$$A_{ij} = n_i n_j - \delta_{ij}, \quad W_{ij} = \delta_{ij} \mu_0 \epsilon_j = \frac{\delta_{ij}}{v_j^2}$$

$$\Rightarrow A\bar{E} = -v^2 W \bar{E} \quad \text{or} \quad (A + v^2 W)\bar{E} = 0$$

Find the velocities by solving

$$0 = \det (A + v^2 W)$$

(cont'd)

7.16 b), cont'd

$$\det(A + v^2 W) = 0$$

$$= \det\begin{bmatrix} n_1^2 - 1 + v^2/v_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 - 1 + v^2/v_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 - 1 + v^2/v_3^2 \end{bmatrix}$$

$$= \left(n_1^2 + \left(\frac{v^2}{v_1^2} - 1\right)\right)\left(\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) + n_2^2\left(\frac{v^2}{v_3^2} - 1\right) + n_3^2\left(\frac{v^2}{v_2^2} - 1\right)\right)$$
$$- n_1 n_2\left(n_1 n_2\left(\frac{v^2}{v_3^2} - 1\right)\right) + n_1 n_3\left(-n_1 n_3\left(\frac{v^2}{v_2^2} - 1\right)\right)$$

$$= \left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right)$$

$$+ n_1^2\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) + n_2^2\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) + n_3^2\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)$$

$$+ \cancel{n_1^2 n_2^2\left(\frac{v^2}{v_3^2} - 1\right)} + \cancel{n_1^2 n_3^2\left(\frac{v^2}{v_2^2} - 1\right)}$$

$$- \cancel{n_1^2 n_2^2\left(\frac{v^2}{v_3^2} - 1\right)} - \cancel{n_1^2 n_3^2\left(\frac{v^2}{v_2^2} - 1\right)}$$

$$= \left[\prod_i\left(\frac{v^2}{v_i^2} - 1\right)\right]\left[1 + \sum_i \frac{n_i^2}{v^2/v_i^2 - 1}\right] \Rightarrow 1 + \sum_i \frac{n_i^2}{v^2/v_i^2 - 1} = 0$$

Now, use the fact that since
n_i = i^{th} component of unit-vector $\hat{n}$
(not index of refraction in i^{th} direction),

$$\sum n_i^2 = 1$$

7.16 b), cont'd

since $\sum n_i^2 = 1$,

$$1 + \sum_i \frac{n_i^2}{v^2/v_i^2 - 1}$$

$$= \sum_i n_i^2 \left[1 + \frac{1}{v^2/v_i^2 - 1} \right]$$

$$= \sum_i n_i^2 \left[\frac{v^2/v_i^2 \cancel{-1} \cancel{+1}}{v^2/v_i^2 - 1} \right]$$

$$= \sum_i n_i^2 \left[\frac{v^2}{v^2 - v_i^2} \right]$$

$$= v^2 \sum_i \frac{n_i^2}{v^2 - v_i^2}$$

Thus,

$$1 + \sum_i \frac{n_i^2}{v^2/v_i^2 - 1} = 0 \Rightarrow \boxed{\sum_i \frac{n_i^2}{v^2 - v_i^2} = 0}$$

7.16, cont'd

c) Show that $\bar{D}_a \cdot \bar{D}_b = 0$, where $\bar{D}_a, \bar{D}_b$ are the displacement associated with the two modes of propagation.

Recall from part (b) that

$$(\hat{n}\cdot\bar{E})\hat{n} - \bar{E} + \mu_0 v^2 \bar{D} = 0 \quad \text{or} \quad (A + v^2 W)\bar{E} = 0$$

In particular,

$$(A + v_a^2 W)\bar{E}_a = 0, \qquad (A + v_b^2 W)\bar{E}_b = 0$$

where v_a, v_b are the principal velocities associated with $\bar{E}_a, \bar{E}_b$.

$$\Rightarrow E_b^T (A + v_a^2 W) E_a = 0, \qquad E_a^T (A + v_b^2 W) E_b = 0$$

Since A, W are symmetric, may transpose & take difference:

$$\Rightarrow (v_a^2 - v_b^2) E_a^T W E_b = 0$$

If $v_a \neq v_b$, then $E_a^T W E_b = 0$

If $v_a = v_b$, then can use Gram-Schmidt to make the vectors orthogonal.

Since $W = \mu_0 \Sigma$ for $\Sigma = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, & $\bar{D} = \Sigma \bar{E}$,

$$E_a^T W E_b = 0 \Rightarrow \bar{E}_a \cdot \bar{D}_b = 0 \text{ or } \bar{E}_b \cdot \bar{D}_a = 0$$

$$\bar{D}_a \cdot \bar{D}_b = E_a^T \Sigma^2 E_b = \frac{1}{\mu_0^2} E_a^T W^2 E_b = \frac{1}{\mu_0^2 v_a^2 v_b^2} E_a^T A^2 E_b$$

Note $A^2 = -A$

$$\Rightarrow \bar{D}_a \cdot \bar{D}_b = -\frac{1}{\mu_0^2 v_a^2 v_b^2} E_a^T A E_b = +\frac{1}{\mu_0 v_a^2} E_a^T W E_b = 0$$

$$\Rightarrow \boxed{\bar{D}_a \cdot \bar{D}_b = 0}$$ (However, $\bar{E}_a \cdot \bar{E}_b \neq 0$ in general.)

7.28 A circularly polarized plane wave moving in the z direction has a finite extent in the x and y directions. Assuming that the amplitude modulation is slowly varying (the wave is many wavelengths broad), show that the electric and magnetic fields are given approximately by

$$\bar{E}(x,y,z,t) = \left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3\right] e^{ikz-i\omega t}$$

$$\bar{B} \simeq \mp i\sqrt{\mu\varepsilon}\,\bar{E}$$

where $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are unit vectors in the x, y, z directions.

Because this is a circularly polarized plane wave, expect that $\bar{E}$ is of the form

$$\bar{E} = \left[E_0(\hat{e}_1 \pm i\hat{e}_2) + F(x,y)\,\hat{e}_3\right] e^{i(kz-\omega t)}$$

From Maxwell's equations, $\nabla\cdot\bar{E} = 0$

$$\Rightarrow \left[\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y} + ikF\right] e^{i(kz-\omega t)} = 0$$

$$\Rightarrow F = \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right).$$ Result follows.

Similarly, from Maxwell's eqns,

$$\frac{\partial \bar{B}}{\partial t} = -\nabla\times\bar{E}$$

$$= -\hat{e}_1\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{e}_2\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) - \hat{e}_3\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

7.28, cont'd

$$\frac{\partial \bar{B}}{\partial t} = -\hat{e}_1\left(\frac{\partial F}{\partial y} - (\pm i)\cancel{\partial}E_0(ik)\right)_{\wedge}^{e^{i(kz-\omega t)}} + \hat{e}_2\left(\frac{\partial F}{\partial x} - E_0(ik)\right)e^{i(kz-\omega t)}$$

$$-\hat{e}_3\left(\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)e^{i(kz-\omega t)}$$

Since we are told the amplitude modulation is slowly varying, we will neglect $\partial F/\partial x$, $\partial F/\partial y$, as these are second derivatives of E_0.

$$\Rightarrow \frac{\partial \bar{B}}{\partial t} \cong -\hat{e}_1\left(\pm kE_0\right)e^{i(kz-\omega t)} - \hat{e}_2\,(ik)E_0\,e^{i(kz-\omega t)}$$

$$-\hat{e}_3\left(\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)e^{i(kz-\omega t)}$$

$$\Rightarrow \bar{B} \cong -\frac{ik}{\omega}\left(\pm E_0\hat{e}_1 + iE_0\hat{e}_2 + \frac{1}{k}\left(\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)\hat{e}_3\right)e^{i(kz-\omega t)}$$

$$= \mp\frac{ik}{\omega}\left(E_0\hat{e}_1 \pm iE_0\hat{e}_2 + \frac{1}{k}\left(i\frac{\partial E_0}{\partial x} \mp \frac{\partial E_0}{\partial y}\right)\hat{e}_3\right)e^{i(kz-\omega t)}$$

$$= \mp i\sqrt{\mu\varepsilon}\left[E_0\left(\hat{e}_1 \pm i\hat{e}_2\right) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3\right]e^{i(kz-\omega t)}$$

$$= \mp i\sqrt{\mu\varepsilon}\,\bar{E}$$