8.2 A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ is filled with a uniform lossless dielectric (μ, ε). A TEM is propagated along this line

a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\varepsilon}}\,\pi a^2 |H_0|^2 \ln(b/a)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

From Gauss's law,

$$\bar{E} = \frac{\lambda}{2\pi\varepsilon}\frac{\hat{\rho}}{\rho} \quad \text{where } \lambda = \text{charge density / length along inner conductor}$$

$$\bar{H}_t = \left(\frac{\varepsilon}{\mu}\right)^{1/2} \hat{z}\times\bar{E} \qquad (7.11')$$

$$= \left(\frac{\varepsilon}{\mu}\right)^{1/2} \frac{\lambda}{2\pi\varepsilon}\frac{\hat{\phi}}{\rho}$$

At the surface of the conductor,

$$|\bar{H}_t| = \left(\frac{\varepsilon}{\mu}\right)^{1/2} \frac{\lambda}{2\pi\varepsilon}\frac{1}{a} \equiv H_0$$

$$\Rightarrow \frac{\lambda}{2\pi\varepsilon} = \left(\frac{\mu}{\varepsilon}\right)^{1/2} a H_0$$

$$\Rightarrow \bar{E} = \left(\frac{\mu}{\varepsilon}\right)^{1/2} a H_0 \frac{\hat{\rho}}{\rho}, \quad \bar{H} = \frac{a}{\rho} H_0 \hat{\phi}$$

(cont'd)

8.2 a), cont'd

Poynting vector $\bar{S} = \frac{1}{2} \bar{E} \times \bar{H}^*$

$$= \frac{1}{2}\left(\frac{\mu}{\varepsilon}\right)^{1/2} |H_0|^2 \left(\frac{a}{\rho}\right)^2$$

Power flow

$$P = \int_A \hat{z} \cdot \bar{S} \, da$$

$$= \frac{1}{2}\left(\frac{\mu}{\varepsilon}\right)^{1/2} |H_0|^2 \int_0^{2\pi} d\phi \int_a^b \rho \, d\rho \left(\frac{a}{\rho}\right)^2$$

$$= \frac{1}{2}\left(\frac{\mu}{\varepsilon}\right)^{1/2} |H_0|^2 \, 2\pi a^2 \ln(b/a)$$

$$\boxed{P = \left(\frac{\mu}{\varepsilon}\right)^{1/2} \pi a^2 |H_0|^2 \ln(b/a)}$$

8.2, cont'd

b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where $\gamma = \frac{1}{2\sigma\delta}\left(\frac{\varepsilon}{\mu}\right)^{1/2} \frac{\frac{1}{a}+\frac{1}{b}}{\ln(b/a)}$

The attenuation constant $\gamma = -\frac{1}{2P}\frac{dP}{dz}$ (8.57)

The power loss per unit length is given by

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_c |\hat{n}\times\overline{H}|^2 \, dl \qquad (8.58)$$

$$= \frac{1}{2\sigma\delta}|H_0|^2 \oint_c \left(\frac{a}{\rho}\right)^2 dl$$

Note there are two boundary component: one at $\rho=a$, the other at $\rho=b$.

$$\Rightarrow -\frac{dP}{dz} = \frac{1}{2\sigma\delta}|H_0|^2\left[(2\pi a)(1)^2 + (2\pi b)\left(\frac{a}{b}\right)^2\right]$$

$$= \frac{1}{2\sigma\delta}|H_0|^2(2\pi a)\left(\frac{1}{b}\right)\left[b+a\right]$$

Using the expression for P from (a),

$$\boxed{\gamma = \frac{-1}{2P}\frac{dP}{dz} = \frac{1}{\cancel{2}}\left(\frac{\varepsilon}{\mu}\right)^{1/2}\frac{1}{\cancel{\pi} a^2}\frac{1}{2\sigma\delta}(\cancel{2\pi})\left(\frac{a}{b}\right)\frac{a+b}{\ln(b/a)}}$$

$$\boxed{= \frac{1}{2\sigma\delta}\left(\frac{\varepsilon}{\mu}\right)^{1/2}\frac{\frac{1}{a}+\frac{1}{b}}{\ln(b/a)}}$$

8.2 cont'd

c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line,

$$Z_0 = \frac{1}{2\pi}\left(\frac{\mu}{\varepsilon}\right)^{1/2} \ln(b/a)$$

$$\text{Voltage} = -\int_a^b \bar{E}\cdot d\bar{l} = -\left(\frac{\mu}{\varepsilon}\right)^{1/2} aH_0 \int_a^b \frac{d\rho}{\rho} = -\left(\frac{\mu}{\varepsilon}\right)^{1/2} aH_0 \ln(b/a)$$

Current:

for the inner conductor, the surface current density is

$$\bar{K} = \hat{n}\times\bar{H} = \hat{\rho}\times\left(\frac{a}{a}H_0\hat{\phi}\right) = H_0\hat{z}$$

so the total current flowing down the inner conductor is

$$I = \oint_C |\bar{K}|\, dl = H_0(2\pi a)$$

$$\boxed{Z_0 = \frac{|\text{Voltage}|}{I} = \left(\frac{\mu}{\varepsilon}\right)^{1/2} \frac{\cancel{a}\cancel{H_0}\ln(b/a)}{\cancel{H_0}(2\pi\cancel{a})} = \frac{1}{2\pi}\left(\frac{\mu}{\varepsilon}\right)^{1/2}\ln(b/a)}$$

8.2, cont'd

d) Show that the series resistance and inductance per unit length of the line are

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c \delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

The power loss $-\frac{dP}{dz} = \frac{1}{2}|I|^2 R$

$$\Rightarrow \boxed{R = \frac{2}{|I|^2}\left(-\frac{dP}{dz}\right) = \frac{2}{(2\pi a)^2 |H_0|^2}\,\frac{1}{2\sigma\delta}\,|H_0|^2 (2\pi a)\left(\frac{1}{b}\right)(a+b)}$$

using results from (b), (c).

$$\boxed{= \frac{1}{2\pi\sigma\delta}\,\frac{1}{ab}(a+b) = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Energy / length in magnetic field inside waveguide =

$$U_{inner} = \frac{\mu}{4}\int_A |H|^2\, da = \frac{\mu}{4}|H_0|^2 \int_0^{2\pi} d\phi \int_a^b \rho\, d\rho \left(\frac{a}{\rho}\right)^2$$

$$= \frac{\mu}{4}|H_0|^2 (2\pi)\, a^2 \ln(b/a)$$

(cont'd)

8.2 d), cont'd

~~Contribution~~
Contribution to energy from $\bar{H}$ penetrating into walls:

$$\bar{H} = \bar{H}_{\parallel} e^{-\xi/\delta} e^{i\xi/\delta} \qquad (8.9)$$

Assuming $\delta \ll$ thickness of conductor, we can approximate

$$U_{wall} = \frac{\mu_c}{4} C \int_0^\infty d\xi \, |\bar{H}|^2 \qquad \text{for } C = \text{circumference}$$

$$= \frac{\mu_c}{4} C |\bar{H}_{\parallel}|^2 \int_0^\infty d\xi \, e^{-2\xi/\delta}$$

$$= \frac{\mu_c}{4} C |\bar{H}_{\parallel}|^2 \left(-\frac{\delta}{2} e^{-2\xi/\delta} \Big|_0^\infty \right)$$

$$= + \frac{\mu_c \delta}{8} C |\bar{H}_{\parallel}|^2$$

Inner wall: $\bar{H}_{\parallel} = H_0 \hat{\phi}$, $\quad C = 2\pi a$

Outer wall: $\bar{H}_{\parallel} = H_0 \frac{a}{b} \hat{\phi}$, $\quad C = 2\pi b$

so total contribution from both walls is approximately

$$\frac{\mu_c \delta}{8} (2\pi a) |H_0|^2 + \frac{\mu_c \delta}{8} (2\pi b) |H_0|^2 \left(\frac{a}{b}\right)^2$$

$$= \frac{\pi \mu_c \delta}{4} |H_0|^2 \frac{a}{b}(a+b)$$

(cont'd)

8.2 d), cont'd

Use $\frac{1}{4} L |I|^2 = U_{inner} + U_{walls}$

$$\Rightarrow L = \frac{4}{|I|^2}\left(U_{inner} + U_{walls}\right)$$

$$= \frac{\cancel{4}}{(2\pi a)^2 \cancel{|H_0|^2}}\left(\frac{\mu}{\cancel{4}} \cancel{|H_0|^2} (2\pi)\, a^2 \ln(b/a) + \frac{\pi \mu_c \delta}{\cancel{4}} \cancel{|H_0|^2} \left(\frac{a}{b}\right)(a+b)\right)$$

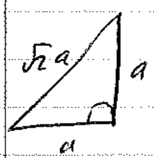
$$= \frac{\mu}{2\pi a}\, a \ln(b/a) + \frac{\cancel{a}\mu_c \delta}{4\pi a}\left(\frac{a}{b} + 1\right)$$

$$\boxed{L = \frac{\mu}{2\pi} \ln(b/a) + \frac{\mu_c \delta}{4\pi}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

(a) only

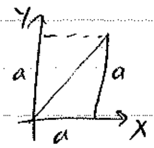
8.5 A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$.
The medium inside has $\mu = \varepsilon = 1$.

a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.



We're going to need to apply D or N boundary conditions & solve for eigenfunctions, but this shape is not something we've worked with previously.

Here is a trick: start with the eigenfunctions for a square of side a,



and then look at the behavior of the eigenfunctions under $x \leftrightarrow y$.

If $f(x,y) = +f(y,x)$, then has vanishing normal derivative on the diagonal. $\rightsquigarrow$ N b.c.

If $f(x,y) = -f(y,x)$, then $f=0$ along diagonal $\rightsquigarrow$ D b.c.

TM modes:

Solve $(\nabla_t^2 + \gamma^2)\psi = 0$ on square.

Sol'ns: $\psi_{mn} \propto \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$, $\quad \gamma_{mn} = \frac{\pi}{a}(m^2+n^2)^{1/2}$

with cutoff frequencies $\boxed{\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\mu_0\varepsilon_0}} = \frac{\pi c}{a}(m^2+n^2)^{1/2}}$

To satisfy D b.c. on the diagonal, take $\mathbb{Z}_2$-odd combination

$$\psi_{mn} \propto \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}$$

Note $\psi_{mn}(x,0) = \psi_{mn}(a,y) = \psi_{mn}(x,x) = 0$, $\qquad m,n > 0$

8.5 a), cont'd

<u>TE modes</u>:

The analysis is similar except that we use cosine's & $\mathcal{X}_2$-even combinations.

Hence, $\psi_{mn} \propto \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} + \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a}$

with ~~@~~ cutoff frequencies $\boxed{\omega_{mn} = \gamma_{mn} c = \frac{\pi c}{a}(m^2+n^2)^{1/2}}$

where $\underline{m, n \geq 0}$ (whereas for TM, $m=0, n=0$ not allowed)