9.1 A common textbook example of a radiating system is a configuration of charges fixed relative to one another but in rotation. The charge density is a function of time, but not of the form $\rho(\vec{x})e^{-i\omega t}$.

a) Show that for rotating charges one alternative is to calculate real time-dependent multipole moments using $\rho(\vec{x},t)$ directly and then compute the multipole moments for a given harmonic frequency with the conventions of (9.1) by ~~exp~~ inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{\ell m}(t)$ to form linear combinations that are real before making the connection.

~~Write~~

Assume the rotation is about the z axis.

Write

$$\rho(\vec{x},t) = \rho(r,\theta,\phi-\omega t,\ t)$$

Multipole moments:

$$q_{\ell m}(t) = \int d^3x\ r^\ell Y^*_{\ell m}(\theta,\phi)\ \rho(r,\theta,\phi-\omega t)$$

$$= \int d^3x\ r^\ell Y^*_{\ell m}(\theta,\phi'+\omega t)\ \rho(r,\theta,\phi') \qquad \text{for } \phi' = \phi-\omega t$$

Now, $Y_{\ell m}(\theta,\phi'+\omega t) = Y_{\ell m}(\theta,\phi')e^{im\omega t}$ since $Y_{\ell m} \propto e^{im\phi}$

$$\Rightarrow q_{\ell m}(t) = \bar{q}_{\ell m}\, e^{-im\omega t}$$

where $\bar{q}_{\ell m} = \int d^3x\ r^\ell Y^*_{\ell m}(\theta,\phi)\ \rho(r,\theta,\phi)$

As a result, the $q_{\ell m}$ contribution will radiate at a frequency $m\omega$.

9.1 a), cont'd

For $m=0$, there is no time dependence.

$m \neq 0$: Note

$$q_{\ell m} Y_{\ell m} + q_{\ell,-m} Y_{\ell,-m} = \mathrm{Re}\left(2\,\bar{q}_{\ell m} Y_{\ell m}\, e^{-im\omega t}\right)$$

Hence we have an effective $q_{\ell m}$ to match harmonic conventions:

$$q_{\ell m\,\text{effective}} = \begin{cases} 2\bar{q}_{\ell m} & m > 0 \\ \bar{q}_{\ell 0} & m = 0 \end{cases}$$

9.1, cont'd

b) Consider a charge density $\rho(\bar{x},t)$ that is periodic in time with period $T = 2\pi/\omega$. By making a Fourier series expansion, show that it can be written as

$$\rho(\bar{x},t) = \rho_0(\bar{x}) + \sum_{n=1}^{\infty} \mathrm{Re}\left(2\rho_n(\bar{x})e^{-in\omega t}\right)$$

where $\rho_n(\bar{x}) = \frac{1}{T}\int_0^T \rho(\bar{x},t)\, e^{in\omega t}\, dt$.

Fourier series:

$$\rho(\bar{x},t) = \sum_{n=-\infty}^{\infty} \rho_n(\bar{x})\, e^{-in\omega t}$$

$$\rho_n(\bar{x}) = \frac{1}{T}\int_0^T \rho(\bar{x},t)\, e^{in\omega t}\, dt$$

Since $\rho(\bar{x},t)$ is real, $\rho_{-n}(\bar{x}) = \rho_n(\bar{x})^*$

$$\Rightarrow \rho(\bar{x},t) = \rho_0(\bar{x}) + \sum_{n=1}^{\infty}\left(\rho_n(\bar{x})\, e^{-in\omega t} + \rho_{-n}(\bar{x})\, e^{+in\omega t}\right)$$

$$= \rho_0(\bar{x}) + \sum_{n=1}^{\infty}\left(\rho_n(\bar{x})\, e^{-in\omega t} + \rho_n(\bar{x})^* \left(e^{-in\omega t}\right)^*\right)$$

$$= \rho_0(\bar{x}) + \sum_{n=1}^{\infty} \mathrm{Re}\left(2\rho_n(\bar{x})\, e^{-in\omega t}\right)$$

9.1, cont'd

c) For a single charge q rotating about the origin in the xy plane in a circle of radius R at constant angular speed ω_0, calculate the $l=0$ and $l=1$ multipole moments by the methods of parts (a), (b) and compare. In method (b) express the charge density $\rho(\bar{x})$ in cylindrical coordinates. Are there higher multipoles? At what frequencies?

For a single rotating charge q,

$$\rho(\bar{x}, t) = \frac{q}{R^2}\,\delta(r-R)\,\delta(\cos\theta)\,\delta(\phi-\omega_0 t)$$

Using the methods of part (a),

$$\bar{q}_{lm} = \int d^3x\, r^l\, Y_{lm}(\theta,\phi)^* \underbrace{\bar{\rho}(r,\theta,\phi)}_{=\frac{q}{R^2}\delta(r-R)\,\delta(\cos\theta)\,\delta(\phi)}\; r^2\,dr\,d\cos\theta\,d\phi$$

$$= R^l q\, Y_{lm}^*\left(\tfrac{\pi}{2}, 0\right)$$

$$\cancel{= qR^l\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}}$$

$$= qR^l\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(0)$$

$$\Rightarrow \bar{q}_{00} = q\frac{1}{\sqrt{4\pi}}, \quad \bar{q}_{11} = -qR\left[\frac{3}{4\pi}\frac{1}{2}\right]^{1/2}, \quad \bar{q}_{10} = 0$$

$$\Rightarrow q_{00}^{\text{eff}} = \frac{q}{\sqrt{4\pi}}, \quad q_{11}^{\text{eff}} = -2qR\left[\frac{3}{8\pi}\right]^{1/2}, \quad q_{10}^{\text{eff}} = 0$$

9.1, c), cont'd

Using the methods of part (b),

$$\rho_n(\vec{x}) = \frac{\omega_0}{2\pi}\int_0^T \rho(\vec{x},t)\, e^{in\omega_0 t}\, dt$$

$$= \frac{\omega_0}{2\pi}\int_0^T dt\, e^{in\omega_0 t}\left[\frac{q}{R^2}\,\delta(r-R)\,\delta(\cos\theta)\,\underbrace{\delta(\phi-\omega_0 t)}_{\frac{1}{\omega}\delta(t-\phi/\omega_0)}\right]$$

$$= \frac{1}{2\pi}\,\frac{q}{R^2}\,\delta(r-R)\,\delta(\cos\theta)\, e^{in\phi}$$

$$q_{\ell m}[\rho_n] = \int r^\ell\, Y_{\ell m}(\theta,\phi)^*\,\rho_n(r,\theta,\phi)\; r^2 dr\, d\cos\theta\, d\phi$$

$$= \frac{q}{2\pi R^2} R^{\ell+2}\underbrace{\int d\phi\; Y_{\ell m}\left(\tfrac{\pi}{2},\phi\right)^* e^{in\phi}}_{2\pi\,\delta_{mn}\,Y_{\ell m}\left(\frac{\pi}{2},0\right)^*}$$

$$= q R^\ell\,\delta_{mn}\left[\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(0)$$

→ essentially matches the result from part (a).

Note higher multipoles will be present whenever $P_\ell^m(0) \neq 0$.

From parity, $P_\ell^m(0) = 0$ for $\ell+m$ odd,
but can be nonzero for $\ell+m$ even.
~~so expect that the ℓ^{th} multipole will radiate at frequencies~~
~~$\ell\omega, (\ell-2)$~~
ℓ even ⇒ frequencies $\{\ell\omega_0, (\ell-2)\omega_0, (\ell-4)\omega_0, \ldots\}$
ℓ odd ⇒ " $\{(\ell-1)\omega_0, (\ell-3)\omega_0, \ldots\}$

from possible values of $m\omega_0$.

9.3 Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap.
An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

In the ~~long~~ long-wavelength = low-frequency limit, the leading ($\equiv$ dipole) contribution should dominate.
Since the frequency is small, we can derive the dipole moment from electrostatics, then use results for radiation from harmonically varying dipoles.

Recall from §2.7 & §3.3 that for a conducting sphere with hemispheres at potential $\pm V$, the potential outside the sphere is

$$\Phi(r,\theta) = \frac{V}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-)^{j-1} \frac{(2j-\frac{1}{2})\,\Gamma(j-\frac{1}{2})}{j!} \left(\frac{R}{r}\right)^{2j} P_{2j-1}(\cos\theta)$$

$$= \frac{3}{2} V \left(\frac{R}{r}\right)^2 \cos\theta + \cdots \qquad (2.27)$$

& recall the potential for a dipole is

$$\frac{1}{4\pi\varepsilon_0} \frac{\bar{p}\cdot\bar{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2} \qquad (4.10)$$

$$\Rightarrow \bar{p} = \frac{3}{2} V (4\pi\varepsilon_0) R^2 \hat{z} = (6\pi\varepsilon_0) V R^2 \hat{z}$$

9.3, cont'd

In the radiation zone, fields from a harmonically-varying dipole are

$$\left.\begin{aligned} \bar{H} &= \frac{ck^2}{4\pi}(\hat{n}\times\bar{p})\frac{e^{ikr}}{r} \\ \bar{E} &= Z_0\,\bar{H}\times\hat{n} \end{aligned}\right\} \quad (9.19) \qquad \text{(w/ factor of } e^{-i\omega t} \text{ understood)}$$

Here,

$$\bar{H} = \frac{ck^2}{4\pi}(6\pi\epsilon_0)VR^2(\hat{n}\times\hat{z})\frac{e^{ikr}}{r}$$

$$= \frac{3}{2}cV\varepsilon_0\,k^2R^2(-\hat{\phi}\sin\theta)\frac{e^{ikr}}{r}$$

$$\boxed{\bar{H} = -\frac{3}{2}cV\varepsilon_0\,k^2R^2\frac{e^{ikr}}{r}\sin\theta\,\hat{\phi}}$$

$$\bar{E} = Z_0\,\bar{H}\times\hat{n} = -\frac{3}{2}cV\underbrace{\varepsilon_0 Z_0}_{\sqrt{\mu_0\varepsilon_0}\,=\,1/c}k^2R^2\frac{e^{ikr}}{r}\sin\theta\,\underbrace{(\hat{\phi}\times\hat{n})}_{=\hat{\theta}}$$

$$\boxed{\bar{E} = -\frac{3}{2}Vk^2R^2\frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}}$$

9.3, cont'd

The time-averaged distribution of radiated power is

$$\frac{dP}{d\Omega} = \frac{1}{2} \, Re\left(r^2 \hat{n} \cdot (E \times H^*) \right)$$

$$= \frac{1}{2} \left(-\frac{3}{2} V \right)^2 (h^2 R^2)^2 \, \frac{\sin^2\theta}{r^2} \, \varepsilon_0 c \; Re\left(r^2 \hat{n} \cdot (\underbrace{\hat{\theta} \times \hat{\phi}}_{= \hat{n}}) \right)$$

$$= \frac{1}{2} \frac{9}{4} V^2 h^4 R^4 \frac{\sin^2\theta}{\cancel{r^2}} \cancel{r^2} \, \varepsilon_0 c$$

$$= \frac{9}{8} V^2 h^4 R^4 (\varepsilon_0 c) \sin^2\theta$$

Note $\varepsilon_0 c = \frac{1}{\mu_0 c}$

The total radiated power is

$$P = \int d\Omega \, \frac{dP}{d\Omega}$$

$$= \frac{9}{8} V^2 h^4 R^4 (\varepsilon_0 c) \, (2\pi) \underbrace{\int_{-1}^{1} d(\cos\theta) \, (1 - \cos^2\theta)}_{\cos\theta - \frac{1}{3}\cos^3\theta \Big]_{-1}^{1} \; = \frac{4}{3}}$$

$$= \frac{\cancel{9}^{\,3}}{\cancel{8}_{\,2}} \frac{\cancel{4}}{\cancel{3}} V^2 h^4 R^4 \varepsilon_0 c \, (2\pi)$$

$$= 3\pi V^2 h^4 R^4 \varepsilon_0 c$$

9.5 a) Show that for harmonic time variation at frequency ω, the electric dipole scalar and vector potentials in Lorenz gauge in the long-wavelength limit are

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r^2} \hat{n}\cdot\bar{p}\,(1-ikr)$$

$$\bar{A}(\bar{x}) = -i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\bar{p}$$

where $k = \omega/c$, $\hat{n} = \bar{r}/r$, $\bar{p}$ is the dipole moment, and time-dependence $e^{-i\omega t}$ is understood.

In chapter 6, the wave eq'ns for $\bar{A}$ & Φ were solved w/ retarded Green f'ns to get

$$\left.\begin{aligned}\bar{A}(\bar{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\int dt'\,\frac{\bar{J}(\bar{x}',t')}{|\bar{x}-\bar{x}'|}\,\delta\!\left(t'+\frac{|\bar{x}-\bar{x}'|}{c}-t\right)\\ \Phi(\bar{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\int dt'\,\frac{\rho(\bar{x}',t')}{|\bar{x}-\bar{x}'|}\,\delta\!\left(t'+\frac{|\bar{x}-\bar{x}'|}{c}-t\right)\end{aligned}\right\}\quad(6.48)$$

Assume $\rho(\bar{x},t) = \rho(\bar{x})e^{-i\omega t}$, $\quad\bar{J}(\bar{x},t) = \bar{J}(\bar{x})e^{-i\omega t}$

$$\Rightarrow\ \bar{A}(\bar{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}\exp\!\left(-i\omega\left(t-\frac{|\bar{x}-\bar{x}'|}{c}\right)\right)$$

$$\Phi(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\exp\!\left(-i\omega\left(t-\frac{|\bar{x}-\bar{x}'|}{c}\right)\right)$$

Write
$$\bar{A}(\bar{x},t) = \bar{A}(x)e^{-i\omega t},\quad \Phi(\bar{x},t) = \Phi(\bar{x})e^{-i\omega t}$$

9.5 a), cont'd

then

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}\, e^{ik|\bar{x}-\bar{x}'|}$$

$$\bar{\Phi}(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\, e^{ik|\bar{x}-\bar{x}'|}$$

Expand $|\bar{x}-\bar{x}'| \simeq r - \hat{n}\cdot\bar{x}'$ (9.7)

$$\Rightarrow \bar{A}(\bar{x}) \simeq \frac{\mu_0}{4\pi}\int d^3x' \, \frac{e^{ikr}}{r}\left(1+\frac{\hat{n}\cdot\bar{x}'}{r}\right)\bar{J}(\bar{x}')\left(1 - ik\,\hat{n}\cdot\bar{x}'\right)$$

$$\bar{\Phi}(\bar{x}) \simeq \frac{1}{4\pi\varepsilon_0}\,\frac{e^{ikr}}{r}\int d^3x' \left(1+\frac{\hat{n}\cdot\bar{x}'}{r}\right)\rho(\bar{x}')\left(1-ik\,\hat{n}\cdot\bar{x}'\right)$$

The terms that contribute to the electric dipole are

$$\bar{A}(\bar{x}) \simeq \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int d^3x' \, \bar{J}(\bar{x}') \qquad (9.13)$$

$$\bar{\Phi}(\bar{x}) \simeq \frac{1}{4\pi\varepsilon_0}\,\frac{e^{ikr}}{r^2}\int d^3x' \, \rho(\bar{x}')\left(\hat{n}\cdot\bar{x}' - ikr\,\hat{n}\cdot\bar{x}'\right)$$

(where we have omitted the electric monopole term $\propto \int d^3x' \, \rho(\bar{x}')$).

9.5 a), cont'd

As in §9.2 of the text, in the expression for $\bar{A}$ we integrate by parts:

$$\int d^3x' \, \bar{J}(\bar{x}') = -\int \bar{x}' (\nabla' \cdot \bar{J}) \, d^3x' = -i\omega \int \bar{x}' \rho(\bar{x}') \, d^3x' \qquad (9.14)$$

$$= -i\omega \bar{p}$$

using the continuity eqn $\nabla \cdot \bar{J} = i\omega\rho$.

$$\Rightarrow \boxed{\bar{A}(\bar{x}) \simeq \frac{\mu_0}{4\pi} (-i\omega) \frac{e^{ikr}}{r} \bar{p}} \qquad (9.16)$$

For the scalar potential,

$$\Phi(\bar{x}) \simeq \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r^2} \int d^3x' \, \rho(\bar{x}') \, \bar{x}' \cdot \hat{n} \, (1 - ikr)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r^2} \hat{n} \cdot \bar{p} \, (1 - ikr)$$

9.5, cont'd

b) Calculate the electric and magnetic fields from the potentials and show that they are given by (9.18).

$$\bar{H} = \frac{1}{\mu_0} \nabla\times\bar{A} = -i\frac{\cancel{\mu_0}\,\omega}{4\pi}\,\nabla\times\left(\frac{e^{ikr}}{r}\bar{p}\right)$$

$$= -i\frac{\cancel{\mu_0}\,\omega}{4\pi}\left(\nabla\frac{e^{ikr}}{r}\right)\times\bar{p}$$

$$= -i\frac{\cancel{\mu_0}\,\omega}{4\pi}(ik)\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\hat{n}\times\bar{p}$$

$$= \frac{\cancel{\mu_0}\,\omega k}{4\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\hat{n}\times\bar{p}$$

$$\boxed{= \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\hat{n}\times\bar{p}}$$

matching (9.18)

$$\bar{E} = -\nabla\Phi - \frac{\partial A}{\partial t}$$

$$= -\nabla\left[\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\,\hat{n}\cdot\bar{p}\,(1-ikr)\right] - (-i\omega)\bar{A}$$

Write

$$\nabla\left[\frac{e^{ikr}}{r^2}\,\hat{n}\cdot\bar{p}\,(1-ikr)\right] = \nabla\left[\left(\frac{e^{ikr}}{r^3}(1-ikr)\right)\bar{x}\cdot\bar{p}\right]$$

$$= \nabla\left(\frac{e^{ikr}}{r^3}(1-ikr)\right)\bar{x}\cdot\bar{p} + \frac{e^{ikr}}{r^3}(1-ikr)\left(\bar{p} + \bar{x}\,\cancel{\nabla\bar{p}}^{\,0} + i\,\cancel{\nabla\times\bar{p}}^{\,0}\right)$$

9.5 b), cont'd

$$\nabla\left(\frac{e^{ikr}}{r^3}(1-ikr)\right) = \hat{n}\left[ik\frac{e^{ikr}}{r^3}(\cancel{1}-ikr) - 3\frac{e^{ikr}}{r^4}(1-ikr) - ik\cancel{\frac{e^{ikr}}{r^3}}\right]$$

$$= \hat{n}\left[\frac{e^{ikr}}{r^3}\right]\left[k^2 r - 3\left(\frac{1}{r}-ik\right)\right]$$

$$\Rightarrow \nabla\left[\frac{e^{ikr}}{r^2}\hat{n}\cdot\bar{p}(1-ikr)\right]$$

$$= \frac{e^{ikr}}{r^2}\left(k^2 r - 3\left(\frac{1}{r}-ik\right)\right)\hat{n}(\hat{n}\cdot\bar{p}) + \frac{e^{ikr}}{r^3}(1-ikr)\bar{p}$$

$$= \frac{e^{ikr}}{r}k^2\hat{n}(\hat{n}\cdot\bar{p}) - 3e^{ikr}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\hat{n}(\hat{n}\cdot\bar{p}) + \frac{e^{ikr}}{r^3}(1-ikr)\bar{p}$$

hence

$$\bar{E} = -\nabla\Phi - \frac{\partial\bar{A}}{\partial t}$$

$$= -\frac{1}{4\pi\varepsilon_0}\left\{\frac{e^{ikr}}{r}k^2\hat{n}(\hat{n}\cdot\bar{p}) - 3e^{ikr}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\hat{n}(\hat{n}\cdot\bar{p}) + \frac{e^{ikr}}{r^3}(1-ikr)\bar{p}\right\}$$

$$+ i\omega(-i\omega)\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\bar{p}$$

(cont'd)

9.5 b), cont'd

$$\bar{E} = -\frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\hat{n}(\hat{n}\cdot\bar{p}) + \frac{c^2k^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\bar{p}$$

$$+\frac{e^{ikr}}{4\pi\varepsilon_0}\left\{3\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\hat{n}(\hat{n}\cdot\bar{p}) - \frac{1}{r^3}(1-ikr)\bar{p}\right\}$$

Use $c^2\mu_0 = \frac{1}{\varepsilon_0}$, ~~xxx~~

$$(\hat{n}\times\bar{p})\times\hat{n} = -\hat{n}\times(\hat{n}\times\bar{p}) = -(\hat{n}\cdot\bar{p})\hat{n} + \bar{p}$$

$$\boxed{\bar{E} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}(\hat{n}\times\bar{p})\times\hat{n} + \frac{e^{ikr}}{4\pi\varepsilon_0}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\left(3\hat{n}(\hat{n}\cdot\bar{p}) - \bar{p}\right)}$$

matching (9.18)