(a) only

9.17 (a) Treat the linear antenna of problem 9.16,
namely a thin linear antenna of length d excited so that
the sinusoidal current makes one full ~~wa~~ wavelength of oscillation,
by the multipole expansion method.

a) Calculate the multipole moments (electric dipole, magnetic dipole,
and electric quadrupole) exactly and in the long-wavelength
approximation.

Source current density is $\bar{J} = \hat{z} I \sin(kz)\, \delta(x)\delta(y)\, \Theta(\frac{d}{2}-|z|)$,

for $k = \frac{2\pi}{d}$ & $\Theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

Exact expressions for multipole moments: (for the case $\bar{m}=0$)

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x\, Y_{lm}^* \left[c\rho \frac{\partial}{\partial r}\left(r j_l(kr)\right) + ik(\bar{r}\cdot\bar{J})\, j_l(kr) \right] \quad (9.167)$$

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x\, Y_{lm}^*\, \nabla\cdot(\bar{r}\times\bar{J})\, j_l(kr) \quad (9.168)$$

Rewrite current density in spherical coordinates:

$$\bar{J} = \hat{z} I \sin(kz)\, \delta(x)\delta(y)\, \Theta(\tfrac{d}{2}-|z|)$$

$$= \hat{r} \frac{I}{2\pi r^2} \sin(kr)\left[\delta(\cos\theta-1) + \delta(\cos\theta+1)\right] \Theta(\tfrac{d}{2}-r)$$

From the continuity eqn,

$$\rho = \frac{1}{i\omega}\nabla\cdot\bar{J} = \frac{-iI}{2\pi r^2 \omega} k\cos(kr)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]\Theta(\tfrac{d}{2}-r)$$

(+ contribution from δ's & endpts which we omit)

& use $k = \frac{\omega}{c}$

9.17 a), cont'd

For later use,

$$\bar{r}\cdot\bar{J} = \frac{I}{2\pi r}\sin(kr)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

$$\bar{r}\times\bar{J} = 0$$

$$\Rightarrow \boxed{a_M(l,m)=0} \text{ for all } l,m$$

~~$a_E(l,m) = \frac{k^2 I}{2\pi c\sqrt{l(l+1)}}\int d^3x\, Y$~~

$$a_E(l,m) = -\frac{k^2 I}{\sqrt{l(l+1)}}\int d^3x\left[\frac{\cos kr}{2\pi r^2}\frac{\partial}{\partial r}\left(r j_l(kr)\right) - \frac{k}{2\pi r}\sin(kr)\, j_l(kr)\right]$$
$$\hookrightarrow \left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] Y^*_{lm}\, r^2\, dr\, d\Omega$$

$$= -\frac{k^2 I}{\sqrt{l(l+1)}}\delta_{m0}\left[Y^*_{l0}(0,0)+Y^*_{l0}(\pi,0)\right]$$
$$\hookrightarrow \int_0^{d/2} dr\left[\underbrace{\cos kr\,\frac{\partial}{\partial r}\left(r j_l(kr)\right) - kr\sin(kr)\, j_l(kr)}_{=\frac{\partial}{\partial r}\left(r\cos(kr)\, j_l(kr)\right)}\right]$$

$$= -\frac{k^2 I}{\sqrt{l(l+1)}}\delta_{m0}\left[Y^*_{l0}(0,0)+Y^*_{l0}(\pi,0)\right]\left[r\cos(kr)\, j_l(kr)\right]\Big|_0^{d/2}$$

$$= -\frac{k^2 I}{\sqrt{l(l+1)}}\delta_{m0}\left[Y^*_{l0}(0,0)+Y^*_{l0}(\pi,0)\right]\frac{d}{2}\cos(\pi)\, j_l(\pi) \quad \text{using } k=\frac{2\pi}{d}$$

9.17 a), cont'd

Use $Y_{\ell 0}(\theta,\phi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} P_\ell(\cos\theta)$ (3.57)

& $P_\ell(0)=1$, $P_\ell(-x) = (-)^\ell P_\ell(x)$

$$\Rightarrow \boxed{a_E(\ell,m) = -\left(\frac{2\pi}{d}\right)^2 \frac{d}{2} I \left(\frac{2\ell+1}{\ell(\ell+1)}\right)^{1/2} \frac{1}{\sqrt{4\pi}} \left(1+(-)^\ell\right) (-)^\ell j_\ell(\pi)\, \delta_{m0} = +\frac{2\pi}{d} I \left[\frac{(2\ell+1)\pi}{\ell(\ell+1)}\right]^{1/2} \left(\frac{1}{2}\right)\left(1+(-)^\ell\right) j_\ell(\pi)\, \delta_{m0}}$$

(exact answer)

Note that $a_E(\ell,m)=0$ for ℓ odd,

so there is no electric dipole contribution.

Next, consider long-wavelength approximation.
Here,

$$a_E(\ell,m) \simeq \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}\left(Q_{\ell m} + Q'_{\ell m}\right) \qquad (9.169)$$

where

$$\left.\begin{aligned} Q_{\ell m} &= \int d^3x\, r^\ell \rho\, Y^*_{\ell m} \\ Q'_{\ell m} &= \frac{-ik}{(\ell+1)c}\int d^3x\, r^\ell Y^*_{\ell m}\, \nabla\cdot(\vec{r}\times\vec{m}) \end{aligned}\right\} \qquad (9.170)$$

9.17 a), cont'd

Here, $Q'_{\ell m} = 0$ since $\overline{m} = 0$

$$Q_{\ell m} = \int d^3x\, Y^*_{\ell m} r^\ell \frac{(-i)I}{2\pi r^2 c} \cos(kr) \left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right] \Theta\left(\tfrac{d}{2} - r\right)$$

$$= \frac{-iI}{c} \delta_{m0} \left[Y^*_{\ell 0}(0,0) + Y^*_{\ell 0}(\pi, 0) \right] \int_0^{d/2} dr \left(\cos kr\right) r^\ell$$

$$= -i \frac{I}{c} \delta_{m0} \left(\frac{2\ell+1}{4\pi}\right)^{1/2} \left(P_\ell(1) + P_\ell(-1)\right) \int_0^\pi dx \left(\cos x\right) x^\ell\ k^{-\ell-1}$$

$$= -i \frac{I}{c} \delta_{m0} \frac{1}{k^{\ell+1}} \left(\frac{2\ell+1}{4\pi}\right)^{1/2} \left(1 + (-)^\ell\right) \int_0^\pi dx\ x^\ell (\cos x)$$

For $\ell = 0$, $\int_0^\pi dx\ x^\ell \cos x = +\sin x \Big|_0^\pi = 0$

so the first nonvanishing contribution is at $\ell = 2$, consistent with the exact result.

For $\ell = 2$,

$$\int_0^\pi dx\ \underbrace{x^2}_{u} \underbrace{\cos x}_{dv} = +x^2 \sin x \Big|_0^\pi - \int_0^\pi dx\ 2x\ (+\sin x)$$

$$= -2 \int_0^\pi \underbrace{x}_{u} \underbrace{\sin x\ dx}_{dv}$$

$$= -2\left[x(-\cos x)\Big|_0^\pi - \int_0^\pi dx (-\cos x) \right]$$

$$= -2\left[\pi + \sin x \Big|_0^\pi \right]$$

$$= -2\pi$$

$$\Rightarrow Q_{2m} = -i \frac{I}{c} \delta_{m0} \frac{1}{k^3} \left(\frac{5}{4\pi}\right)^{1/2} (2)\ (-2\pi)$$

9.17 a), cont'd

In any event, in the long-wavelength approximation,

$$\boxed{a_E(\ell,m) \simeq \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}(-i)\frac{I}{ck^{\ell+1}}\delta_{m0}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}(1+(-)^\ell)\int_0^\pi dx\, x^\ell \cos x}$$

$$= -\frac{kI}{(2\ell+1)!!}\left[\frac{(\ell+1)(2\ell+1)}{4\pi\ell}\right]^{1/2}(1+(-)^\ell)\int_0^\pi dx\, x^\ell \cos x$$

$$= -\frac{2\pi}{d}\frac{I}{(2\ell+1)!!}\left[\frac{(\ell+1)(2\ell+1)}{4\pi\ell}\right]^{1/2}(1+(-)^\ell)\int_0^\pi dx\, x^\ell \cos x$$

Similarly, in the long-wavelength approximation,

$$a_M(\ell,m) \simeq \frac{ik^{\ell+2}}{(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}(M_{\ell m} + M'_{\ell m}) \qquad (9.171)$$

where

$$\left.\begin{aligned} M_{\ell m} &= -\frac{1}{\ell+1}\int d^3x\, r^\ell Y^*_{\ell m}\, \nabla\cdot(\bar{r}\times\bar{J}) \\ M'_{\ell m} &= -\int d^3x\, r^\ell Y^*_{\ell m}\, \nabla\cdot\bar{m} \end{aligned}\right\} \qquad (9.172)$$

but since both $\bar{m} = 0$ & $\bar{r}\times\bar{J} = 0$,

we see

$\boxed{a_M(\ell,m) = 0}$ in the long-wavelength approximation,

consistent with the exact answer.

9.22 A spherical hole of radius a in a conducting medium can serve as an electromagnetic resonant cavity.

a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies $\omega_{\ell m}$ of the cavity for TE and TM modes.

TE: $\bar{r}\cdot\bar{E}=0$ (= magnetic multipole)

$$\left.\begin{aligned}\bar{E}^{(M)}_{\ell m} &= Z_0\, j_\ell(kr)\,\bar{X}_{\ell m}\\ \bar{H}^{(M)}_{\ell m} &= -\frac{i}{kZ_0}\nabla\times\bar{E}^{(M)}_{\ell m}\end{aligned}\right\}(9.116)$$

w/ g_ℓ taken to be j_ℓ, to ensure well-behaved at origin

$$= -\frac{i}{k}\nabla\times\left(j_\ell(kr)\,\bar{X}_{\ell m}\right)$$

Perfect conductor $\Rightarrow$ $H_\perp=0$, $E_\parallel=0$ at $r=a$.

$$\Rightarrow \bar{r}\cdot\bar{H}\big|_{r=a}=0=\bar{r}\times\bar{E}\big|_{r=a}$$

Both of these $\propto j_\ell(kr)$, hence $j_\ell(ka)=0$

$\Rightarrow$ $k=\frac{x_{\ell n}}{a}$, where $x_{\ell n}$ is the n^{th} root of j_ℓ

$\Rightarrow$ $\boxed{\text{TE frequencies are } \omega=\frac{x_{\ell n}c}{a}}$ where $j_\ell(x_{\ell n})=0$

(note each ℓ frequency has degeneracy $2\ell+1$, from possible m values)

9.22 a), cont'd

<u>TM</u>: $\bar{r}\cdot\bar{H}=0$ (= electric multipoles)

$$\left.\begin{aligned} \bar{H}^{(E)}_{\ell m} &= j_\ell(kr)\,\bar{X}_{\ell m} \\ \bar{E}^{(E)}_{\ell m} &= \frac{iZ_0}{k}\,\nabla\times\bar{H}^{(E)}_{\ell m} \end{aligned}\right\} \quad (9.118)$$

Perfect conductor $\Rightarrow$ $H_\perp=0$, $E_\parallel=0$ at $r=a$

$H_\perp=0$ automatic since $\bar{r}\cdot\bar{X}_{\ell m}=0$

$E_\parallel=0 \Rightarrow \bar{r}\times\left(\nabla\times\left(j_\ell(kr)\,\bar{X}_{\ell m}\right)\right)=0$ at $r=a$

Write

$$\bar{r}\times(\nabla\times\bar{A}) = \nabla(\bar{r}\cdot\bar{A}) - (\bar{r}\cdot\nabla)\bar{A} - \underbrace{(\bar{A}\cdot\nabla)\bar{r}}_{=\bar{A}} - \bar{A}\times\underbrace{(\nabla\times\bar{r})}_{=0} \quad \text{(front covers)}$$

$$= \nabla(\bar{r}\cdot\bar{A}) - (\bar{r}\cdot\nabla)\bar{A} - \bar{A}$$

$$= \nabla(\bar{r}\cdot\bar{A}) - \left(1 + r\frac{\partial}{\partial r}\right)\bar{A}$$

$$= \nabla(\bar{r}\cdot\bar{A}) - \frac{\partial}{\partial r}(r\bar{A})$$

Here, $\bar{A} = j_\ell(kr)\,\bar{X}_{\ell m}$

Since $\bar{r}\cdot\bar{X}_{\ell m}=0$, $\bar{r}\cdot\bar{A}=0$

$$\Rightarrow \bar{r}\times\left(\nabla\times\left(j_\ell(kr)\,\bar{X}_{\ell m}\right)\right) = -\frac{\partial}{\partial r}\left(r j_\ell(kr)\right)\bar{X}_{\ell m}$$

9.22 a), cont'd

Thus, $E_{11} = 0 \Rightarrow$ $\boxed{\omega = \frac{y_{en}\, c}{a}}$ are TM frequencies

where $\frac{d}{dx}\left[x\, j_\ell(x)\right]\Big|_{x = y_{\ell n}} = 0$ defines $y_{\ell n}$